Symmetry breaking in an initially curved pre-stressed micro beam loaded by a distributed electrostatic force

Lior Medina a,⇑, Rivka Gilat b, Slava Krylov a

a Faculty of Engineering, School of Mechanical Engineering, Tel Aviv University, Ramat Aviv 69978, Israel
b Department of Civil Engineering, Faculty of Engineering, Ariel University, Ariel 44837, Israel

ABSTRACT

The symmetric and asymmetric buckling of an initially curved micro beam subjected to an axial pre-stressing load and transversal distributed electrostatic force is studied. The analysis is based on a reduced order (RO) model resulting from the Galerkin decomposition with buckling modes of a straight beam used as the base functions. The criteria of symmetric limit point buckling and of non-symmetric bifurcation are derived in terms of the geometric parameters of the beam and the axial load. Two symmetry breaking conditions, defining the relations between the axial load and the geometric parameters of beams for which an asymmetric response bifurcates from the symmetric one, are obtained. The necessary criterion establishes the conditions for the appearance of bifurcation points on the unstable branch of the symmetric response curve; the sufficient criterion assures a realistic asymmetric buckling bifurcating from the stable branches of the symmetric response curve. A comparison between the RO model results and those obtained by direct numerical analysis shows good agreement between the two and indicates that the obtained criteria can be used to predict symmetric and non-symmetric buckling in electrostatically actuated curved pre-stressed micro beams. It is shown that while the symmetry breaking conditions are affected by the nonlinearity of the electrostatic force, its influence is less pronounced than in the case of the symmetric snap-through criterion. The nature of the latter and the relations between it and the symmetry breaking criteria are found to go through a prominent qualitative change as the initial distance between the beam and the electrode, characterizing the electrostatic force, changes.

1. Introduction

Curved beams (arches) loaded by concentrated or distributed transverse forces may exhibit bistability, namely the existence of two different stable equilibria under the same loading. The transition between two stable states in these structures is commonly referred to as a snap-through buckling. The behavior of beams liable to the snap-through buckling, due to prescribed deflection-independent “mechanical” loads, is a well understood topic in structural mechanics (Simitses, 1989; Villagio, 1997; Pi et al., 2002; Simitses and Hodges, 2006) which continues to attract attention of researchers (Chandra et al., 2013; Moghaddasie and Stanculescu, 2013).

In the case of the electrostatic actuation, the snap-through behavior is affected by the nonlinearity of the electrostatic force parameterized by the initial distance between the beam and the electrode. The criteria for symmetric (limit-point) snap-through and for asymmetric (bifurcation) snap-through of an initially stress-free bell-shaped beam were obtained in Krylov et al. (2008) and Medina et al. (2012b), respectively; symmetric and asymmetric snap-through in an initially straight buckled beam was analyzed in Medina et al. (2012a). It was shown, that in the case of the electrostatic loading, both symmetric and asymmetric snap-through may take place in beams with lower initial elevation/curvature when compared to the case of “mechanical” deflection-independent loading.

However, in microfabricated beams attached to fully constrained (unmovable) anchors, an axial force is almost always present. This force originates in a residual axial stress resulting from the fabrication process (e.g., see Kaajajari et al., 2009), due to a temperature variation yielding thermal strains (Zhu and Espinosa, 2004; Moghaddasie and Stanculescu, 2013; Stanculescu et al., 2012) or applied intentionally in order to control the natural frequency of the structure (Gabbay and Senturia, 2000; Ruzziconi et al., 2013). This implies that the assumption of an initially stress-free state may not be adequate, and thus a formulation, including the effect of an axial load on the stability, is needed. To

⇑ Corresponding author. Tel.: +972 036405299.
E-mail address: liormedi@post.tau.ac.il (L. Medina).

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this end, we extend the stability analysis of electrostatically actuated curved micro beams to incorporate an axial load, which changes the initial curvature of the beam. Our goal in this work is to investigate the influence of the pre-loading on the beam’s stability and to establish general criteria for the symmetric snap-through and the symmetry breaking. These criteria broaden the ones obtained for stress-free initially curved beams and for buckled initially straight beams given in Medina et al. (2012b) and Medina et al. (2012a) which are special cases of the presently obtained results. The main contribution of the present work, when compared to Krylov et al. (2008, 2011), Pane and Asano (2008), Zhang et al. (2007), Ouakad and Younis (2010), Das and Batra (2009a,b), Intaraprasonk and Fan (2011) and Alkharabsheh and Younis (2013) is in a detailed analytic description of the symmetric and non-symmetric buckling of electrostatically actuated beams and in the development of an explicit symmetric buckling and symmetry breaking criteria, which allow prediction of the beam’s behavior based on its geometric characteristics and the axial load it carries. Note that in the present work, neither the initial elevation of the beam nor the axial force are considered as imperfections (as in Zhu and Espinosa, 2004; Ruzziconi et al., 2013) and the magnitude of the initial elevation (as well as the elevation caused by the axial load) is comparable with the beam’s deflection under electrostatic load. Each or both of these parameters may have a dominant influence on the beam stability.

2. Formulation

We consider a flexible initially curved, axially loaded, double clamped prismatic micro beam of length $L$ having a rectangular cross-section of width $b$ and thickness $d$ as shown in Fig. 1. The beam is made of homogeneous isotropic linearly elastic material with Young’s modulus $E$. Since the width $b$ of a micro-beam is typically larger than its thickness $d$, an effective (plain strain) modulus of elasticity $E_E = E/(1 - \nu^2)$ is used, where $\nu$ is Poisson’s ratio. The initial shape of the stress-free beam is described by the function $w_0(x) = h_0 z_0(x)$, where $h_0$ is the initial elevation of the beam’s central point above its ends, and $z_0(x)$ is a non-dimensional function such that $\max_{x \in [0;L]} |z_0(x)| = 1$. The main assumption of the model is that an axial load appearing upon fabrication changes the beam’s initial elevation $h_0$ to a different elevation designated as $h$. The beam is subjected to a distributed electrostatic force provided by an electrode located at a distance $g_0$ (the gap) from the beam’s ends and extended beyond its ends (Krylov et al., 2008).

We assume that $d \ll L$, $h \ll L$ and that the deflections are small with respect to the beam’s length. Under these assumptions, the beam’s behavior is described in the framework of the Euler-Bernoulli theory combined with the shallow arch approximation. The potential energy of the beam consists of the mechanical (strain) energy of the beam associated with bending and stretching, the work of the axial load (Villagio, 1997; Washizu, 1974) and the electrostatic co-energy calculated based on the parallel capacitor approximation. Note that while the influence of the fringing fields on the electrostatic force acting on the curved beam could be taken into considerations (e.g., Das and Batra, 2009b; Krylov et al., 2008), we use an electrostatic force presented in the form of a simple parallel capacitor formula for the sake of simplicity and transparency of the development.

By using the principle of stationary potential energy, we obtain the equilibrium equations of the beam (Villagio, 1997 for the case of “mechanical” loading) as follows

$$\frac{\partial}{\partial x} \left( \frac{P - E_A}{EA} \left( \frac{\partial \bar{w}}{\partial x} + \frac{1}{2} \left( \frac{\partial \bar{w}}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial \bar{w}_0}{\partial x} \right)^2 \right) \right) = 0$$  
(1)

$$\bar{E} \eta \left( \frac{\partial \bar{w}}{\partial x} \frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{w}_0}{\partial x} \frac{\partial \bar{w}_0}{\partial x} + \frac{P - E_A}{2L} \left( \frac{\partial \bar{w}}{\partial x} - \frac{1}{2} (\partial \bar{w}_0 / \partial x)^2 \right) \right) \frac{\partial \bar{w}}{\partial x} = 0$$  
(2)

where $\eta$ is the applied electrostatic load given as (note that the sign is taken to be consistent with Fig. 1)

$$\bar{j}^* = \frac{e_0 \bar{V}^2}{2(g_0 + w)^2}$$  
(4)

and $e_0 = 8.854 \times 10^{-12} \text{F/m}$ is the permittivity of the free space and $V$ is the voltage difference between the beam and the electrode.

In accordance with Eq. (1), the axial force is constant along the beam (with $P$ positive for compressive), hence Eqs. (1) and (2) can be reduced to the following single equation (e.g., Villagio, 1997 for a “mechanically” loaded beam)

$$\bar{E} \eta \left( \frac{\partial \bar{w}}{\partial x} \frac{\partial \bar{w}}{\partial x} + \frac{P - E_A}{2L} \int_0^1 \left( \frac{\partial \bar{w}}{\partial x} - \frac{1}{2} (\partial \bar{w}_0 / \partial x)^2 \right) \right) \frac{\partial \bar{w}}{\partial x} = 0$$  
(5)

Fig. 1. Model of an initially curved axially loaded double-clamped beam actuated by distributed electrostatic force. The dashed line corresponds to the deformed configuration. Positive directions of the beam’s deflection and of the loading are shown.

which is subjected to the boundary conditions given by the last two of Eqs. (3). In essence, the integral term in Eq. (5) represents the average of the axial force induced by the beam’s deflection.

For convenience, we re-write Eqs. (1) and (2) in a non-dimensional form

$$\left( P - 2x \left( u' + \frac{1}{2} (w')^2 - \frac{1}{2} (w_0')^2 \right) \right)' = 0$$  
(6)

$$w'' - w_0'' + \left( w' \left( P - 2x \left( u' + \frac{1}{2} (w')^2 - \frac{1}{2} (w_0')^2 \right) \right)' + \frac{\beta}{(1 + w)^2} = 0$$  
(7)

with homogeneous boundary conditions, where $' = \text{' denotes derivative with respect to the non-dimensional coordinate $0 \leq x \leq 1$ and $P$ is the non-dimensional axial force. The non-dimensional form of Eq. (5) is

$$w'' - w_0'' + \left( P - x \int_0^1 \left( (w')^2 - (w_0')^2 \right) \right) \right) w' + \frac{\beta}{(1 + w)^2} = 0$$  
(8)

The non-dimensional quantities used in Eqs. (6)-(8) are defined in Table 1.
3. Reduced order model

In order to analyze the snap-through and pull-in behavior of the beam, a reduced order (RO) model based on the Galerkin decomposition is constructed. The deformed shape of the beam is approximated by the series

\[ w(x) \approx \sum_{i=1}^{n} q_i \phi_i(x) \]  

(9)

where \( \phi_i \) represent the buckling eigenmodes of a straight double-clamped beam and are given by the expression

\[ \phi_i(x) = C_i \left( \frac{\cos(\lambda_i x)}{\sin(\lambda_i)} - \frac{1}{\lambda_i} \sin(\lambda_i x) - \frac{1}{\lambda_i} + \frac{1 - \cos(\lambda_i x)}{\sin(\lambda_i)} \right) \]  

(10)

Here \( C_i \) are constants, which are chosen such that \( \max_{x \in [0, L]} (\phi_i(x)) = 1 \) and \( \lambda_i \) are the eigenvalues which are found as a solution of the equation \( \cos(\lambda_i) + (\lambda_i/2) \sin(\lambda_i) = 1 \).

It is possible to represent the initial shape of the beam in the following form

\[ w_0(x) = \sum_{i=1}^{n} q_0 \phi_i(x) \]  

(11)

Substitution of Eq. (9) and Eq. (11) into Eq. (8), multiplication by \( \phi_i \) and integration in conjunction with the orthogonality of the eigenmodes, produce a system of coupled nonlinear algebraic equations

\[ B(q - q_0) - (P - \alpha(q - q_0)\bar{S}q)\bar{S}q + \beta \bar{Q} = 0 \]  

(12)

where \( (\cdot)^T \) denotes the matrix transpose, \( q = \{q_i\} \) and \( q_0 = \{q_{0i}\} \). The elements of the generalized force vector \( \bar{Q} = \{Q_i\} \) and of the matrices \( B = \{b_{ij}\} \) and \( S = \{s_{ij}\} \), which are associated with the bending and stretching stiffness of the beam, respectively, are given by the expressions

\[ Q_i = \int_0^1 \frac{\phi_i}{\left( 1 + \sum_{j=1}^{n} q_j \phi_j(x) \right)^2} dx \]  

(13)

\[ b_{ij} = \delta_{ij} \int_0^1 \phi_i \phi_j^\prime dx, \quad s_{ij} = \delta_{ij} \int_0^1 \phi_i^\prime \phi_j dx \]  

(14)

with \( \delta_{ij} \) being the Kronecker delta. Note that \(-1 \leq q_i \leq 1\).

In the case of initially straight beams, RO models based on Galerkin decomposition with linear eigenmodes as shape functions are proven to be a reliable tool for the analysis of the static and dynamic pull-in behavior and are widely reported in the literature (Nayfeh et al., 2005; Abdel-Rahman et al., 2002; Krylov, 2007). Validation of the model for the case of a curved beam along with the convergence study can be found in Krylov et al. (2008), Das and Batra (2009b) and Ouakad et al. (2009).

For the investigation of the symmetric and asymmetric snap-through, the RO model should include at least two terms, the first symmetric and the first anti-symmetric ones. By setting \( n = 2 \) in the approximation of the beam’s shape \( w(x) \) in Eq. (9), and by taking the initial shape to have the form of the fundamental buckling mode of the straight beam (i.e. \( w_0(x) = h_0 \phi_0 \)), see Eq. (9) where \( q_0 = h_0 \), the RO model in Eq. (12) is reduced to a system of two coupled nonlinear algebraic equations in terms of the general coordinates \( q_1 \) and \( q_2 \), namely

\[ b_{11}(q_1 - h_0) - \left( P - \alpha \left( q_1 \phi_1^T - h_0^2 \phi_0^T + s_{22} \phi_0^T \right) \right) s_{21} q_1 = -\beta \int_0^1 \frac{\phi_1}{\left( 1 + q_1 \phi_1 + q_2 \phi_2 \right)^2} dx \]  

(15)

\[ b_{22} q_2 - \left( P - \alpha \left( q_1 \phi_1^T - h_0^2 \phi_0^T + s_{22} \phi_0^T \right) \right) s_{22} q_2 = -\beta \int_0^1 \frac{\phi_2}{\left( 1 + q_1 \phi_1 + q_2 \phi_2 \right)^2} dx \]  

(16)

where \( b_{11} = 2 \pi^2, b_{22} = 1667.962 \), \( s_{11} = \pi^2/2 \) and \( s_{22} = 20.653 \). Note that since the base functions are those of exact buckling modes of an initially straight beam, we have \( b_{11}/s_{11} = 4 \pi^2 = P_{eq}^1 \) as the lowest non-dimensional buckling load and \( b_{22}/s_{22} = P_{eq}^2 \) as the second non-dimensional buckling load.

Eqs. (15) and (16) indicate that in the presently studied general model, the beam’s behavior, namely the dependence between the voltage parameter \( \beta \) and the deflection, depends on three parameters, the geometric ones, \( h_0, \alpha \) and the axial force \( P \). It is to note that although the geometric parameter \( \alpha \) is used in the formulation for the sake of generalization, all subsequent figures are given for a rectangular cross section for which \( \alpha = 6/d^2 \).

4. Snap-through criteria

Eqs. (15) and (16) corresponding to the two DOF model, cannot be solved in a closed form due to the presence of the integral terms, which are associated with the electrostatic force. For this reason, three-dimensional response diagrams, mapping all stable and unstable equilibrium configurations in the \( q_1, q_2, \beta \) space are first built numerically. The symmetric branch \( \beta = \beta(q_1) \) of the equilibrium path is built by setting \( q_2 = 0 \) in Eq. (15). For the non-symmetric branch, equating the expressions for \( \beta \) extracted from both Eqs. (15) and (16), we obtain a nonlinear algebraic equation \( F(q_1, q_2) = 0 \), which for prescribed values of \( q_1 \) yields \( F(q_1, q_2) = 0 \). The solution, namely \( q_2 = q_2(q_1) \), when it exists, is obtained numerically by using the solver for nonlinear algebraic equations incorporated into the Maple package, while numerically evaluating the integrals. Finally, the values of the voltage parameter \( \beta \), corresponding to the symmetric and non-symmetric branches of the buckling diagram are obtained by substituting the values of \( q_1 \) and \( q_2(q_1) \) back into Eq. (16).

The result is presented in Fig. 2, illustrating the response of an electrostatically loaded pre stressed initially curved beam with thickness of \( d = 0.2 \). Depending on the values of the parameter characterizing the beam geometry, \( h_0 \), and the pre-stress, \( P \), various behaviors are observed. The response may exhibit a single limit point, \( Pl \) (Fig. 2(a)) at which a symmetric pull-in collapse of the beam to the electrode takes place, or three limits points with the additional ones, \( S \) and \( R \) (Fig. 2(b)–(f)). At the latter points, snap-through and snap-back (or release) between two stable branches happen (Fig. 2(b) and (c)), while under different circumstances a snap through to the electrode occurs at \( S \) (Fig. 2(e)). The response can be symmetric (\( q_2 = 0 \), Fig. 2(a) and (b)), or include bifurcation points, \( AS \) and \( AR \), where asymmetric branches emerge from the symmetric one such that symmetry breaking occurs (Fig. 2(c)–
These bifurcation points may be located on the unstable branch between the S and R points (Fig. 2(c) and (e), or on the stable branches yielding asymmetric snap-through (Fig. 2(d) and (f)). Note that although both snap-through and pull-in instabilities are effected by the mechanical and electrostatic nonlinearities, for the sake of convenience and in order to prevent confusion, in the case when three limit points are present, we refer hereafter to the first limit-point collapse as snap-through (since it may occur also in an initially straight beam). We preserve this terminology also in the case when the second limit point is degenerated, as in Fig. 2(f) where the second pull-in collapse is suppressed.

The expected similarity between the figures presented here and those shown in Medina et al. (2012b) for an initially stress-free curved beam, implies that criteria for the symmetric snap-through and both non-critical (necessary condition) and critical (sufficient conditions) bifurcations should be defined for the present general case. These will actually form a generalization of the previously obtained ones.

In accordance with Fig. 2, the branches of the bifurcation diagram emerging from the equilibrium path representing the symmetric response correspond to the non-symmetric configurations. Hence, in order to find the position of the bifurcation points on the symmetric branch, we linearize Eqs. (15) and (16) in terms of $q_s \ll 1$ around the path $q_s = 0$. Taking into account that the following integrals vanish

$$\int_0^1 \frac{\varphi_1 \varphi_2}{(1 + q_s \varphi_1)} \, dx = 0 \quad \int_0^1 \frac{\varphi_2}{(1 + q_s \varphi_1)} \, dx = 0$$

we obtain

$$b_{11}(q_1 - h_0) - (P - 2s_{11}(q_1^2 - h_0^2))s_{11}q_1 + \beta l_1(q_1) = 0$$

$$\left((b_{22} - (P - 2s_{11}(q_1^2 - h_0^2))s_{22}) - 2\beta l_2(q_1)\right)q_2 = 0$$

where $l_1$ and $l_2$ are

$$l_1(q_1) \triangleq \int_0^1 \frac{\varphi_1}{(1 + q_s \varphi_1)} \, dx \quad l_2(q_1) \triangleq \int_0^1 \frac{\varphi_2}{(1 + q_s \varphi_1)} \, dx$$

Eq. (18) is independent on $q_2$, and hence corresponds to the single DOF model which describes the symmetric response of the beam (Krylov et al., 2008). By expressing $\beta$ in terms of $q_1$ from Eq. (18) and differentiating it with respect to $q_1$, we obtain the following expression

$$\frac{1}{l_1} \left((\varphi_1 h_0^2 - 3q_1^2)s_{11} + s_{11}P - b_{11}\right)l_1$$

$$- \left((q_1^2 - q_1^2 h_0^2)s_{11} - s_{11}q_1 P - b_{11}(h_0 - q_1)\right)l_2 = 0$$

Fig. 2. Buckling diagrams of the electrostatically loaded beam (two DOF RO model, Eqs. (15) and (16)) for $d = 0.2$ and different initial elevations and axial loads: (a) $h_0 = 0.1$ and $P/P_0 = 0.5$, (b) $h_0 = 0.1$ and $P/P_0 = 1$, (c) $h_0 = 0.24$ and $P/P_0 = 1$, (d) $h_0 = 0.295$ and $P/P_0 = 1$, (e) $h_0 = 0.2$ and $P/P_0 = 2$, (f) $h_0 = 0.3$ and $P/P_0 = 2$. Points S and R are the snap-through and release limit points; points AS and AR are the bifurcation points of the asymmetric snap-through and release and point PI is the pull-in point. The dashed line in (e) and (f) represent the $\beta = 0$ plane.
which defines the locations $q_i$ of the limit points. Here $l_i = \frac{dI}{dq_i}$. It is clear that the singular point $q_i = -1$ (see Eq. (22)), represents a contact between the beam and the electrode, leaving the other feasible solution to be confined to $q_i > -1$. By taking into account that $l_i > 0$ for $q_i > -1$ Eq. (21) is satisfied when the nominator vanishes. With the adopted base functions given by Eq. (10), analytical expression for $l_i$ is obtained as follows

$$l_i(q_i) = \frac{1}{2\sqrt{(1+q_i)^3}}$$

(22)

Substitution of the above result into Eq. (21) yields

$$(1 + q_i)^2 \left( \frac{9}{2} q_i^2 + 3q_i^2 - \frac{5}{2} s_{11} + 2h_0^2 - \frac{b_{11}}{s_{11}} q_i - 2h_0^2 - \frac{b_{11}}{s_{11}} \left( \frac{3}{2} h_0 - 1 \right) \right) = 0$$

(23)

Since $q_i > -1$, the condition Eq. (23) can be re-written in the form

$$\frac{9}{2} q_i^2 + 3q_i^2 - \frac{5}{2} s_{11} + 2h_0^2 - \frac{b_{11}}{s_{11}} q_i - 2h_0^2 - \frac{b_{11}}{s_{11}} \left( \frac{3}{2} h_0 - 1 \right) = 0$$

(24)

where $h \equiv s_{11} x / P E$. The three roots of this equation correspond to the symmetric snap-through (S), symmetric release (R) and pull-in (PI) points and are given in terms of the system's parameters, i.e. the beam's elevation, thickness and the applied axial load.

Note that the non-dimensional thickness $d = d_0 / g_0$ is parameterized by the distance $g_0$ between the beam and the electrode and actually reflects also the influence of the electrostatic force.

In order to find the location of the bifurcation points, the following procedure is carried out. By substituting $l_i(q_i)$ resulting from Eq. (18) into Eq. (19), one obtains that the eigenvalue problem defined by Eq. (19) has a non-trivial solution when the following equation is satisfied

$$2 \left( q_i^2 x - q_i h_0^2 \right) s_{11} - s_{11} q_i P - b_{11} (h_0 - q_i) \right) I_2 - \left( s_{22} x \left( h_0^2 - q_i^2 \right) s_{11} + s_{22} P - b_{22} \right) I_1 = 0$$

(25)

which can also be written in the following form

$$2 \frac{b_{11}}{b_{22}} \left( q_i^2 x - \left( h_0^2 - q_i^2 \right) \right) q_i - h_0 \right) I_2 - \left( x \frac{P}{P F_x} \left( h_0^2 - q_i^2 \right) + P \frac{P}{P F_x} - 1 \right) I_1 = 0$$

(26)

This equation maps the location, $q_i$, of the asymmetric snap-through (AS) and the asymmetric release (AR) bifurcation points on the symmetric branch.

As was previously observed, in the present general case, the location of the limit and bifurcation points depends not only on the geometric parameters $\tilde{x}$ (namely $d$) and $h_0$ but also on the applied axial load $P$. This dependency is shown in Figs. 3 and 4 for a specific value of thickness $d$ (namely $\hat{x}$). Setting $\hat{x}$ to have a constant value, Eqs. (24) and (26) define surfaces in the $q_i$, $h_0$, $P / P_F$ space. These surfaces are presented in Fig. 3(a) and (b), which show the location of the symmetric limit points and the bifurcation points, respectively, as a function of both the initial elevation and the axial force. In addition, Fig. 3(c) and (d) depict the corresponding voltage parameter given by Eq. (18) in which suitable critical values of $q_i$ have been substituted. Note that a numerical evaluation of the integral included in Eqs. (24) and (26) is required. Hence, the surfaces are actually built by the sets of two dimensional curves, $q_i = q_i(h_0)$ for various axial loads and $q_i = q_i(P / P_e)$ for various initial elevations, presented in Fig. 4(a), (c) and (b), (d), respectively. We re-iterate that negative $P / P_e$ represents applied axial pre-tension.

Figs. 3 and 4 indicate that in the presence of a compression axial load, a smaller initial elevation is required for evoking the bi-stability and the symmetry breaking, and the operating voltage at which the buckling occurs is smaller. Similarly, as the initial elevation increases, the axial load assuring bi-stability and asymmetric response decreases, and the corresponding critical voltage increases. Note that for $P = 0$, one receives the bifurcation plot for the special case of an initially curved stress-free beam (obtained in Medina et al., 2012b with different base functions) and for $h = 0$, the special case of a pre-stressed initially straight beam (Medina et al., 2012a).

It is interesting to note that in the framework of the present model, the character of the buckling map (e.g. Fig. 4) depends on the values of the beam's thickness and the axial load. As the axial load decreases and the beam's thickness increases, the pull-in and the release branches come closer to each other and eventually merge. Such a situation is shown in Fig. 5 indicating that the region of bi-stability is not always determined solely by a lower bound value of the initial elevation, $h_0$, as in the "mechanical" case. According to the Fig. 5, there are two distinct regions of bi-stability, one for $h_0 < h_0 < h_0$ and one for $h_0 > h_0$. As the axial compression decreases, the confined region $[h_0, h_0]$ diminishes and the value of $h_0$ increases, namely the region of bi-stability becomes smaller. This effect will be reflected also by the symmetric snap-through criterion presented in the following section.

4.1. Symmetric snap-through criterion

The symmetric snap-through criterion is obtained by the approach presented in Krylov et al., 2008. It follows from Figs. 3(a) and 4(a) and (b) that in order to have bi-stability, Eq. (24), which is cubic in terms of $q_i$, must possess three different real roots. Hence, its discriminant must be positive. A vanishing discriminant, namely

$$\begin{align*}
\frac{125}{19658} & \frac{1}{\tilde{x}} \left( \left( h_0^2 + 4 \tilde{x} \right) \tilde{x} + \frac{P}{P F_x} - 1 \right)^3 - \frac{1}{2916} \frac{1}{\tilde{x}} \left( h_0^2 - \frac{16}{27} \tilde{x} \right) \\
& + \frac{P}{P F_x} + 9h_0 - 1 \right)^2 = 0
\end{align*}$$

(27)

defines a surface dividing the $d$, $h_0 / d$, $P / P_F$ space into two regions, in one of which, symmetric snap-through exists. Two dimensional plots representing sections of the mentioned surface at three different values of $P / P_F$, three different values of $h_0 / d$ and three different values of $d$ are depicted in Fig. 6. The surface itself, given by the closed form expression Eq. (27), is shown in Fig. 7. This surface visualizes the criterion for symmetric snap-through. Hence, beams for which the geometric parameters $d$, $h_0 / d$ and the axial load parameter $P / P_F$ are represented by a point in the domain bounded by the surface and the planes $h_0 = 0$ and $d = 0$ will not exhibit bi-stable behavior. Beams characterized by points located outside of the above mentioned domain will experience a symmetric response exhibiting three limit points. The presence of three different limit points within the feasible working region means that the symmetric equilibrium path has two stable branches required for bi-stability.

From Fig. 6, one can observe that as long as the axial pre-compression is smaller than the critical load, i.e. $P / P_F < 1$, and for small values of $d$ ($d < \sim 0.4$), the higher the axial compression, the lower the initial elevation required for the appearance of symmetric snap-through. Under axial compressive load greater than the critical load, even a straight beam will be bi-stable, as the surface in Fig. 7 ends at $P / P_F = 1$ for $h_0 / d = 0$. However, a closer look
reveals that for certain values of \( d \), the \( h_0 \) region bounded by the minimal \( h_0 \) required for bi-stability, contains a confined region over which bi-stability does not exit. Such a region can be seen in Fig. 6(a) confined between the gray line depicted for \( P/P_E = 1.5 \) and the \( d = 1 \) plane and at around \( d = 0.5 \) for \( P/P_E = -1.5 \) and 0, and also in Fig. 6(c) for \( d = 0.5 \) and \( P/P_E \) between 0 and about 0.5. Consequently, in contrast to the “mechanical” case, electrostatically actuated beams pre-stresses by an axial load exceeding the critical buckling load, do not always (independently on the initial elevation) exhibit bi-stability. This observation is in accordance with the previously discussed Fig. 5. The reason for this behavior is that for prescribed \( P/P_E, h_0 \) and for \( d = d_0 \) larger than a certain value, i.e., for relatively closely located electrode, the second stable configuration, which would exist in the "mechanically" loaded beam, is not realized and the snap-through is followed by the collapse to the electrode under nonlinearly increasing electrostatic force.

It is interesting to note that as expected, the criterion for \( P = 0 \) is the one obtained in Medina et al. (2012b) for an initially stress-free curved beam, and the criterion for \( h_0 = 0 \) coincides with the one obtained in Medina et al. (2012a) for a straight pre-stressed beam. Furthermore, for \( d = 0 \) and \( P \leq P_E \) the criterion coincides with the symmetric snap-through criterion of an initially curved pre-stressed beam of a rectangular section subjected to displacement independent “mechanical” transverse load

\[
\frac{h_0}{d} \geq \frac{1}{\pi} \left[ \frac{P_E}{3} \left( 1 - \frac{P}{P_E} \right) \right]^{\frac{1}{2}}
\]

4.2. Asymmetric snap-through criteria

Having found the condition guaranteeing symmetric snap, consider now the criteria for the non-symmetric snap-through. Close examination of Figs. 3 and 4 indicates that two non-symmetric bifurcation criteria should be formulated. The first defines the condition required for the appearance of the bifurcation (the necessary condition) and the second defines the conditions for the appearance of a bifurcation point on the stable branch of the symmetric response (sufficient condition).
4.2.1. Necessary conditions

The necessary condition is obtained on the basis of Eq. (26) which defines the location of the bifurcation points and which is depicted in Figs. 3(b) and 4(a) and (b). From the latter, it is clear that to this end, the minimum of \( h_0 \) and \( P = P_E \) with respect to \( q_1 \) should be sought. Expressing Eq. (26) in the form of \( P = P_E = f(q_1, x_0, h_0) \), the value of \( q_1 \) corresponding to the minimum of \( P/E \) is obtained as a solution of \( \partial P/\partial q_1 = 0 \). Substituting this value in Eq. (26) yields the implicit relations between the thickness, \( d \), the initial elevation, \( h_0 \), and the pre-stressing load, \( P \), required for the appearance of the bifurcation points, namely, the non-symmetric response. Note that the very same relations are obtained when using the \( q_1 \) minimizing \( h_0(q_1, x_0, P) \). These relations can be represented by the two dimensional iso-axial load and iso-thickness curves depicted in Fig. 8.

These curves are actually lying on the three dimensional surface shown in Fig. 9. This surface, forming the necessary criterion for non-symmetric snap-through, divides the \( d \), \( h_0/d \), \( P/P_E \) space such that beams characterized by points located above the surface exhibit non-symmetric response bifurcating from the symmetric buckling curve. Notice that, similarly to what was found for the symmetric criterion, the elevation to thickness ratio required for

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**Fig. 4.** (a) and (b) Location of the critical points of the electrostatically loaded beam (c) and (d) corresponding critical values of the voltage parameter for \( d = 0.2 \). The dashed lines represent the limit points of the buckling diagram given by Eq. (24) and correspond to the symmetric snap-through (S), symmetric release (R) and pull-in (PI) points, and the solid lines represent the bifurcation points of the buckling diagram given by Eq. (26) and correspond to the asymmetric snap-through (AS), and release (AR) points.

**Fig. 5.** Location of the critical points of the electrostatically loaded beam for \( d = 0.38 \) and \( P/P_E = -1.5 \). Two bistability regions are confined within \( h_{01} < h_0 < h_{02} \) and \( h_{03} < h_0 \); no bistability is observed for \( h_{02} < h_0 < h_{03} \).
the appearance of bifurcation points decreases as the axial compression increases, and vice versa.

We again draw attention to the fact that the present general necessary criterion reduces to the previously derived criteria for the special cases: \( P/PE = 0 \), namely a stress-free initially curved beam; \( h_0 = 0 \), namely a pre-stressed straight beam. Furthermore, for \( d \to 0 \) and \( P \leq P_{E}^{(2)} \) the necessary relation between the initial elevation and the pre-stress load, that assures the appearance of bifurcation points on the buckling curve of a beam subjected to transverse displacement-independent "mechanical" load is obtained. This relation, which can be derived in the manner presented here for an electrostatic load, has the following form for a rectangular cross-section

\[
\frac{h_0}{d} \geq \sqrt{\frac{P_{E}^{(2)}}{29.61 \left(1 - \frac{P}{P_{E}^{(2)}}\right)}}
\]  

Note that Figs. 8 and 9 indicate a weak dependency of the necessary condition on the thickness \( d \) implying that the effect of the nonlinear electrostatic force on this criterion is less pronounced than in the case of the symmetric snapping criterion.
In order to establish a fit for the necessary condition, we make use of the above observations and further note that according to Fig. 8(a), the various $h_0/d = F(d)$ curves for various values of axial load are almost parallel to each other. Hence, it is possible to use the fit for $P = 0$ from Medina et al. (2012b) with the free term replaced by the necessary condition for a “mechanically” loaded model, Eq. (29). This yields the following fit

$$\frac{h_0}{d} \geq 0.0297d^2 - 0.2781d + \sqrt{\frac{P^2_E}{2961T} \left(1 - \frac{P}{P_E}\right)} \quad (30)$$

that defines a surface which approximates the surface shown in Fig. 9.

### 4.2.2. Sufficient conditions

In order to establish a fit for the necessary condition, we make use of the above observations and further note that according to Fig. 8(a), the various $h_0/d = F(d)$ curves for various values of axial load are almost parallel to each other. Hence, it is possible to use the fit for $P = 0$ from Medina et al. (2012b) with the free term replaced by the necessary condition for a “mechanically” loaded model, Eq. (29). This yields the following fit

$$\frac{h_0}{d} \geq 0.0297d^2 - 0.2781d + \sqrt{\frac{P^2_E}{2961T} \left(1 - \frac{P}{P_E}\right)} \quad (30)$$

that defines a surface which approximates the surface shown in Fig. 9.

In order to formulate the sufficient condition for the non-symmetric snapping we examine again Fig. 4(a) and (b). According to these figures, it is clear that the sufficient condition corresponds to the points where the (S), (R) curves intersect the (AS), (AR) curves respectively. This may yield two separate criteria, one guaranteeing a non-symmetric snap-through and the other, an asymmetric snap-back.

Equating the expressions for $P/P_E$ extracted from both Eqs. (24) and (26), one obtains an equation $F(q_1, \tilde{a}, P) = 0$, which has two solutions $q_1$ defining the location of the intersection point of (S) and (AS) and (R) and (AR). Substitution of each of those two locations back into Eq. (24) or (26) provides two relations of the form $F(\tilde{a}, h_0, P) = 0$ constituting the sufficient conditions for the appearance of the critical non-symmetric snap-through and snap-back. These two three-dimensional conditions are derived via numerical calculations establishing two-dimensional conditions given as a relation between two of the system’s parameters while the third is kept constant. Such, iso-pre-stress and iso-thickness curves, representing the sufficient conditions, are shown in Figs. 10 and 11, for non-symmetric snap-through and snap-back, respectively. The corresponding three dimensional surfaces are depicted in Figs. 12 and 13. Each of these surfaces, forming the sufficient criterion for non-symmetric snap-through or snap-back, divides the $h_0/d, P/P_E$ space such that only beams characterized by points located above the surface exhibit non-symmetric response bifurcating from the symmetric buckling curve.

A comparison between Figs. 12, 13 and Fig. 9 shows that, as expected, the sufficient conditions for the existence of critical asymmetric response are higher (in terms of $h_0$ and/or $P$) than the necessary conditions described in the latter. Moreover, one can notice from Figs. 11(a) and 13 that the snap-back condition is presented only for small values of thickness to gap ratio $d$. Specifically, the curves in Fig. 11(a), lying on the surface in Fig. 13, approach the symmetric snap-through criterion (shown in Figs. 6(a) and 7) but do not cross it. Crossing the symmetric bound surface of Fig. 7 brings us into a region for which there is no bi-stability, namely there is no snapping through and back between two stable branches. In this region, only symmetric or asymmetric pull-in occurs such that the asymmetric release criterion is not relevant.

As was previously observed for both symmetrical and necessary conditions, both sufficient conditions show that the higher the axial compression load, the lower is the required initial elevation to thickness ratio. This trend is common to all criteria since an applied compressive axial pre-load amplifies the beam’s elevation,
Fig. 10. Two dimensional phase diagrams of the sufficient condition for the critical snap-through for various cases: (a) three different axial loads, (b) three different thicknesses. The area above each line represents the conditions for which the beam’s response exhibits an asymmetric snap-through.

Fig. 11. Two dimensional phase diagrams of the sufficient condition for the critical snap-back for various cases: (a) three different axial loads, (b) three different thicknesses. The area above each line represents the conditions for which the beam’s response exhibits an asymmetric snap-back.

Fig. 12. Phase diagram of the sufficient condition for the critical snap-through. The space above the surface represents the conditions for which an asymmetric snap-through manifest itself in the beam’s response.

Fig. 13. Phase diagram of the sufficient condition for the critical snap-back. The space above the surface represents the conditions for which an asymmetric snap-back manifest itself in the beam’s response.
thus enabling the decrease of the initial elevation induced by fabrication.

As was shown for the necessary condition, both sufficient criteria converge at $d \to 0$ (and for $P \leq (3P_f^2 - P_e)/2$) to the “mechanical” criterion. The latter, which unlike the electrostatic criterion can be analytically derived, has the following form for a rectangular cross-section

$$\frac{h_0}{d} \geq \sqrt[6]{\frac{3P_f^2}{2P - P_e} - \frac{C_0}{6\pi^2}} \quad (31)$$

While the “mechanical” model exhibits only one sufficient condition, two sufficient conditions are formulated for the electrostatically loaded beam. This is due to the non-linear nature of the electrostatic force which, being displacement dependent, changes drastically as the beam snaps to a post-buckled state. Since at this state the electrostatic force is significantly larger than in the pre-buckled state, different sufficient conditions for snapping from the pre-buckled and post-buckled states are manifested.

In contrast to the necessary condition, the nature of both sufficient conditions changes as the thickness parameter rises for...
different axial loads. Therefore, in order to create a reasonable fit, it is imperative to take into account the variation of each criterion with the axial load. A close inspection shows that the dependence of the snap-through criterion on the axial load is less pronounced than that of the snap-back criterion. Therefore the approach employed for the necessary condition is employed for the derivation of the former. Specifically, a fit in which the free term was replaced with the “mechanical” criterion from Eq. (31), is proposed as follows

\[ \frac{h_0}{d} \geq -0.164d^4 + 0.519d^3 - 0.611d^2 + 0.166d \]

plus

\[ \sqrt{\frac{3p^{(2)} - 2P - P_E}{6\pi^2}} \]

The sufficient condition for the snap-back presents a pronounced sensitivity to the axial load. In order to take this into account, all terms of the fit should depend on \( P \). Furthermore, due to the divergence between the two sufficient conditions, it is imperative to develop a more accurate fit. To this end, various fits were formulated for a variety of specific values of axial load. Those were then used to formulate an expression approximating the dependence of each term of the fit (excluding the free one which consists on the “mechanical” criterion) on the axial load. The result of this process is presented by the following expression

\[ \frac{h_0}{d} \geq f_1(P)d^4 + f_2(P)d^3 + f_3(P)d^2 + f_4(P)d \]

\[ + \sqrt{\frac{3p^{(2)} - 2P - P_E}{6\pi^2}} \]

(33)

where \( f_1, f_2, f_3, f_4 \) are

\[ f_1(P) = 42.299 - 22.867 \frac{P}{P_E} + 1.375 \left( \frac{P}{P_E} \right)^2 \]

\[ f_2(P) = -31.449 + 14.849 \frac{P}{P_E} \]

\[ f_3(P) = 8.327 - 3.488 \frac{P}{P_E} - 0.297 \left( \frac{P}{P_E} \right)^2 \]

![Fig. 15. Buckling diagrams (w_m - midpoint elevation) for beams with axial load of \( P/P_E = -1.5 \) (a) \( d = 0.2 \), \( h_0 = 0.4 \) (b) \( d = 0.2 \), \( h_0 = 0.5 \) (c) \( d = 0.7 \), \( h_0 = 1.75 \) and for beams with axial loads of \( P/P_E = 1.5 \) (d) \( d = 0.2 \), \( h_0 = 0.2 \) (e) \( d = 0.2 \), \( h_0 = 0.3 \) (f) \( d = 0.7 \), \( h_0 = 0.7 \). Solid lines – two DOF Galerkin RO model, diamonds – seven DOF Galerkin RO model and the diagonal crosses – finite differences.](image-url)
\[ f_4(P) = 0.064 - 0.007\left(\frac{P}{P_E}\right)^3 - 0.002\left(\frac{P}{P_E}\right)^4 \]

which is relevant for beams exhibiting bi-stability namely in the domain defined by the symmetric snapping criterion.

4.2.3. Summary

In order to illustrate the relations between the four obtained criteria and see the overall picture, we present all of them on the \( h_d/d, d \) plane, for two distinct values of applied axial pre-load. The two phase diagrams are shown in Fig. 14 which includes also a gallery of responses demonstrating the behavior of various beams with the parameters represented by points located in various areas of the phase diagram.

Fig. 14 clearly demonstrates several features of the presently investigated system. For small \( d \), the symmetric criterion is placed below all the others which thus define the symmetry breaking in bi-stable beams, namely the appearance of asymmetric bi-stable behavior. As expected, the necessary condition is placed below the sufficient condition which splits into two. The sufficient condition for the release sets itself apart from its counterpart in such a manner that a case in which an asymmetric snap-through and a symmetric release can be realized. For higher values of \( d \), the picture is reversed, as the symmetric buckling criterion is placed above the other ones. Noting that larger values of \( d \) correspond to the situation in which the beam is closer to the electrode, it can be deduced that only a large enough distance between the beam and the electrode enables bi-stability. Beams which are placed close to the electrode will experience collapse to the electrode following the snap-through buckling. For intermediate values of \( d \), a transition zone, in which the symmetric criterion is not uni-valued, exists. Over this region, not all beams having an initial elevation larger than the minimal one (defined by the lower branch of the symmetric criterion) exhibit bi-stable behavior.

In order to demonstrate the applicability of the derived criteria to realistic devices, two beams which had been experimentally examined by Krylov et al. (2008) are used. One beam was initially straight with \( d = 2.1 \mu m, g_0 = 7.7 \mu m \) and \( h_0 = 0 \mu m \), namely had non-dimensional geometric parameters \( d = 0.27, h_0/d = 0 \), and the second had \( d = 2.5 \mu m, g_0 = 10 \mu m \) and \( h_0 = 4.8 \mu m \), namely \( d = 0.25, h_0/d = 1.92 \). Both beams were assumed to be initially stress free, namely had \( P = 0 \). The point \( (d, h_0/d, P/P_E) = (0.27, 0.0) \) representing the first beam, is located under the surface forming the symmetric snap-through criterion (Figs. 6 and 7), in the region where no bi-stability is expected. This is in accordance with the experimentally observed response of this beam (shown in Fig. 16(a) in Krylov et al., 2008) implying that it exhibits only pull-in buckling. The experimental response of the second beam (Fig. 16(b), Krylov et al., 2008), shows a snap-through occurring at a voltage which is lower than that corresponding to the limit point. This scenario can be interpreted to be an asymmetric snap-through at a bifurcation point. This is in accordance with the fact that the point \( (0.25, 1.92, 0) \), representing the second beam, is located above the surface forming the symmetric asymmetric snap-through criterion (Figs. 11 and 12), and is thus expected to experience asymmetric snap-through.

5. Numerical validation

The snap-through points and symmetry breaking criteria obtained in the previous section, were developed using an approximate two degrees of freedom RO model. In order to validate the approximation, two additional solutions were obtained numerically: the solution based on the RO model with the first seven modes, and the finite differences (FD) solution of Eqs. (6) and (7). For the latter, second order central differences with 30 intervals along the beam were used. To describe the unstable branches of the equilibrium curves, both approaches were implemented in conjunction with the arc-length method (Crisfield, 1997).

A comparison between the responses as predicted by the two DOF RO model, the seven DOF RO model and the FD solution is presented in Fig. 15, where variation of the midpoint deflection of the beam with the applied voltage is shown for two values of prestress and several initial elevations. It can be seen that for the micro beam considered here, the two DOF RO model provides a reasonable accuracy for the position of both the limit and bifurcation points. However, the error in the voltage parameter increases as the beam's initial elevation rises.

In order to further validate the symmetry breaking criteria obtained on the basis of the two DOF RO model, the location of the snap-through, release and pull-in points predicted by the latter is compared to the location as extracted from the numerical analyzes described above. This comparison, which is shown in Fig. 16,
indicates that the present approximation and the numerical results are in a good agreement. This implies that the presently obtained criteria for symmetric snap-through and symmetry-breaking in electrostatically actuated curved prestressed micro beams are reliable.

6. Conclusions

In this work, the symmetric and non-symmetric buckling of an initially curved beam loaded by a constant axial load and a transverse nonlinear, configuration dependent, distributed electrostatic force was analyzed. Focusing on curved beams having an initial elevation introduced by fabrication, prior to the application of the axial load, the presently studied case is a generalization of those investigated by Medina et al. (2012b,a) where an initially curved stress-free beam and an initially straight beam buckled due to an axial compression load are considered.

The approximate reduced order model of the beam was built by means of Galerkin decomposition with buckling eigenmodes of an associated straight beam as base functions. Then, the criterion for a symmetric limit point snap-through buckling along with the criteria for a non-symmetric buckling were developed using the RO model, with two DOF, based on the first symmetric and antisymmetric modes. The verification of the obtained results was performed by a comparison with results derived by a seven DOF RO Galerkin model and by a direct finite difference solution of the governing differential equations. The comparison indicates that the established criteria can be used for the prediction of both the symmetric snap-through and the symmetry breaking in electrostatically actuated prestressed curved micro beams with satisfactory accuracy. It is interesting to note that while the two DOF ROM provides a quite accurate prediction of the position of both the limit and bifurcation points, the predicted corresponding voltage parameters are much less accurate. Hence, it is concluded that the two DOF ROM is sufficient for the determination of the limit and bifurcation positions, and the buckling criteria derived from this mapping, but is insufficient for the determination of the corresponding voltage parameter, the prediction of which requires more DOF.

Since the bifurcation points associated with the non-symmetric buckling may be located on stable or unstable branches of the symmetric equilibrium curve, depending on the system’s parameters, two symmetry breaking criteria were established. The necessary non-symmetric buckling criterion provides the conditions required for the appearance of non-symmetric solutions which emerge from points located on an unstable branch of the symmetric buckling path. In such a case, the non-symmetric configurations are not realized under quasi static force control loading, and the beam experiences limit-points snapping with symmetric configurations. In contrast, the sufficient symmetry breaking criterion establishes the condition for the critical non-symmetric buckling, when snapping with asymmetric configuration occurs at loading and deflection smaller than those corresponding to the limit-point.

From the acquired criteria it is possible to deduce that as the axial compression increases, the minimal values of the initial elevation guaranteeing bi-stability and symmetry breaking reduce. Namely, increasing axial compressions move all the criteria such that the regions of $h_0/d, d$, which represent beams exhibiting symmetric and asymmetric snapping increase. The initially produced elevation has a similar effect on the axial compression promising bi-stability and symmetry breaking, for beams of small thickness. Moreover, it was observed that the effect of the axial load on the form of the necessary and snap-through sufficient condition is rather weak in contrast to the release sufficient condition. These observations enabled the construction of a fit for each of the symmetric braking criteria in the range of $0 \leq d < 1$.

Furthermore, it was found that the surfaces representing the symmetric snap-through criteria and that representing the sufficient conditions for asymmetric snap-through cross each other such that for some beams, the symmetry breaking affects the snapping between two stable branches, while for others, generally placed closer to the electrode, the symmetry breaking affects the pull-in behavior. In contrast to the “mechanical” case, not all electrostatically actuated beams having an initial elevation larger than the minimal one (defined by the lower branch of the symmetric criterion) exhibit bi-stable behavior. Hence, electrostatically actuated beams pre-stressed by an axial load exceeding the critical buckling load, do not always (independently on the initial elevation) exhibit bi-stability.

References

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