Note

On a problem of Hendry

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A graph $G$ is homogeneously traceable if for every vertex $v$ there is a Hamiltonian path beginning at that vertex. Clearly every Hamiltonian graph is homogeneously traceable. Given a homogeneously traceable graph $G$, we can construct a new graph $H(G)$ with the same vertex set $V(G)$, called the Hamiltonian path graph of $G$ in [1], in which two vertices are adjacent if and only if they are connected by a Hamiltonian path in $G$. Hendry [2] asked whether there are any homogeneously traceable graphs for which $|E(G)| > |E(H(G))|$. The following two theorems partially answer this question.

Theorem 1. If $G$ is Hamiltonian then $H(G)$ has a subgraph isomorphic to $G$ and so $|E(G)| \leq |E(H(G))|$. 

Proof. Let the vertices around the Hamiltonian circuit be labelled $v_1, \ldots, v_n, v_1$ where all suffices will be reduced modulo $n$. Since clearly $v_i$ and $v_{i+1}$ are connected by a Hamiltonian path in $G$ then $v_1, \ldots, v_n, v_1$ is a Hamiltonian circuit in $H(G)$ as well. Also, if $v_i$ is adjacent to $v_j$ in $G$ then it is easy to see that $v_{i-1}$ and $v_{j-1}$ are connected by a Hamiltonian path in $G$ so that $v_{i-1}v_{j-1} \in E(H(G))$. This gives us our result.

Theorem 2. If $G$ is a regular homogeneously traceable graph, then $|E(G)| \leq |E(H(G))|$. 

Proof. Let $r$ be the valency of $G$, and let $v_1$ be an arbitrary vertex of $G$. Further, let $v_1, \ldots, v_n$ be a Hamiltonian path of $G$. For any vertex $v_i$ of $G$ which is
adjacent to $v_n$, we can construct a new Hamiltonian path starting at $v_1$: $v_1, \ldots, v_i, v_n, v_{n-1}, \ldots, v_{i+1}$. Therefore, $G$ has at least $r$ Hamiltonian paths starting at $v_1$. Thus every vertex of $H(G)$ has degree at least $r$, giving us our result. □

However, we now show that this result cannot be extended to all homogeneously traceable graphs.

Suppose $G'$ is a non-Hamiltonian graph that contains an edge $e$ such that for every vertex $v$ in $G'$ there is a Hamiltonian path starting at $v$ and using the edge $e$. The Petersen graph is such a graph where any edge will suffice for $e$. Now if we form a new graph $G$ from the disjoint union $G' \cup K_n$ by joining the two end vertices of $e$ by edges to every vertex in $K_n$, then $G$ is non-Hamiltonian but homogeneously traceable. Since $G$ is non-Hamiltonian then $H(G) \subseteq G$. Clearly if $n$ is large enough $|E(G)| > |E(G')|$ and so $|E(G)| > |E(H(G))|$. If we take $G$ to be the Petersen graph then $n \geq 16$ will suffice.

In fact, we can see that for these graphs we have

$$\frac{|E(H(G))|}{|E(G)|} \to 0 \quad \text{as } n \to \infty.$$

In conclusion we have the following conjecture.

**Conjecture.** If $G$ is a homogeneously traceable graph then $H(G)$ is connected.

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**Reference**
