In this paper, we present necessary optimality conditions and sufficient optimality conditions, and weak, strong, and converse duality theorems for a class of continuous-time generalized fractional programming problems with nonlinear operator inequality and linear operator equality constraints. The primal and dual problems considered in this paper contain, as special cases, the continuous-time analogues of various primal-dual pairs of similar problems previously studied in the areas of finite-dimensional linear, quadratic, and nonlinear programming.

1. INTRODUCTION

The purpose of this paper is to establish optimality conditions and duality relations for the following continuous-time fractional minmax programming problem with nonlinear operator inequality and affine equality constraints:

\[
\inf \max_{1 \leq r \leq m} \left\{ \int_0^T f_r(x)(t) \, dt \right\}
\]

subject to \(x \in \Phi\),

where

\[
\Phi = \{ x \in W^m[0, T] : g(x)(t) \leq 0 \text{ for all } t \in [0, T], \quad h(x)(t) = 0 \text{ for all } t \in [0, T] \},
\]

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\]

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\(W^n[0, T] (\equiv W^n_{2, 1}[0, T])\) is the Hilbert space of all absolutely continuous \(n\)-dimensional vector functions \(t \to x(t) \in \mathbb{R}^n\) (\(n\)-dimensional Euclidean space) defined on the compact interval \([0, T] \subset \mathbb{R}\) with Lebesgue square-integrable derivative \(\dot{x}(t) = dx(t)/dt\), \(\int_0^T \|\dot{x}(t)\|^2 dt = \int_0^T \sum_{i=1}^n (\dot{x}_i(t))^2 dt < \infty\), and with inner product \(\langle \cdot | \cdot \rangle\) defined by

\[
(x | y) = \langle x(0), y(0) \rangle + \int_0^T \langle \dot{x}(t), \dot{y}(t) \rangle dt,
\]

with \(\langle a, b \rangle = \sum_{i=1}^n a_i b_i\) for \(a, b \in \mathbb{R}^n\); \(f_r, r = 1, 2, \ldots, m\), and \(g_i\) (the \(i\)th component of \(g\)), \(i = 1, 2, \ldots, p\), are nonlinear operators from \(W^n[0, T]\) into the space \(C[0, T]\) of all continuous functions defined on \([0, T]\), \(h_j, j = 1, 2, \ldots, q\), and \(k_r, r = 1, 2, \ldots, m\), are continuous affine operators from \(W^n[0, T]\) into \(C[0, T]\), and \(\int_0^T k_r(x(t)) dt > 0\) for all \(x \in W^n[0, T]\) and \(r = 1, 2, \ldots, m\).

Static versions of (P1), called \textit{generalized fractional programming problems}, have recently received much attention in the literature of mathematical programming. These problems have been encountered in multiobjective programming [4], approximation theory [5, 6], goal programming [10, 28], and economics [39].

Duality for a generalized linear fractional programming problem subject to only nonnegativity constraints was originally considered by von Neumann [39] in the context of an economic equilibrium problem. More recently, various duality models and results for generalized linear and nonlinear fractional programs have appeared in [9, 11, 12, 20, 22, 27, 35], and some results pertaining to their computational aspects have been reported in [7, 13, 14, 19, 21, 26].

It is clear that (P1) contains as special cases some interesting classes of continuous-time programming problems. In particular, if we let \(m = 1\), then (P1) reduces to the fractional programming problem

\[
\inf \frac{\int_0^T f_1(x(t)) dt}{\int_0^T k_1(x(t)) dt} \quad \text{subject to } x \in \Phi, \quad (P2)
\]

and if we set \(\int_0^T k_r(x(t)) dt \equiv 1\) for \(r = 1, 2, \ldots, m\), we obtain the minmax problem

\[
\inf \max_{1 \leq r \leq m} \int_0^T f_r(x(t)) dt \quad \text{subject to } x \in \Phi. \quad (P3)
\]

The special case which is obtained from (P1) by choosing \(m = 1\) and \(\int_0^T k_1(x(t)) dt \equiv 1\) was studied in [41].
The following problems are important special cases of (P1)-(P3):

\[
\inf_{1 \leq r \leq m} \max \frac{\int_0^T F_r(x(t), t) \, dt}{\int_0^T K_r(x(t), t) \, dt}
\]

subject to \( G(x(t), t) \leq a(t) + \int_0^t H(x(s), t, s) \, ds, \quad t \in [0, T] \),

\[
Q_1
\]

\[
A(t) x(t) = b(t) + \int_0^t B(t, s) x(s) \, ds, \quad t \in [0, T],
\]

\[
Q_2
\]

where \( F_r, r = 1, 2, \ldots, m, \) are real-valued functions defined on \( \mathbb{R}^n \times [0, T] \), \( K_r(x(t), t) = \langle \beta_r(t), x(t) \rangle + \zeta_r(t) \) with \( \beta_r \in C^0[0, T] \) and \( \zeta_r \in C[0, T] \), \( r = 1, 2, \ldots, m \); \( G_i, a_i, \) and \( H_i \) (the \( i \)th components of \( G, a, \) and \( H \)), \( i = 1, 2, \ldots, p \), are real-valued functions defined on \( \mathbb{R}^n \times [0, T] \), \( [0, T] \), and \( \mathbb{R}^n \times [0, T] \times [0, T] \) respectively; \( b \in C^q[0, T] \), and \( A(t) \) and \( B(t, s) \) are \( q \times n \) matrices whose elements are continuous functions defined on \( [0, T] \) and \( [0, T] \times [0, T] \) respectively;

\[
\inf \frac{\int_0^T F_1(x(t), t) \, dt}{\int_0^T K_1(x(t), t) \, dt}
\]

subject to (1.1) and (1.2);

\[
Q_3
\]

\[
\inf_{1 \leq r \leq m} \max \frac{\int_0^T F_r(x(t), t) \, dt}{\int_0^T K_r(x(t), t) \, dt}
\]

subject to (1.1) and (1.2).

Although continuous-time fractional minmax problems of the above type do not seem to have been studied in the related literature, some other classes of continuous-time problems have been investigated under various assumptions. In particular, optimality criteria and duality results for different types of continuous-time nonlinear programs are discussed in [1–3, 8, 17, 18, 23–25, 30–33, 37, 38, 40, 41], among others. For detailed accounts of finite-dimensional minmax theory and applications, the reader is referred to [15, 16], and for an extensive bibliography on fractional programming to [36].

In this paper, we establish optimality principles and duality relations for (P1) under generalized convexity hypotheses. This is accomplished by treating (P1) as a special case of a more general minmax programming model studied in [41]. In preparation for utilizing the results of [41], in Section 2 we introduce an auxiliary problem and prove its equivalence to (P1). Subsequently, in Sections 3 and 4, we use this equivalence result in conjunction with the results of [41] to obtain optimality conditions and duality statements for (P1).

The results presented here are applicable to certain general classes of constrained variational and optimal control problems. In particular, they
can be applied, under appropriate assumptions, to the following optimal control problem with linear dynamics and with nonlinear integral inequality constraints on the state and control variables:

\[
\inf_{1 \leq r \leq m} \max_{0 \leq t \leq T} \int_0^T \eta_r(x(t), u(t), t) \, dt
\]

subject to

\[
M(t) x(t) + N(t) u(t) + \int_0^t K_r(t, \tau) x(t) \, d\tau + \int_0^t K_c(t, \tau) u(t) \, d\tau = a(t),
\]

\[
\mu(x(t), u(t), t) \leq b(t) + \int_0^t v(x(\tau), u(\tau), t, \tau) \, d\tau,
\]

\[
t \in [0, T], \quad x \in X, \quad u \in U.
\]

2. AN AUXILIARY PROBLEM

Before we can utilize the optimality and duality results of [41] for deriving similar results for (P1), it is necessary to somehow transform (P1) to the same minmax format as that of the principal problem studied in [41], that is, we must show that (P1) is equivalent to a problem of the form:

\[
\inf_{x \in W^n[0, T]} \sup_{y \in Y} \psi(x, y),
\]

by properly specifying the function \( \psi \) and the set \( Y \). To this end, we begin with the following simple observation.

**Lemma 2.1.** For all \( x \in W^n[0, T] \), we have that

\[
\phi(x) \equiv \max_{1 \leq r \leq m} \frac{\int_0^T f_r(x)(t) \, dt}{\int_0^T k_r(x)(t) \, dt} = \max_{u \in \Pi} \frac{\int_0^T \sum_{r=1}^m u_r f_r(x)(t) \, dt}{\int_0^T \sum_{r=1}^m u_r k_r(x)(t) \, dt} \equiv \gamma(x),
\]

where

\[
\Pi = \left\{ u \in R^m : u \geq 0, \sum_{r=1}^m u_r = 1 \right\}.
\]

**Proof.** Since

\[
\phi(x) = \frac{\int_0^T f_r(x)(t) \, dt}{\int_0^T k_r(x)(t) \, dt} \quad \text{for} \quad r = 1, 2, ..., m,
\]
it follows that

\[ \phi(x) \int_0^T \sum_{r=1}^m u, k_r(x(t)) dt \geq \int_0^T \sum_{r=1}^m u, f_r(x(t)) dt, \]

and therefore \( \phi(x) \geq \gamma(x) \). On the other hand, it is easily seen that

\[ \gamma(x) \geq \frac{\int_0^T f_r(x(t)) dt}{\int_0^T k_r(x(t)) dt} \quad \text{for } r = 1, 2, \ldots, m, \]

and so \( \gamma(x) \geq \phi(x) \); hence \( \phi(x) = \gamma(x) \).

With the help of the above lemma, we next show that (P1) is equivalent to the following problem:

\[ \inf_{x \in W^m[0, T]} \sup_{(u, v, w) \in \Gamma} \psi(x, u, v, w), \quad \text{(AP)} \]

where

\[ \psi(x, u, v, w) = \frac{\int_0^T \left( \langle u, f(x(t)) \rangle + \langle v(t), g(x(t)) \rangle + \langle w(t), h(x(t)) \rangle \right) dt}{\int_0^T \langle u, k(x(t)) \rangle dt}, \]

\[ \Gamma = \Pi \times W^p_+[0, T] \times W^q[0, T], \]

\[ W^p_+[0, T] = \{ v \in W^p[0, T] : v(t) \geq 0 \text{ for all } t \in [0, T] \}, \]

\[ f = (f_1, f_2, \ldots, f_m), \]

and

\[ k = (k_1, k_2, \ldots, k_m). \]

**Lemma 2.2.** (P1) is feasible if and only if \( \inf_{x \in W^m[0, T]} \sup_{(u, v, w) \in \Gamma} \psi(x, u, v, w) < +\infty \), in which case the optimal values of (P1) and (AP) are equal; that is, \( \inf_{x \in W^m[0, T]} \sup_{(u, v, w) \in \Gamma} \psi(x, u, v, w) = \inf_{x \in \Phi} \phi(x) \). In this case, (P1) and (AP) are equivalent in that a one-to-one correspondence exists between their optimal solutions.

**Proof.** Since

\[ \sup_{(u, v, w) \in \Gamma} \psi(x, u, v, w) = \begin{cases} \sup_{u \in \Pi} \int_0^T \langle u, f(x(t)) \rangle dt \equiv \gamma(x) & \text{if } x \in \Phi, \\ +\infty & \text{otherwise,} \end{cases} \quad (2.1) \]

and since by Lemma 2.1, \( \gamma(x) = \phi(x) \) for all \( x \in W^m[0, T] \), it follows that there are two mutually exclusive possibilities:
(i) (P1) is feasible, and \( \inf_{x \in \Phi} \phi(x) = \inf_{x \in W^n[0, T]} \sup_{(u, v, w) \in \Gamma} \psi(x, u, v, w) < +\infty \),

(ii) (P1) is not feasible, and \( \inf_{x \in W^n[0, T]} \sup_{(u, v, w) \in \Gamma} \psi(x, u, v, w) = +\infty \).

This establishes the first assertion. Now let \( x^* \) be an optimal solution of (P1). Then \( x^* \in \Phi \) and hence in view of (2.1) and (i), \((x^*, u^*, 0, 0)\) with \( u^* \) chosen such that \( \gamma(x^*) = \phi(x^*) \), is an optimal solution of (AP). Conversely, let \((x^*, u^*, v^*, w^*)\) be an optimal solution of (AP). If \( \gamma(x^*, u^*, v^*, w^*) = +\infty \), then (P1) is infeasible. If \( \gamma(x^*, u^*, v^*, w^*) < +\infty \), then from (2.1) and (i) it follows that \( x^* \) is optimal for (P1).

Now the results of [4] can be applied to (AP) and hence to (P1) if we can ensure that the following conditions are fulfilled:

(C1) The set \( \Gamma \) is closed and convex;

(C2) The function \( \gamma(\cdot, u, v, w) \) is strictly quasiconvex on \( W^n[0, T] \) for every fixed \((u, v, w) \in \Gamma\), and \( \phi(x, \cdot, \cdot, \cdot) \) is strictly quasiconcave on \( \Gamma \) for every fixed \( x \in W^n[0, T] \).

Clearly, the set \( \Gamma \), being a product of closed and convex sets, is itself closed and convex; hence (C1) is satisfied. In order to make sure that (C2) also holds, we henceforth require that the following assumption be satisfied:

(A) The operators \( f_r, r = 1, 2, \ldots, m, \) and \( g_i, i = 1, 2, \ldots, p, \) are convex and continuously Fréchet differentiable on \( W^n[0, T] \).

Using (A), one can easily verify, in a manner similar to the finite-dimensional case [29], that the function \( \psi(\cdot, u, v, w) \) is pseudoconvex on \( W^n[0, T] \) for every fixed \((u, v, w) \in \Gamma\) and \( \psi(x, \cdot, \cdot, \cdot) \) is pseudoconcave on \( \Gamma \) for every fixed \( x \in W^n[0, T] \). Since a pseudoconvex (pseudoconcave) function is strictly quasiconvex (strictly quasiconcave), we see that (C2) is also satisfied.

Incidentally, if differentiability is not assumed, then (A) can be utilized to show that \( \psi(\cdot, u, v, w) \) is quasiconvex. However, since a quasiconvex function is not in general strictly quasiconvex, in this case (C2) may not be ensured.

3. Optimality Conditions

Making use of the equivalence result established for (P1) and (AP), we now specialize the optimality results of [4] for (P1). For the definitions of the regularity conditions (constraint qualifications) which appear in the
Theorem 3.1. Let $x^* \in \Phi$ be an optimal solution of $(P1)$, let $u^* \in \Pi$ be chosen such that $\gamma(x^*) = \phi(x^*)$ (so that $(x^*, u^*, 0, 0)$ is an optimal solution of $(AP)$), and assume that either one of the following regularity conditions is satisfied:

(a) $\psi$ has the low-value property at $(x^*, u^*, 0, 0)$;

(b) There exists a closed ball $B(u^*, 0, 0; \varepsilon)$ (centered at $(u^*, 0, 0)$ with radius $\varepsilon > 0$) in $\Gamma$ such that $\psi(x^*, u^*, 0, 0) > \psi(x^*, u, v, w)$ for all $(u, v, w)$ on the boundary of $B(u^*, 0, 0; \varepsilon) \cap \Gamma$.

Then there exists $(u^0, v^0, w^0) \in \Gamma$ such that

(i) $\psi(x^*, u, v, w) < \psi(x^*, u^0, v^0, w^0)$ for all $x \in W^m[0, T]$ and all $(u, v, w) \in \Gamma$;

(ii) $\int_0^T \left[ \sum_{r=1}^m u_r^0 Df_r(x^*) + \sum_{i=1}^p v_i^0(t) Dg_i(x^*) + \sum_{j=1}^q w_j^0(t) Dh_j(x^*) \right] z(t) dt$

$- \psi(x^*, u^0, v^0, w^0) \int_0^T \sum_{r=1}^m u_r^0 Dk_r(x^*) z(t) dt = 0$

for all $z \in W^m[0, T]$, (3.2)

$\langle v^0(t), g(x^*)(t) \rangle = 0$ for all $t \in [0, T]$, (3.3)

where $D\sigma(x^*) z(t)$ denotes the Fréchet derivative of $\sigma$ at $x^*$ evaluated at $z(t)$.

Proof. By Lemma 2.2 of the preceding section and Theorem 2.1 (if (a) holds) and Theorem 2.4 (if (b) holds) of [41], there exists $(u^0, v^0, w^0) \in \Gamma$ such that (3.1) is satisfied. From the second inequality of (3.1) it follows that $x^*$ minimizes the function $\psi(\cdot, u^0, v^0, w^0)$ over $W^m[0, T]$ and hence we must have $D\psi(x^*, u^0, v^0, w^0) z = 0$ for all $z \in W^m[0, T]$, which yields (3.2). Since $x^*$ is feasible and $v^0 \in W^p_\sigma[0, T]$, it follows that $\langle v^0(t), g(x^*)(t) \rangle \leq 0$ for all $t \in [0, T]$. However, if strict inequality holds for a $t \in (0, T)$, then because of the continuity of the function $t \mapsto \langle v^0(t), g(x^*)(t) \rangle$ on $[0, T]$, there is an interval $J \subset [0, T]$ (containing $t$) such that $\langle v^0(t), g(x^*)(t) \rangle < 0$ for all $t \in J$ and thus $\int_J \langle v^0(t), g(x^*)(t) \rangle dt < 0$, which contradicts the first inequality of (3.1) with $u = u^0, v = 0, w = w^0$: hence (3.3) holds.

We next present two sufficiency results; the first one (Theorem 3.2) is valid without any convexity or differentiability assumptions, while
the second (Theorem 3.3) depends on the pseudoconvexity property of \( \psi(\cdot, u, v, w) \).

**Theorem 3.2.** Let \( x^* \in W^n[0, T] \) and assume that there exists \((u^*, v^*, w^*) \in \Gamma \) such that

\[
P \int_0^T \langle u^*, f(x)(t) \rangle \, dt = \sup_{u \in \Pi} \int_0^T \langle u, f(x)(t) \rangle \, dt \tag{3.4}
\]

for all \( x \in \Phi \), and

\[
\psi(x^*, u^*, v^*, w^*) \leq \psi(x, u^*, v^*, w^*) \leq \psi(x, u^*, v^*, w^*) \tag{3.5}
\]

for all \( x \in W^n[0, T] \) and all \((u, v, w) \in \Gamma \). Then \( x^* \) is an optimal solution of (P1).

**Proof.** Using the first inequality of (3.5), it can be shown as in the proof of Theorem 3.1 of [41] that \( x^* \) is a feasible solution of (P1) and \( \langle v^*(t), g(x^*)(t) \rangle = 0 \) for all \( t \in [0, T] \). Now using this conclusion, the second inequality of (3.5), and Lemma 2.1, it is easily seen that \( x^* \) is optimal for (P1).

**Theorem 3.3.** Let \( x^* \in \Phi \) and assume that there exists \( u^* \in \Pi \) such that (3.4) holds, and there exist \( v^* \in W^p[0, T] \) and \( w^* \in W^n[0, T] \) such that (3.2) and (3.3) are satisfied with \((x^*, u^0, v^0, w^0) \) replaced by \((x^*, u^*, v^*, w^*) \). Then \( x^* \) is an optimal solution of (P1).

**Proof.** Let \( x \) be an arbitrary feasible solution of (P1). Since (3.2) is equivalent to \( D\psi(x^*, u^*, v^*, w^*)(x - x^*) = 0 \) with \( z = x - x^* \), from the pseudoconvexity property of \( \psi(\cdot, u^*, v^*, w^*) \) it follows that \( \psi(x^*, u^*, v^*, w^*) \leq \psi(x, u^*, v^*, w^*) \). Because of (3.3) and feasibility of \( x \), this inequality reduces to

\[
P \int_0^T \langle u^*, f(x^*)(t) \rangle \, dt \leq \int_0^T \langle u^*, f(x)(t) \rangle \, dt \leq \int_0^T \langle u^*, k(x)(t) \rangle \, dt.
\]

Now the assertion follows from (3.4) and Lemma 2.1.

We conclude this section with a brief discussion of the explicit form of the foregoing optimality conditions in terms of the data of (Q1). For the purpose of computing (3.2), we assume that the functions \( F_r(\cdot, t), r = 1, 2, \ldots, m, G(\cdot, t), \) and \( H(\cdot, t, s) \) are continuously differentiable on \( R^n \) for all \( t, s \in [0, T] \), and that the functions \( t \rightarrow \nabla F_r(x(t), t) \) (gradient of \( F_r(\cdot, t) \) at \( x(t) \)), \( r = 1, 2, \ldots, m, t \rightarrow \nabla G_i(x(t), t) \), \( s \rightarrow \nabla H_i(x(s), t, s) \), and \( t \rightarrow \nabla H_i(x(s), t, s) \) \( i = 1, 2, \ldots, p, \) are continuous on \([0, T]\).

Now we can express (3.2) for (Q1) as follows:
\[ \int_0^T \left[ \sum_{r=1}^m u_r^0 \langle \nabla F_r(x^*(t), t) - \vec{\psi} \beta_r(t), z(t) \rangle 
\right. \\
+ \sum_{i=1}^p \langle v_i^0(t) \nabla G_i(x^*(t), t), z(t) \rangle \\
- \int_0^t \sum_{i=1}^p \langle v_i^0(t) \nabla H_i(x^*(s), t, s), z(s) \rangle \, ds \\
+ \left. \left\langle w^0(t), A(t) z(t) - \int_0^t B(t, s) z(s) \, ds \right\rangle \right] \, dt = 0 \]

for all \( z \in W^n[0, T] \),

where \( \vec{\psi} \) is the same as \( \psi \) expressed in terms of the data of (Q1) evaluated at \( (x^*, u^0, v^0, w^0) \). Applying Fubini's theorem [34] to the double integrals in the above expression, we obtain

\[ \int_0^T \left[ \sum_{r=1}^m u_r^0 \langle \nabla F_r(x^*(t), t) - \vec{\psi} \beta_r(t), z(t) \rangle 
\right. \\
+ \sum_{i=1}^p \langle v_i^0(t) \nabla G_i(x^*(t), t), z(t) \rangle \\
- \int_T^t \sum_{i=1}^p \langle v_i^0(s) \nabla H_i(x^*(t), s, t), z(s) \rangle \, ds + A'(t) w^0(t) \\
- \int_0^T B'(s, t) w^0(s) \, ds \right] \, dt = 0 \quad \text{for all} \quad z \in W^n[0, T], \]

where prime denotes transposition. Clearly, this expression implies that the right-hand factor inside the inner product sign must be equal to zero for all \( t \in [0, T] \). Hence, (3.2) and (3.3) for (Q1) become

\[ \sum_{r=1}^m u_r^0 \langle \nabla F_r(x^*(t), t) - \vec{\psi} \beta_r(t), z(t) \rangle \\
+ \sum_{i=1}^p \langle v_i^0(t) \nabla G_i(x^*(t), t), z(t) \rangle \\
- \int_t^T \sum_{i=1}^p \langle v_i^0(s) \nabla H_i(x^*(t), s, t), z(s) \rangle \, ds + A'(t) w^0(t) \\
\int_t^T B'(s, t) w^0(s) \, ds = 0 \quad \text{for all} \quad t \in [0, T], \]

\[ \left\langle v^0(t), G(x^*(t), t) - a(t) - \int_0^t H(x^*(s), t, s) \, ds \right\rangle = 0 \quad \text{for all} \quad t \in [0, T]. \]
4. Duality

In this section, we identify a dual problem for (P1) and obtain appropriate direct and converse duality relations. As in the preceding section, we utilize the intermediate problem (AP) for specializing the duality results of [41] for (P1).

According to [41, Theorem 2.1], the following problem is dual to (AP) and hence to (P1):

\[
\sup_{(u,v,w) \in \Gamma} \inf_{x \in W^0[0,T]} \psi(x, u, v, w). \tag{DP1}
\]

Comparing (P1) with (DP1), we observe that these problems are not of the same type in that (P1) is a “discrete” infsup problem whose objective function contains a finite number of ratios, while (DP1) is a “continuous” supinf problem in which the outer optimization process takes place on an infinite set.

We first establish a weak duality relationship between (P1) and (DP1).

**Theorem 4.1 (Weak Duality).** Let \( x^* \) and \((x_0^*, u_0^*, v_0^*, w_0^*)\), with \( \psi(x_0^*, u_0^*, v_0^*, w_0^*) = \inf_{x \in W^0[0,T]} \psi(x, u_0^*, v_0^*, w_0^*) \), be arbitrary feasible solutions of (P1) and (DP1), respectively. Then \( 4(x^*) \geq \psi(x_0^*, u_0^*, v_0^*, w_0^*) \).

**Proof.** Since \( g(x(t)) < 0 \) and \( h(x^*)(t) = 0 \) for all \( t \in [0, T] \), we have

\[
\psi(x_0^*, u_0^*, v_0^*, w_0^*) = \inf_{x \in W^0[0,T]} \psi(x, u_0^*, v_0^*, w_0^*) \leq \psi(x^*, u_0^*, v_0^*, w_0^*)
\]

and hence the desired inequality follows from Lemma 2.1.

In view of Lemma 2.2, the following duality results for the pair (P1)–(DP1) are special cases of Theorems 2.1–2.4 of [41].

**Theorem 4.2 (Strong Duality).** Let \( x^* \in \Phi \) be an optimal solution of (P1), let \( u^* \in \Pi \) be chosen such that \( \phi(x^*) = \psi(x^*, u^*, 0, 0) \) is an optimal solution of (AP), and assume that either (a) or (b) of Theorem 3.1 is satisfied. Then there exists \( (u_0^*, v_0^*, w_0^*) \in \Gamma \) such that \( \phi(x^*) = \psi(x^*, u_0^*, v_0^*, w_0^*) = \psi(x^*, u^*, 0, 0) \).

**Theorem 4.3 (Converse Duality).** Let \( (x_0^*, u_0^*, v_0^*, w_0^*) \) be an optimal solution of (DP1) and assume that either one of the following regularity conditions is satisfied:
(a) \( \psi \) has the high-value property at \((x^0, u^0, v^0, w^0)\);

(b) There exists a closed ball \(B(x^0; \epsilon)\) such that \(\psi(x^0, u^0, v^0, w^0) < \psi(x, u^0, v^0, w^0)\) for all \(x\) on the boundary of \(B(x^0; \epsilon)\).

Then there exists \(x^* \in W^n[0, T]\) such that \(x^*\) is an optimal solution of \((P1)\) and \(\psi(x^0, u^0, v^0, w^0) = \psi(x^*, u^0, v^0, w^0) = \phi(x^*)\).

We next briefly discuss some special cases of \((DP1)\).

First, we note that the dual problems for \((P2)\) and \((P3)\) take the following forms:

\[
\sup_{x \in W^n[0, T]} \inf_{T} \left[ \int_0^T [f_1(x)(t) + \langle v(t), g(x)(t) \rangle + \langle w(t), h(x)(t) \rangle] \, dt \right] \frac{\int_0^T k_1(x)(t) \, dt}{T}
\]

subject to \(v \in W^n_p[0, T], w \in W^n[0, T]\); \((DP2)\)

\[
\sup_{(u, v, w) \in \Gamma} \inf_{x \in W^n[0, T]} \left[ \int_0^T \left[ \langle u, f(x)(t) \rangle + \langle v(t), g(x)(t) \rangle \right] \, dt + \langle w(t), h(x)(t) \rangle \right] \, dt.
\]

\((DP3)\)

In view of our differentiability assumptions, it is clear that the Lagrangian dual problem \((DP1)\) leads to the following Wolfe-type duality formulation:

\[
\sup \int_0^T \left[ \left\{ \sum_{r=1}^m u_r, Df_r(x) \right\} + \sum_{i=1}^n v_i(t) \, Dg_i(x) + \sum_{j=1}^q w_j(t) \, Dh_j(x) \right] z(t) \, dt
\]

subject to

\[
\int_0^T \left[ \sum_{r=1}^m u_r, Dk_r(x) \right] z(t) \, dt = 0 \quad \text{for all } z \in W^n[0, T],
\]

\(u \geq 0, \sum_{r=1}^m u_r = 1, v \in W^n_d[0, T], w \in W^n[0, T], x \in W^n[0, T].\)

Obviously, similar Wolfe-type dual problems can be formulated for \((DP2)\) and \((DP3)\).
The duality formulations (DP1) and (DP1) contain as special cases the continuous-time versions of a large number of duality models previously proposed and investigated for finite-dimensional nonlinear programming problems. These special cases can easily be specified by appropriate choices of \( f_r, k_r, m, g, \) and \( h. \) Here, for simplicity, we consider a special case of (P1) which involves only quadratic and linear functions, and determine the explicit forms of the resulting pairs of primal and dual problems.

Consider the following problem:

\[
\inf \max \frac{\int_0^T \left[ \frac{1}{2} \langle x(t), P_r(t) x(t) \rangle + \langle x_r(t), x(t) \rangle + \xi_r(t) \right] dt}{\int_0^T \left[ \langle \beta_r(t), x(t) \rangle + \zeta_r(t) \right] dt}
\]

subject to

\[
C(t) x(t) \geq e(t) + \int_0^t D(t, s) x(s) ds \quad \text{for all } t \in [0, T],
\]

\[
x(t) \geq 0 \quad \text{for all } t \in [0, T],
\]

where \( P_r(t), r = 1, 2, \ldots, m, \) are \( n \times n \) symmetric positive semidefinite matrices whose elements are continuous functions defined on \([0, T]\), \( \alpha_r, \beta_r \in C^\infty[0, T], \zeta_r, \zeta \in C[0, T], \) \( r = 1, 2, \ldots, m, e \in C^\infty[0, T], \) and \( C(t) \) and \( D(t, s) \) are \( p \times n \) matrices whose elements are continuous functions defined on \([0, T] \times [0, T]\), respectively.

According to (DP1), the dual of (SP1) can be stated as follows:

\[
\sup \frac{1}{\int_0^T \sum_{r=1}^m u_r \left[ \langle \beta_r(t), x(t) \rangle + \zeta_r(t) \right] dt}
\]

\[
\times \int_0^T \left\{ \sum_{r=1}^m u_r \left[ \frac{1}{2} \langle x(t), P_r(t) x(t) \rangle + \langle \alpha_r(t), x(t) \rangle + \zeta_r(t) \right] \right. \\
\left. + \langle v(t), -C(t) x(t) + e(t) + \int_0^t D(t, s) x(s) ds \rangle + \langle w(t), -x(t) \rangle \right\} dt
\]

subject to

\[
\int_0^T \left\{ \sum_{r=1}^m u_r \left[ x(t) P_r(t) \right] \right. \\
\left. + \langle v(t), -C(t) x(t) + \int_0^t D(t, s) z(s) ds \rangle + \langle w(t), -z(t) \rangle \right\} dt
\]

\[
\times \int_0^T \sum_{r=1}^m u_r \left[ \langle \beta_r(t), x(t) \rangle + \zeta_r(t) \right] dt
\]

\[
- \int_0^T \left\{ \sum_{r=1}^m u_r \left[ \frac{1}{2} \langle x(t), P_r(t) x(t) \rangle + \langle \alpha_r(t), x(t) \rangle \right] + \zeta_r(t) \right\}
\]
\[+ \left< v(t), -C(t) x(t) + e(t) + \int_0^t D(t, s) x(s) \, ds \right>\]

\[+ \left< w(t), -x(t) \right> \, dt \]

\[\times \int_0^T \left< \sum_{r=1}^m u_r \beta_r(t), z(t) \right> \, dt = 0 \quad \text{for all } z \in W^n[0, T],\]

\[u \geq 0, \quad \sum_{r=1}^m u_r = 1, \quad v \in W^p_+(0, T), \quad w \in W^n_+(0, T).\]

Evidently, (SP1) and (SD1) contain several interesting pairs of primal and dual problems. However, we specify only two special cases in which the dual problems take somewhat simpler forms:

\[\inf \max_{1 \leq r \leq m} \int_0^T \left[ \frac{1}{2} \left< x(t), P_r(t) x(t) \right> + \left< \alpha_r(t), x(t) \right> \right] \, dt \quad (SP2)\]

subject to (4.1) and (4.2);

\[\sup \int_0^T \left\{ \sum_{r=1}^m \left[ \frac{1}{2} \left< x(t), P_r(t) x(t) \right> + \xi_r(t) \right] + \left< e(t), v(t) \right> \right\} \, dt\]

subject to

\[C(t) v(t) \leq \sum_{r=1}^m \left[ P_r(t) x(t) + \alpha_r(t) \right] + \int_t^T D'(s, t) v(s) \, ds \quad \text{for all } t \in [0, T],\]

\[v(t) \geq 0 \quad \text{for all } t \in [0, T].\]

\[\inf \max_{1 \leq r \leq m} \int_0^T \left< \alpha_r(t), x(t) \right> \, dt \quad (SP3)\]

subject to (4.1) and (4.2);

\[\sup \int_0^T \left< e(t), v(t) \right> \, dt\]

subject to

\[C(t) v(t) \leq \sum_{r=1}^m \alpha_r(t) + \int_t^T D'(s, t) v(s) \, ds \quad \text{for all } t \in [0, T],\]

\[v(t) \geq 0 \quad \text{for all } t \in [0, T].\]
Letting $m=1$ in (SP2)–(SD2) and (SP3)–(SD3), we obtain primal-dual pairs of continuous-time quadratic and linear programming problems. Duality for these categories of problems has been treated previously in the literature by different methods. Most of the publications dealing with continuous-time linear and quadratic programs are listed in [40].

5. CONCLUDING REMARKS

As a consequence of a general minmax approach developed in [41], in this paper we have established optimality conditions and duality theorems for a continuous-time minmax programming problem with Fréchet differentiable convex operator inequality and linear operator equality constraints. As pointed out earlier, this problem contains a number of important special cases which, in turn, may be viewed as continuous-time analogues of similar problems previously studied in the area of finite-dimensional nonlinear programming.

Although the principal problem (P1) was formulated on the Hilbert space $W^n[0, T]$, as discussed in [41], the results of this paper, with the exception of Theorem 3.1(b) and Theorem 4.3(b), are valid for any Banach function space. Consequently, in view of the minmax method employed in [41], it is possible to use different appropriate function spaces for the original space in (P1), for the range spaces of the constraint operators, and for the space of multiplier functions. In particular, instead of $W^n[0, T]$ in (P1) one can use the space $L^n_\infty[0, T]$ of all (equivalence classes of) Lebesgue measurable essentially bounded $n$-dimensional vector functions defined on $[0, T]$. Obviously, this latitude has important implications for the scope and applicability of the results presented in this paper.

REFERENCES