Normalized system of functions with respect to the Laplace operator and its applications

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Abstract

A system of functions 0-normalized with respect to the operator $\Delta$ in some domain $\Omega \subset \mathbb{R}^n$ is constructed. Application of this system to boundary value problems for the polyharmonic equation is considered. Connection between harmonic functions and solutions of the Helmholtz equation is investigated.

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1. Introduction

Notion of $f$-normalized system of functions with respect to a partial differential operator was introduced earlier by author in [9] for construction and investigation of polynomial solutions to a linear PDE with constant coefficients. It is a generalization of normalized systems of functions introduced by Bondarenko in [6]. These notions allowed to construct basic systems of polynomial solutions to a general form of linear PDE with constant coefficients. It means that the notion of $f$-normalized system of functions works well for investigation of polynomial solutions to PDE but what about nonpolynomial $f$-normalized systems of functions. Is it possible to construct an explicit formula for 0-normalized system of functions with respect to a simple PDO having nonpolynomial components? In present paper we are trying to answer this question and to construct an 0-normalized system of

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functions with respect to the operator $\Delta$ in some domain $\Omega \subset \mathbb{R}^n$ (Section 2) and to apply this system for investigation of the Riquier’s problem [12] (Section 3) and Helmholtz equation (Section 4).

Let $L_1, L_2$ be commuting linear partial differential operators acting on functions which belong to the vector space $\mathcal{X}$ such that $L_k \mathcal{X} \subset \mathcal{X}$ ($k = 1, 2$) and defined in some domain $\Omega \subset \mathbb{R}^n$.

**Definition** [9]. An infinite ordered system of functions $\{f_k(x) | k = 0, 1, 2, \ldots \}$ ($f_k \in \mathcal{X}$) is called $f$-normalized with respect to $L_1$ in the domain $\Omega$, having a base $f_0(x)$ if everywhere in that domain

$$L_1 f_0(x) = f(x), \quad L_1 f_k(x) = f_{k-1}(x), \quad k \in \mathbb{N}.$$ 

The main property of a system of functions $f$-normalized with respect to $L_1$ in the domain $\Omega$ is the following: the series 

$$u(x) = \sum_{k=0}^{\infty} L_k^2 f_k(x) \quad (\ast)$$

is a formal solution to the equation $L_1 u(x) - L_2 u(x) = f(x)$ in $\Omega$.

Idea of a $f$-normalized system of functions is not absolutely new. If we consider 0-normalized system of functions with respect to the operator $L - \lambda$ then we get an eigenfunction (a base of the system) and associated functions for the operator $L$. Moreover, application of $f$-normalized systems of functions is mainly based on using formula (\ast). This formula is well known for some special types of operators $L_1$ and $L_2$. For example, let $L_2$ be any first order linear PDO with coefficients not depending on $t$ and $L_1 = \partial / \partial t$ then we can choose 0-normalized system of functions with respect to the operator $L_1$ in the form $f_k(t, x) = (t^k / k!) g(x)$. In this case in (\ast) we get a Lie series [7].

Another well-known example of special case of (\ast) is a representation of harmonic functions [2,5]. To get this case we should take, in (\ast), $L_1 = \partial^2 / \partial t^2$, $L_2 = -\Delta$, and

$$f_k(t, x) = \frac{t^{2k}}{2k!} g_1(x) + \frac{t^{2k+1}}{(2k+1)!} g_2(x).$$

Moreover, if $n = 1$ and $g_2 = g'_1$ then (\ast) can be written in the form

$$u(t, x) = \cos \left( t \frac{\partial}{\partial x} \right) g_1(x) + \sin \left( t \frac{\partial}{\partial x} \right) g_3(x),$$

which was known even by Euler. Of course, the obtained harmonic functions have a very special form and there are a lot of more general and useful representations of harmonic and polyharmonic functions [2,13]. The operator $G_k(x; u)$ in (1) giving polyharmonic functions is similar to an operator used in [2] (Theorem 2.2) for Almansi expansion of holomorphic functions.

Suppose we would like to apply formula (\ast) for construction a solution to a linear PDE with operator $L(D)$. First of all we need to represent the operator $L(D)$ in the form $L_1(D) - L_2(D)$, where $L_1(D)$ is a “simple” operator, that is to decompose the operator $L(D)$. On this point of view using (\ast) is an application of a decomposition method [1].
2. Normalized system of functions

In this section we are going to construct 0-normalized system of functions with respect to the Laplace operator \( \Delta \) in some bounded domain \( \Omega \subset \mathbb{R}^n \) possessing the starlike property, \( \forall t \in [0, 1], x \in \Omega \Rightarrow tx \in \Omega \). For this we define on \( \Omega \) the following sequence of functions:

\[
G_0(x; u) = u(x) \\
G_k(x; u) = \frac{1}{4^k k!} \int_0^1 (1 - \alpha)^{k-1} \alpha^{n/2-1} \frac{u(\alpha x)}{(k-1)!} d\alpha,
\]

where \( u(x) \) is a harmonic function in \( \Omega \).

**Theorem 1.** The system of functions \( \{G_k(x; u) | k = 0, 1, \ldots\} \) is 0-normalized with respect to the operator \( \Delta \) in \( \Omega \).

**Proof.** First we prove that the system of polynomials \( \{F_k(x; u_m) | k = 0, 1, \ldots\} \), where

\[
F_k(x; u_m) = \frac{|x|^{2k}}{(2, 2)_k (n + 2m, 2)_k} u_m(x),
\]

\( (a, b)_k = a(a + b) \ldots (a + bk - b) \) with the convention \( (a, b)_0 = 1 \), and \( u_m(x) \) is a homogeneous harmonic polynomial of order \( m \), is a 0-normalized system with respect to the operator \( \Delta \) in \( \mathbb{R}^n \). These polynomials were considered in [8] for construction of a system of harmonic polynomials. For this reason according to the definition we need to check that

\[
\Delta F_k(x; u_m) = F_{k-1}(x; u_m).
\]

Indeed it is not hard to see that

\[
\Delta (u_m(x)|x|^l) = l \left( (l + n - 2)u_m(x) + 2 \sum_{i=1}^n x_i D_{x_i} u_m(x) \right) |x|^{l-2}
\]

for \( l \geq 2 \). Using homogeneity of the polynomial \( u_m(x) \) we get

\[
\Delta (u_m(x)|x|^l) = l(l + 2m + n - 2)u_m(x)|x|^{l-2}, \quad x \in \mathbb{R}^n.
\]

If we take \( l = 2k \) and divide both sides of (3) by \( (2, 2)_k (n + 2m, 2)_k \) we get the desired equality (2).

Finally for \( k = 0 \) we have \( \Delta F_0(x; u_m) = \Delta u_m(x) = 0 \). So, the system of polynomials \( \{F_k(x; u_m) | k = 0, 1, \ldots\} \) is 0-normalized with respect to the operator \( \Delta \) in \( \mathbb{R}^n \).

Obviously \( 0 \in \Omega \) and therefore we can expand the harmonic function \( u(x) \) in some neighborhood of the origin \( \Omega_0 \) in terms of homogeneous harmonic polynomials \( u_m(x) \).

Let us transform polynomials \( F_k(x; u_m) \) for \( k > 0 \). Using equalities

\[
\frac{1}{(n + 2m, 2)_k} = \frac{(n, 2)_k}{(n, 2)_{k+m}}, \quad \frac{(n, 2)_k}{(n, 2)_{k+m}} = 2^k \Gamma(n/2 + s) \Gamma(n/2),
\]

where \( \Gamma(n) \) is Euler’s \( \Gamma \)-function, we get

\[
\frac{1}{(n + 2m, 2)_k} = \frac{1}{2^k \Gamma(k)} \frac{\Gamma(n/2 + m + k) \Gamma(k)}{\Gamma(n/2 + m + k)}.
\]
If we apply here the well-known connection between $\Gamma$- and $B$-function [4] we get
\[
\frac{1}{(n+2m, 2k)} = \frac{1}{2} \int_0^1 \alpha^m (1 - \alpha)^{k-1} \frac{\alpha^{n/2 - 1}}{(2k - 2)!!} d\alpha,
\]
where $2k!! = 2 \cdot 4 \ldots 2k$.

Taking into account that $(2, 2)_k = 2k!!$ we can write the polynomial $F_k(x; u_m)$ in the form
\[
F_k(x; u_m) = \left| x \right|^{2k} \int_0^1 \frac{(1 - \alpha)^{k-1}}{(2, 2)_k (2k - 2)} u_m(\alpha x) \frac{\alpha^{n/2 - 1}}{(k-1)!} d\alpha.
\]

It is not hard to see that it implies that $F_k(x; u_m) = G_k(x; u_m)$ for $k > 0$ and therefore for every $u_m(x)$ the system $\{G_k(x; u_m) \mid k = 0, 1, \ldots\}$ is 0-normalized with respect to $\Delta$ in $\mathbb{R}^n$. Let us sum the polynomials $G_k(x; u_m)$ on $m$ for $x \in \Omega_0$. We get
\[
\sum_{m=0}^{\infty} G_k(x; u_m) = G_k(x; u) = \frac{\left| x \right|^{2k}}{4^k k!} \int_0^1 \frac{(1 - \alpha)^{k-1} \alpha^{n/2 - 1}}{(k-1)!} u(\alpha x) d\alpha.
\]

Because of the possibility to differentiate and to integrate under the summation sign in $\Omega_0$ we can write $\Delta G_k(x; u) = 0$ and
\[
\Delta G_k(x; u) = \sum_{m=0}^{\infty} \Delta G_k(x; u_m) = \sum_{m=0}^{\infty} \Delta G_{k-1}(x; u_m) = G_{k-1}(x; u), \quad x \in \Omega_0.
\]

Therefore the system of functions $\{G_k(x; u) \mid k = 0, 1, \ldots\}$ defined by (1) is 0-normalized with respect to operator $\Delta$ in $\Omega_0$.

It is obvious that the functions $G_k(x; u)$ are defined on the whole of $\Omega$ and moreover $G_k(x; u) \in C^2(\Omega)$ because of the ability to differentiate with respect to a parameter under the integration sign.

Let us check directly that the equality $\Delta G_k(x; u) = G_{k-1}(x; u)$ is fulfilled not only in $\Omega_0$ but at any point of $\Omega$. Taking into account identities
\[
\Delta(vw) = v\Delta w + w\Delta v + 2 \sum_{i=1}^n \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i}, \quad \Delta \sum_{i=1}^n \frac{\partial u}{\partial x_i}(\alpha x) = \frac{\partial u}{\partial \alpha},
\]
and equality (3) for $l = 2k$ and $m = 0$ we get
\[
\Delta G_k(x; u) = \frac{2 \left| x \right|^{2k-2}}{4^k (k-1)!} \int_0^1 \frac{(1 - \alpha)^{k-1} \alpha^{n/2 - 1}}{(k-1)!} \times \left( (2k + n - 2)u(\alpha x) + 2\alpha \frac{\partial}{\partial \alpha} u(\alpha x) \right) d\alpha.
\]

Let $k > 1$. Applying integration by parts we obtain
\[
\Delta G_k(x; u) = \frac{2}{4^k (k-1)!} \int_0^1 \left( 1 - \alpha \right)^{k-2} \alpha^{n/2-1} (2k - 2) u(\alpha x) \, d\alpha
\]
\[
= \frac{2}{4^k (k-1)!} \int_0^1 \left( 1 - \alpha \right)^{k-2} \alpha^{n/2-1} (2k - 2) u(\alpha x) \, d\alpha = G_{k-1}(x; u).
\]

If \( k = 1 \) then according to (4) we can write
\[
\Delta G_k(x; u) = \frac{1}{2} \int_0^1 \left( nu(\alpha x) + 2\alpha \frac{\partial}{\partial \alpha} u(\alpha x) \right) \, d\alpha
\]
\[
= \int_0^1 \alpha^{n/2} \frac{\partial}{\partial \alpha} u(\alpha x) + \frac{\partial}{\partial \alpha} (\alpha^{n/2}) u(\alpha x) \, d\alpha
\]
\[
= \alpha^{n/2} u(\alpha x)|_0^1 = u(x) \equiv G_0(x; u).
\]

Therefore the system of functions \( \{G_k(x; u) \mid k = 0, 1, \ldots \} \) is 0-normalized with respect to the operator \( \Delta \) in \( \Omega \). \( \square \)

3. The polyharmonic equation

Consider the following boundary value problem for the polyharmonic equation called the Riquier’s problem [6,11]. Find a function \( u \in C^{2m}(\Omega) \) such that
\[
\Delta^k u \in C(\overline{\Omega}) \text{ for } k = 0, \ldots, m-1 \text{ and }
\Delta^m u = 0, \quad x \in \Omega,
\]
\[
\Delta^k u|_{\partial \Omega} = f_k(s), \quad s \in \partial \Omega, \quad k = 0, \ldots, m-1.
\]

**Theorem 2.** Let the functions \( f_k \) be continuous on \( \partial \Omega \). Then problem (5) has a solution.

**Proof.** Let us explicitly construct the solution to problem (5). We represent it in the form
\[
u(x) = \sum_{k=0}^{m-1} G_k(x; u(k)),
\]where \( u(k)(x) \) is a solution of the following Dirichlet problem:
\[
\Delta u(k) = 0, \quad x \in \Omega, \quad u(k) \in C(\overline{\Omega}),
\]
\[
u(k)(x)|_{\partial \Omega} = f_k(s) - \sum_{i=1}^{m-k-1} G_i(x; u(i+k+1))|_{\partial \Omega}.
\]
Because \( u_{(k)}(x) \) are harmonic functions, from (1) it follows that \( u \in C^{2m}(\Omega) \) and moreover we can see that \( \Delta^m G_k(x; v) = 0 \) for any harmonic \( v \) and \( 0 \leq k \leq m - 1 \). Therefore, if we take \( v \) such that \( 0 \leq v \leq m - 1 \) then we get

\[
\Delta^v u(x) = \sum_{k=v}^{m-1} G_{k-v}(x; u_{(k)}) = u_{(v)}(x) + \sum_{i=1}^{m-v-1} G_i(x; u_{(i+v)}). \tag{8}
\]

Letting \( x \to \partial\Omega \) and using (7) for \( k = v \) we obtain \( \Delta^v u(x)|_{\partial\Omega} = f_v(s) \).

The above consideration is valid because all functions from the right-hand side of (8) are continuous on \( \Omega \).

So, to find a solution of problem (5) we need to resolve for \( k = m - 1, \ldots, 0 \) the Dirichlet problems (7), and then to plug the obtained solutions \( u_{(k)}(x) \) into (6). Functions \( u_{(k)}(x) \) exist because all functions \( G_i(x; u_{(j)}) \) under the summation in (7) are continuous on \( \partial \Omega \) since \( f_j \in C(\partial \Omega) \) for \( k + 1 \leq j \leq m - 1 \). Hence continuity of functions \( f_k \) for \( k = 0, \ldots, m - 1 \) is sufficient for existence of solution (6) to the considered problem (5).

\[ \blacksquare \]

Remark. If we consider the Dirichlet boundary value problem for the polyharmonic equation \([13]\) then continuity of the boundary functions \( f_k \) is not enough for existence of solution \( f_k \in C^{m-k}(\partial \Omega) \) see \([10]\)). Such is the case because smoothness of solution we claim in the problem \( \Delta^k u \in C(\Omega) \) for \( k = 0, \ldots, m - 1 \) differ from the smoothness of solution for the standard Dirichlet problem \( u \in C^{m-1}(\Omega) \).

4. The Helmholtz equation

In this section we investigate the connection between solutions of the Helmholtz equation

\[
\Delta v(x) + \lambda v(x) = 0, \quad x \in \Omega, \; \lambda \in \mathbb{R}, \tag{9}
\]

and harmonics in \( \Omega \). Let us introduce the function \( g_m(t) \) depending on the real parameter \( m \),

\[
g_m(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2,2)_k(m,2)_k} \frac{t^k}{k!}.
\]

It is obvious that \( g_m(t) \) is an entire function for \( m \notin -2\mathbb{N} \cup \{0\} \). If \( m \geq 0 \) is an integer and \( J_m(t) \) is the first order Bessel function then we can get the representation

\[
J_m(t) = \frac{t^m}{2^m \Gamma(m+1)} g_{2m+2}(t^2). \tag{10}
\]

Denote by \( \Omega_0 \subset \Omega \) some neighborhood of the origin.
**Lemma 1.** There exists such $\Omega_0$ that for any $v \in C^2(\Omega)$ satisfying in $\Omega$ Eq. (9) there exists a function $u(x)$ harmonic in $\Omega_0$ and satisfying in $\Omega_0$ the equality

$$v(x) = u(x) - \frac{\lambda}{4} \int_0^1 g_4(\lambda(1-\alpha)|x|^2)u(\alpha x)\alpha^{n/2-1} d\alpha. \quad (11)$$

**Proof.** Because the function $v(x) \exp(\sqrt{\lambda} x_{n+1})$ is an harmonic function in $\Omega \times \mathbb{R}$, $v(x)$ is an analytic function in $\Omega$. According to [2,3] we can find unique harmonic functions $u^{(0)}(x), \ldots, u^{(k)}(x), \ldots$ such that the following equality holds in $\Omega_0$:

$$v(x) = \sum_{k=0}^{\infty} |x|^{2k} u^{(k)}(x). \quad (12)$$

Let us find the functions $u^{(k)}(x)$. If we use the 0-normalized system of functions constructed in Theorem 1 then we get

$$v(x) = \sum_{k=0}^{\infty} (-\lambda)^k G_k(x; u) = u(x) - \frac{\lambda}{4} \sum_{k=0}^{\infty} (-\lambda)^k \frac{(1-\alpha)^k |x|^{2k}}{(2,2)_k(2,2)_k} u(\alpha x)\alpha^{n/2-1} d\alpha$$

$$= u(x) - \frac{\lambda}{4} \int_0^1 g_4(\lambda(1-\alpha)|x|^2)u(\alpha x)\alpha^{n/2-1} d\alpha.$$

Taking $u(x) = \sum_{m=0}^{\infty} x_{n+1}$ we can rewrite it in the form

$$v(x) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-\lambda)^k |x|^{2k}}{(2,2)_k(n+2m,2)_k} u_m(x) = \sum_{m=0}^{\infty} g_{2m+n} (\lambda |x|^2) u_m(x). \quad (13)$$

On the other hand if we substitute $v(x)$ from (12) into Eq. (9) and equate terms having the same power of $|x|$ we can easily obtain for $k \geq 0$,

$$\lambda u^{(k)}(x) = -(2k+2)((2k+n)u^{(k+1)}(x) + 2\Lambda u^{(k+1)}(x)),$$

where $\Lambda = x_1 \partial / \partial x_1 + \cdots + x_n \partial / \partial x_n$. Taking here $u^{(k)}(x)$ as a series of homogeneous polynomials $u^{(k)}_m(x)$ and using the equality $\Lambda u^{(k)}_m = mu^{(k)}_m$ we get

$$u^{(k)}_m(x) = \frac{\lambda u^{(k-1)}_m(x)}{2k(2m+n+2k-2)} = \frac{(-\lambda)^k u^{(0)}_m(x)}{(2,2)_k(n+2m,2)_k}.$$

Substituting $u^{(k)}_m(x)$ to (12) and changing the summation order we get (13), i.e., formula (11) holds in $\Omega_0$. \hfill \Box

**Theorem 3.** For any $v \in C^2(\Omega)$ satisfying in $\Omega$ Eq. (9) there exists a function $u(x)$ harmonic in $\Omega$ and such that (11) holds.
Proof. Let \( v(x) \) satisfy the hypothesis. We show that the local result obtained in Lemma 1 also holds in case \( \Omega_0 = \Omega \).

Let us substitute (11) into (9) assuming that \( u \in C^2(\Omega) \). We can easily check that for \( \varphi \in C^2(\Omega) \) the following formula is true:

\[
\Delta (\varphi(\lambda|x|)u(\alpha x)) = \alpha^2 \varphi(\lambda|x|) \Delta u(\alpha x) \\
+ \frac{\lambda}{|x|} \varphi'(\lambda|x|) ((n - 1)u(\alpha x) + 2\alpha \Delta u(\alpha x)) \\
+ \lambda^2 \varphi''(\lambda|x|) u(\alpha x).
\]

We will use it for calculating \( \Delta v(x) \).

Let \( \lambda > 0 \). Then denoting \( f(t) = t^2/4 g_4(t^2) \) from (11) we obtain

\[
\Delta v = \Delta u - \int_0^1 \left[ \alpha^2 f\left(\sqrt{\lambda(1 - \alpha)} \frac{|x|}{|\alpha x|}\right) \Delta u(\alpha x) \\
+ \frac{\sqrt{\lambda(1 - \alpha)}}{|x|} ((n - 1)u(\alpha x) + 2\alpha u'(\alpha x)) f'\left(\sqrt{\lambda(1 - \alpha)} \frac{|x|}{|\alpha x|}\right) \\
+ \lambda(1 - \alpha) f''\left(\sqrt{\lambda(1 - \alpha)} \frac{|x|}{|\alpha x|}\right) u(\alpha x) \right] \frac{\alpha^{n/2-1}}{1 - \alpha} \, d\alpha.
\]

Therefore using integration by parts,

\[
2 \sqrt{\lambda} \frac{1}{|x|} \int_0^1 f'\left(\sqrt{\lambda(1 - \alpha)} \frac{|x|}{|\alpha x|}\right) u'(\alpha x) \frac{\alpha^{n/2-1}}{1 - \alpha} \, d\alpha \\
= \lambda u - \frac{\sqrt{\lambda}}{|x|} \int_0^1 \left[ \left( n \frac{\alpha}{\sqrt{1 - \alpha}} + \frac{\alpha}{\sqrt{1 - \alpha}} \right) f'\left(\sqrt{\lambda(1 - \alpha)} \frac{|x|}{|\alpha x|}\right) \\
- \alpha \sqrt{\lambda} \frac{|x|}{|\alpha x|} f''\left(\sqrt{\lambda(1 - \alpha)} \frac{|x|}{|\alpha x|}\right) \right] u(\alpha x) \frac{\alpha^{n/2-1}}{1 - \alpha} \, d\alpha,
\]

and so we get

\[
\Delta v = \Delta u - \lambda u - \int_0^1 f\left(\sqrt{\lambda(1 - \alpha)} \frac{|x|}{|\alpha x|}\right) \Delta u(\alpha x) \frac{\alpha^{n/2+1}}{1 - \alpha} \, d\alpha \\
+ \frac{\lambda}{\sqrt{\lambda(1 - \alpha)} |x|} f'\left(\sqrt{\lambda(1 - \alpha)} \frac{|x|}{|\alpha x|}\right) \\
- f''\left(\sqrt{\lambda(1 - \alpha)} |x|\right) u(\alpha x) \frac{\alpha^{n/2-1}}{1 - \alpha} \, d\alpha.
\]
We can simplify the obtained equality with help of the property of the function $f(t)$, $f''(t) - t^{-1} f'(t) + f(t) = 0$ for $t \in \mathbb{R}/\{0\}$, which can be easily verified. So we have

$$\Delta v(x) = \Delta u(x) - \lambda v(x) - \int_0^1 f\left(\sqrt{\lambda(1-\alpha)} |x|\right) \Delta u(ax) \frac{\alpha^{n/2+1}}{1-\alpha} d\alpha.$$  

Therefore assuming that $v(x)$ is a solution to (9) we can easily get the equation for defining $u(x)$. Coming back from the function $f(t)$ to the function $g_4(t)$ we can write that equation in the form

$$\Delta u(x) - \frac{|x|^2}{4} \int_0^1 g_4(\lambda(1-\alpha)|x|^2) \Delta u(ax) \alpha^{n/2+1} d\alpha = 0, \quad x \in \Omega.$$  

The same result can be obtained for $\lambda < 0$. In this case in (11) we need to make the change of variables $\lambda = -\mu$ and $\psi(t) = -(t^2/4)g_4(-t^2)$ and repeat all calculations made for $\lambda > 0$. In this case the following property of function $\psi(t)$ should be used: $\psi''(t) - t^{-1} \psi'(t) - \psi(t) = 0$ for $t \in \mathbb{R}/\{0\}$ and $\psi'(t)/(2t)|_{t=0} = -1$.

Let us consider equality (11) as an equation with respect to the unknown function $u(x)$ and known function $v(x)$.

**Lemma 2.** If $v \in C^2(\Omega)$ then Eq. (11) has unique solution $u \in C^2(\Omega)$.

**Proof.** Let $\Omega^{(c)} = \varepsilon \hat{\Omega}$ for $\varepsilon \in (0, 1)$. It is obvious that $\alpha \Omega^{(c)} \subset \Omega^{(c)}$ for $\alpha \in [0, 1]$ and there exists $C > 0$ such that $|v|, |v_{\alpha}|, |v_{\alpha \alpha}| < C$ for all $0 \leq i, j \leq n$ and $x \in \Omega^{(c)}$.

We define the function $u_k(x)$ by the recurrence relation

$$u_k(x) = v(x) + \int_0^1 g_4(\lambda(1-\alpha)|x|^2) u_{k-1}(\alpha x) \alpha^{n/2-1} d\alpha, \quad x \in \Omega^{(c)},$$

assuming that $u_0 = v$. It is clear that $u_k \in C^2(\Omega^{(c)})$.

Let us estimate the expression $|u_k - u_{k-1}|$. Denoting $d = \sup_{x \in \Omega^{(c)}} |x|$ and $C_1 = |\lambda| g_4(-\lambda)|d^2|/4$ we get

$$|u_k(x) - u_{k-1}(x)| \leq C_1 |x|^2 \int_0^1 |u_{k-1}(\alpha x) - u_{k-2}(\alpha x)| d\alpha,$$

from which we can derive $|u_k(x) - u_{k-1}(x)| \leq C C_1^k |x|^{2k} \int_0^1 \alpha^{2k-2} \ldots \int_0^1 \alpha^2 \int_0^1 \alpha d\gamma \ldots d\alpha$ and therefore

$$|u_k(x) - u_{k-1}(x)| \leq C \frac{(C_1 |x|^2)^k}{(2k-1)!!}.$$  

Noticing that $u_k = \sum_{i=1}^k (u_i - u_{i-1}) + u_0$ and then using estimate (16) we can conclude that there exists a function $u_\varepsilon \in C(\Omega^{(c)})$ such that $u_k(x) \to u_\varepsilon(x)$ for $k \to \infty$ uniformly on $x \in \Omega^{(c)}$. 

Let us show that the function \( u_\varepsilon(x) \) is differentiable on \( \Omega(\varepsilon) \). We can calculate that

\[
\frac{\partial u_k}{\partial x_i} = \frac{\partial v}{\partial x_i} + \lambda \frac{x_i}{2} \int_0^1 \left[ g_4 \left( \lambda (1 - \alpha) |x|^2 \right) + \lambda (1 - \alpha) |x|^2 \partial g_4^{(\lambda (1 - \alpha) |x|^2)} \right] u_{k-1}(\alpha x) \alpha^{n/2 - 1} \, d\alpha \\
+ \lambda \frac{|x|^2}{4} \int_0^1 g_4 \left( \lambda (1 - \alpha) |x|^2 \right) \frac{\partial u_{k-1}}{\partial x_i}(\alpha x) \alpha^{n/2} \, d\alpha.
\]

(17)

If we denote \( C_2 = |\lambda| (g_4(\lambda) - |\lambda| d^2 g_4'(\lambda)) / 2 \) then we obtain

\[
\left| \frac{\partial u_k}{\partial x_i} - \frac{\partial u_{k-1}}{\partial x_i} \right| \leq C_2 |x| \int_0^1 \left| u_{k-1}(\alpha x) - u_{k-2}(\alpha x) \right| \, d\alpha \\
+ C_1 |x|^2 \int_0^1 \left| \frac{\partial u_{k-1}}{\partial x_i}(\alpha x) - \frac{\partial u_{k-2}}{\partial x_i}(\alpha x) \right| \, d\alpha.
\]

(18)

Using here estimate (16) we get

\[
\left| \frac{\partial u_k}{\partial x_i} - \frac{\partial u_{k-1}}{\partial x_i} \right| \leq CC_2 \frac{C_1^{k-1} |x|^{2k-1}}{(2k-2)!!} \\
+ \lambda \frac{|x|^2}{4} \int_0^1 \left[ g_4 \left( \lambda (1 - \alpha) |x|^2 \right) + \lambda (1 - \alpha) |x|^2 \partial g_4^{(\lambda (1 - \alpha) |x|^2)} \right] u_{k-1}(\alpha x) \alpha^{n/2} \, d\alpha
\]

and therefore

\[
\left| \frac{\partial u_k}{\partial x_i} - \frac{\partial u_{k-1}}{\partial x_i} \right| \leq CC_1^{k} |x|^{2k} \int_0^1 \alpha^{2k-2} \ldots \int_0^1 \gamma^2 \, d\gamma \ldots d\alpha.
\]

(19)

The obtained estimate leads to the existence of \( w \in C(\Omega(\varepsilon)) \) such that \( \partial u_k / \partial x_i \rightarrow w(x) \) for \( k \rightarrow \infty \) uniformly on \( x \in \Omega(\varepsilon) \). Hence \( w(x) = \partial u_\varepsilon / \partial x_i \) and therefore \( u_\varepsilon \in C^1(\Omega(\varepsilon)) \).

Now we prove that \( u_\varepsilon \in C^2(\Omega(\varepsilon)) \). Differentiation of (17) gives

\[
\frac{\partial^2 u_k}{\partial x_i \partial x_j} = \frac{\partial^2 v}{\partial x_i \partial x_j} + \lambda \frac{x_i x_j}{2} \int_0^1 \left[ 2(1 - \alpha) g_4^{(\lambda (1 - \alpha) |x|^2)} + \lambda (1 - \alpha)^2 |x|^2 g_4''(\lambda (1 - \alpha) |x|^2) \right] u_{k-1}(\alpha x) \alpha^{n/2 - 1} \, d\alpha \\
+ \lambda \frac{x_i}{2} \int_0^1 g_4(\lambda (1 - \alpha) |x|^2) + \lambda (1 - \alpha) |x|^2 g_4'(\lambda (1 - \alpha) |x|^2) \right] \frac{\partial u_{k-1}}{\partial x_i}(\alpha x) \alpha^{n/2} \, d\alpha
\]

and

\[
\times \frac{\partial u_{k-1}}{\partial x_i}(\alpha x) \alpha^{n/2} \, d\alpha.
\]
\[ + \lambda \frac{x_j}{2} \int_0^1 \left[ g_4(\lambda(1-\alpha)|x|^2) \right. \]

\[ + \lambda(1-\alpha)|x|^2 g_4'(\lambda(1-\alpha)|x|^2) \left. \frac{\partial u_{k-1}}{\partial x_j}(ax)\alpha^{n/2} \right] d\alpha \]

\[ + \lambda \frac{|x|^2}{4} \int_0^1 g_4(\lambda(1-\alpha)|x|^2) \left. \frac{\partial^2 u_{k-1}}{\partial x_i \partial x_j}(ax)\alpha^{n/2+1} \right] d\alpha. \]

If \( i = j \) then to the right-hand side of the above formula it is necessary to add the expression

\[ \frac{\lambda}{2} \int_0^1 \left[ g_4(\lambda(1-\alpha)|x|^2) + \lambda(1-\alpha)|x|^2 g_4'(\lambda(1-\alpha)|x|^2) \right] u_{k-1}(ax)\alpha^{n/2} d\alpha. \]

If we denote \( C_3 = |\lambda|^3 d^4 g_4''(-|\lambda|d^2) - 2|\lambda|^2 d^2 g_4'(-|\lambda|d^2) + C_2 \) then by analogy with (15) and (18) we can obtain

\[ \left| \frac{\partial^2 u_k}{\partial x_i \partial x_j} - \frac{\partial^2 u_{k-1}}{\partial x_i \partial x_j} \right| \leq C_3 \int_0^1 |u_{k-1}(ax) - u_{k-2}(ax)| d\alpha \]

\[ + C_2 |x| \int_0^1 \left| \left( \frac{\partial u_{k-1}}{\partial x_i}(ax) - \frac{\partial u_{k-2}}{\partial x_i}(ax) \right) \right| d\alpha \]

\[ + \left| \frac{\partial u_{k-1}}{\partial x_j}(ax) - \frac{\partial u_{k-2}}{\partial x_j}(ax) \right| \right| d\alpha \]

\[ + C_1 |x|^2 \int_0^1 \left| \frac{\partial^2 u_{k-1}}{\partial x_i \partial x_j}(ax) - \frac{\partial^2 u_{k-2}}{\partial x_i \partial x_j}(ax) \right| d\alpha. \]

Using (16) and (19) we get for \( k \geq 2 \),

\[ \left| \frac{\partial^2 u_k}{\partial x_i \partial x_j} - \frac{\partial^2 u_{k-1}}{\partial x_i \partial x_j} \right| \leq \left( C C_3 + 2(k-1) C C_2^2 / C_1 + C d \right) \frac{C_1^{k-1} |x|^{2k-2}}{(2k-3)!!} \]

\[ + C_1 |x|^2 \int_0^1 \left| \frac{\partial^2 u_{k-1}}{\partial x_i \partial x_j}(ax) - \frac{\partial^2 u_{k-2}}{\partial x_i \partial x_j}(ax) \right| d\alpha. \]

Therefore for \( k \geq 2 \) we have estimate

\[ \left| \frac{\partial^2 u_k}{\partial x_i \partial x_j} - \frac{\partial^2 u_{k-1}}{\partial x_i \partial x_j} \right| \leq C k \left( C C_3 + (k-1) C C_2^2 + C_1 d \right) \frac{C_1^{k-2} |x|^{2k-2}}{(2k-3)!!} \]

\[ + C \frac{(C_1 |x|^2)^k}{(2k-1)!!}. \]
It allows us to conclude that there exists such a function \( w \in C^2(\Omega^{(\varepsilon)}) \) that \( \partial^2 u_k/\partial x_i \partial x_j \to w(x) \) for \( k \to \infty \) uniformly on \( x \in \Omega^{(\varepsilon)} \). Because of \( \partial u_k/\partial x_i \to \partial u_\varepsilon/\partial x_i \) we have \( u(x) = \partial^2 u_\varepsilon/\partial x_i \partial x_j \) and therefore \( u_\varepsilon \in C^2(\Omega^{(\varepsilon)}) \).

Now we can construct the desired function \( u(x) \). It is not difficult to see that \( u_\varepsilon(x) = u_\mu(x) \) for \( x \in \Omega^{(\varepsilon)} \cap \Omega^{(\mu)} \) and therefore we can define the function \( u(x) \) in the form \( u(x) = u_\varepsilon(x) \) for \( x \in \Omega^{(\varepsilon)} \). It is obvious that the constructed function \( u(x) \) satisfies Eq. (11) and possesses the property \( u \in C^2(\Omega) \).

Let us investigate the uniqueness of solution of Eq. (11). Assume that there exists a nonzero function \( u \in C^2(\Omega) \) which is a solution of the homogeneous equation (11). By analogy with (15) it is not difficult to derive

\[
|u(x)| < C_1 |x|^2 \int_0^1 |u(\alpha x)| \, d\alpha, \quad x \in \Omega,
\]

and hence for any natural number \( k \),

\[
|u(x)| \leq \frac{(C_1 |x|^2)^k}{(2k - 1)!!} \sup_{\alpha \in [0,1]} u(\alpha x).
\]

Therefore if \( \sup_{\alpha \in [0,1]} u(\alpha x) < \infty \) for \( x \in \Omega \) then \( u(x) = 0 \).

\[\square\]

**Remark.** The lemma’s result remains true if the number \( n \) in Eq. (11) is not connected with dimension of \( \Omega \) and everywhere in the lemma’s statement instead of \( C^2(\Omega) \) we take \( C(\Omega) \).

Let us go back to Eq. (14). If in (11) we make the change of variables \( u = \Delta u \) and take \( n = n + 4 \), \( v(x) = 0 \) then it turns into (14). Moreover because \( u \in C^2(\Omega) \) we have \( \Delta u \in C(\Omega) \). Therefore in this case we can use Remark to Lemma 2. According to it, since the function \( \Delta u(x) \) is defined on \( \Omega \), \( \Delta u(x) = 0 \) for \( x \in \Omega \), i.e., \( u(x) \) is a harmonic function in \( \Omega \).

Thus let \( v(x) \) be an arbitrary solution to Eq. (9). Substitute it into (11). According to Lemma 2 there exists a function \( u \in C^2(\Omega) \) such that (11) turns into the identity. Moreover since \( v(x) \) is a solution of (9) then \( u(x) \) is a harmonic function in \( \Omega \).

We can rewrite formula (11) in terms of Bessel functions. From (10) we derive

\[
g_4(t^2) = \frac{2}{t} J_1(t)
\]

and therefore for \( \lambda > 0 \), (11) has the form

\[
v(x) = u(x) - \sqrt{\lambda} \frac{|x|}{2} \int_0^1 J_1(\sqrt{\lambda(1 - \alpha)} |x|) u(\alpha x) \frac{\alpha^{n/2 - 1}}{\sqrt{1 - \alpha}} \, d\alpha.
\]

\[\text{Theorem 4.} \quad \text{Any function } u(x) \text{ harmonic in } \Omega \text{ can be uniquely represented in the form}
\]

\[
u(x) = u(x) + \lambda \frac{|x|^2}{4} \int_0^1 g_4(-\lambda \alpha (1 - \alpha)|x|^2) u(\alpha x) \alpha^{n/2 - 1} \, d\alpha,
\]

where \( v(x) \) is a solution of Eq. (9).
Proof. First we prove (20) in some neighborhood of the origin \( \Omega_0 \subset \Omega \). Let \( v(x) = \sum_{m=0}^{\infty} v_m(x) \) be a solution of Eq. (9) in \( \Omega_0 \) such that \( \forall m \in \mathbb{N}, |v_m(x)| < M \) in \( \Omega_0 \). According to the representation \( v(x) \) through the harmonic functions (13) we are going to search for a harmonic function among functions of the form

\[
u(x) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} C_{k,m} |x|^{2k} v_m(x).
\]

It is not difficult to see that

\[
\Delta u(x) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left( C_{k+1,m} (2k+2)(2k+2m+n) - \lambda C_{k,m+2} \right) |x|^{2k} v_m(x).
\]

From here we can derive that \( C_{k,m} = \lambda/(2k(2k+2m+n-2)C_{k-1,m+2} \). Hence,

\[
C_{k,m} = \frac{\lambda^k}{(2,2)_k(2k+2m+n-2,2)_k} C_{0,m+2k}.
\]

Taking \( C_{0,m} = 1 \) we write \( u(x) \) in the form

\[
u(x) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^k |x|^{2k}}{(2,2)_k(2k+2m+n-2,2)_k} v_m(x). \tag{21}
\]

Let us transform the obtained formula. We have

\[
u(x) = v(x) + \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{\lambda^k |x|^{2k}}{k!(k+m+n/2-1,1)_k} v_m(x),
\]

from which

\[
u(x) = v(x) + \frac{|x|^2}{4} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^k |x|^{2k}}{(k+1)!(k+m+n/2,1)_k} v_m(x).
\]

Now using equalities \( (k+m+n/2,1)_k+1 = \Gamma(2k+m+n/2+1)/\Gamma(k+m+n/2) \), \( B(k+m+n/2, k+1) = \Gamma(k+m+n/2)\Gamma(k+1)/\Gamma(2k+m+n/2+1) \) we get

\[
u(x) = v(x) + \frac{|x|^2}{4} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^k |x|^{2k}}{(k+1)!\Gamma(k+1)} B(k+m+n/2, k+1) v_m(x).
\]

Hence

\[
u(x) = v(x) + \frac{|x|^2}{4} \int_0^1 \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^k |x|^{2k}}{(2,2)_k(4,2)_k} (\alpha(1-\alpha))^k v_m(\alpha x) \alpha^{n/2-1} d\alpha.
\]

The obtained formula can be easily transformed into the form (20).

Because \( |v_m(x)| < M, \forall m \in \mathbb{N} \), the double series from (21) converge absolutely in \( \Omega_0 \). Moreover because the series that represents the function \( v(x) \) in \( \Omega_0 \) is infinitely differentiable then the function \( u(x) \) from (21) is also infinitely differentiable function in \( \Omega_0 \) and hence \( u(x) \) is a harmonic function in \( \Omega_0 \).
Let us come back to the domain \( \Omega \). Let \( u(x) \) be a given function harmonic in \( \Omega \). We define function \( u(x) \) as a solution of the integral equation (20). By analogy with Lemma 2 we can prove the following statement.

**Lemma 3.** If \( u \in C^2(\Omega) \) then Eq. (20) has a unique solution \( v \) in \( C^2(\Omega) \).

**Remark.** If everywhere in the lemma’s statement we write \( C(\Omega) \) instead of \( C^2(\Omega) \) then it remains true.

By repeating the reasoning of Theorem 3 but for Eq. (20) and using instead of Lemma 2 the above Lemma 3 we can construct the proof. ✷

**Theorem 5.** Let \( u \in C(\Omega) \) be a solution of Eq. (11) then for \( n \geq 2 \) it can be represented in the form (20).

**Proof.** Let us fix \( u'(x) \) and let \( v(x) \) be a solution of Eq. (20). Denote by \( u(x) \) a solution of Eq. (11) with the given function \( v(x) \). According to Lemmas 2 and 3 for \( u' \in C(\Omega) \) and \( n \geq 2 \) the functions \( v(x) \) and \( u(x) \) exist and \( v, u \in C(\Omega) \).

First we prove the theorem in some neighborhood of the origin \( \Omega_0 \) such that formulas (13) and (21), which correspond in \( \Omega_0 \) to Eqs. (11) and (20), are fulfilled. Substitute \( v(x) \) from (13) to (21) and consider the case when \( u(x) = u_m(x) \). We have

\[
u'(x) = \sum_{k=0}^{\infty} \frac{k^k(|x|/2)^{2k}}{k!(2s + k + m + n/2 - 1, 1)_k} \sum_{s=0}^{\infty} \frac{(-\lambda)^s(|x|/2)^{2s}}{s!(m + n/2, 1)_s} u_m(x).
\]

Hence making summation on \( k + s = p \) and using the equality

\[
\frac{1}{(s + p + m + n/2 - 1, 1)_{p-1}} \frac{1}{(m + n/2, 1)_s} = \frac{(s + m + n/2, 1)_{p-1}}{(m + n/2, 1)_{2p-1}},
\]

we obtain

\[
u'(x) = u_m(x) + \sum_{p=1}^{\infty} \frac{\lambda^p(|x|/2)^{2p}}{p!(m + n/2, 1)_{2p-1}} \sum_{s=0}^{p} (-1)^s \binom{p}{s} (s + m + n/2, 1)_{p-1} u_m(x).
\]

Since \( (s + m + n/2, 1)_{p-1} \) is a polynomial of degree \( p - 1 \) in \( s \) then

\[
\sum_{s=0}^{p} (-1)^s \binom{p}{s} (s + m + n/2, 1)_{p-1} = 0
\]

and therefore \( u'(x) = u_m(x) \). Because \( u(x) \) can be represented in \( \Omega_0 \) by the series in \( u_m(x) \) then in \( \Omega_0 \) the theorem is proved.

Consider the general case. Substitute \( v(x) \) from (11) into (20). After some transformations we obtain

\[
u'(x) = u(x) + \frac{|x|^2}{4} \int_0^1 Q(\alpha, |x|) u(\alpha x) \alpha^{n/2-1} d\alpha,
\]

(22)
where
\[ Q(\alpha, |x|) = ga(-\lambda \alpha (1-\alpha)|x|^2) - g4(\lambda (1-\alpha)|x|^2) \]
\[ -\lambda \frac{|x|^2}{4} \int_\alpha^1 \beta g4(-\lambda \beta (1-\beta)|x|^2) g4(\lambda (\beta - \alpha)\beta|x|^2) d\beta. \]

Since \( u(x) = u_m(x) \Rightarrow u'(x) = u_m(x) \) we get
\[ \forall m \geq 0, \forall x \in \mathbb{R}^n, \int_0^1 Q(\alpha, |x|) \alpha^{m+n/2-1} d\alpha = 0. \]

Because \( Q(\alpha, |x|) \) is entire function in \( \alpha \) and \( |x| \) we can easily obtain
\[ \int_0^1 Q^2(\alpha, |x|) \alpha^{n/2-1} d\alpha = 0. \]

The above integral for \( n > 0 \) is defined and therefore \( Q(\alpha, |x|) = 0 \) for \( \alpha \in [0, 1] \) and \( x \in \mathbb{R}^n \). From (22) we deduce \( u'(x) = u(x) \). \( \square \)

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