A Direct Method for the Regularity of the Gain Term in the Boltzmann Equation

Xuguang Lu*

Department of Applied Mathematics, Tsinghua University, Beijing 100084, P. R. China

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This paper presents an essentially simple and direct method to prove the regularizing properties of the collision gain term in the first equation of the Boltzmann hierarchy. Some corollaries of the main results of this paper are given to improve the previous results on the regularity of the gain term in the Boltzmann equation. An extension of a new fundamental result concerning separated variable kernels is also given to some general kernels.

1. INTRODUCTION

As a mathematical model of the kinetic theory of gases, the Boltzmann equation,

$$\frac{\partial}{\partial t} f + v \cdot \nabla_x f = Q(f, f),$$

(B)

describes the time–space evolution of the one-particle distribution function $f(x, v, t)$ (complemented with appropriate initial and boundary condition) of time $t \in [0, \infty)$, position $x \in \Omega \subset \mathbb{R}^N$, and velocity $v \in \mathbb{R}^N (N \geq 2)$ for a simple monatomic gas of identical particles. The right-hand side of Eq. (B) is the so-called collision term, which describes the rate of change of $f$ due to a binary collision and takes the form $Q(f, g) = Q^+(f, g) - Q^-(f, g)$, where

$$Q^+(f, g)(v) = \int \int_{\mathbb{R}^N \times S^{N-1}} B(v - v_*, \omega) f(v') g(v'_*) \ d\omega \ dv_* \quad (\text{gain term}),$$

(1.1)

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\[ v' = v - \langle v - v_*, \omega \rangle \omega, \quad v'_* = v_* + \langle v - v_*, \omega \rangle \omega, \quad \omega \in \mathbb{S}^{N-1}, \]
\[ Q^-(f, g)(v) = f(v) \int_{\mathbb{R}^N} \left[ \int_{\mathbb{S}^{N-1}} B(v - v_*, \omega) d\omega \right] g(v_*) dv_* \quad \text{(loss term).} \quad (1.2) \]

Of course, in Eq. (B), \( Q(f, f) \) means \( Q(f(x, \cdot, t), f(x, \cdot, t)) \). In these expressions, the collision kernel \( B(z, \omega) \) is a given nonnegative Borel function of \(|z|\) and \( \langle z, \omega \rangle \) only:

\[ B(z, \omega) = b(|z|, \theta), \quad \theta = \arccos(|z|^{-1}\langle z, \omega \rangle). \quad (1.3) \]

Here \( \langle x, y \rangle = x \cdot y \) denotes the inner product in \( \mathbb{R}^N \), \(|x| = \sqrt{\langle x, x \rangle} \). For investigating Eq. (B), we usually adopt an assumption of angular cutoff: \( B \in L^1_{\chi_0}(\mathbb{R}^N \times \mathbb{S}^{N-1}) \) (see [7, 3, 12]). For inverse power potentials, \( B \) takes the form

\[ B(z, \omega) = |z|^\beta b(\theta), \quad -N < \beta \leq 1. \quad (1.4) \]

In this case, the angular cutoff assumption is equivalent to \( b(\theta)(\sin \theta)^{N-2} \in L^1(0, \pi/2) \), which includes the hard sphere model \( b(\theta) = \cos \theta, \beta = 1 \).

Compared with the loss term \( Q^-(f, g) \), the gain term (1.1) has a complicated structure, which appears to be hard to handle. However, in a recent work Lions [8] first found that the gain operator \( Q^+ \) possesses a remarkable advantage in regularity. This advantage is summarized in the following theorem.

**Theorem L [8].** Assume that \( B(z, \omega) = b(|z|, \theta) \) satisfy \( b(r, \theta) \in C^\infty((0, \infty) \times (0, \pi/2)) \left[ C^\infty \text{ functions with compact supports contained in } (0, \infty) \times (0, \pi/2) \right] \), and let \( f \in L^2(\mathbb{R}^N) \) and \( g \in L^2(\mathbb{R}^N) \). Then

\[ \| Q^+(f, g) \|_{H^{N-1/2}(\mathbb{R}^N)}, \| Q^+(g, f) \|_{H^{N-1/2}(\mathbb{R}^N)} \leq C \| f \|_{L^2(\mathbb{R}^N)} \| g \|_{L^2(\mathbb{R}^N)} \]

for some \( C > 0 \) independent of \( f, g \).

Here and below \( H^s(\mathbb{R}^N) \) denotes (for instance, for \( s \geq 0 \)) the usual Sobolev space

\[ \left\{ f \in L^2(\mathbb{R}^N) : \| f \|_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \left( 1 + |\xi|^2 \right)^s |\hat{f}(\xi)|^2 d\xi < \infty \right\} \]

and \( \hat{\cdot} \) denotes the Fourier transform \( \hat{f}(\xi) = \int_{\mathbb{R}^N} f(v) e^{-i\langle \xi, v \rangle} dv \).

Theorem L and its extension [8] are very useful for investigating kinetic equations; some applications have been given in [8–10] and [13]. The proof in [8] for Theorem L is based on (1) the bilinear form of \( Q^+ \), that is, for a fixed \( g \in L^2(\mathbb{R}^N) \), consider the linear operators \( Q^+_g, Q^+_g : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \).
given by $Q^1(f) = Q^+(f, g)$ and $Q^2(f) = Q^+(g, f)$; (2) an observation that the adjoint operators of $Q^1, Q^2$ have a similar structure of the type “generalized Radon transforms” (thanks to the specific nature of the collision geometry); and (3) the method of stationary phase and some facts from the theory of Fourier integral operators. Although the proof in [8] is very complicated, it contains some generality. For instance, following Lions’ approach and combining an analysis on the collision geometry in the relativistic situation, Andreasson [1] extended Theorem L to the gain term in the relativistic Boltzmann equation. Note that in Theorem L the condition on collision kernels does not contain the hard sphere model $b(r, \theta) = r \cos \theta$. Of course, if we follow the method in [8] we may obtain the corresponding result for functions $f, g$ which belong to some spaces smaller than $L^1$ or $L^2$, but the proof still will be complicated. The first simpler proof of Theorem L was given by Wennberg [13] via Carleman’s representation and weighted Radon transforms, which were also used to prove the following generalization of Theorem L:

**Theorem W ([13], $N = 3$).** Let the kernel $B$ in (1.1) be given as

$$B = |v - v_*|^{\beta} (\cos \theta)^\beta \Phi(\theta, v'),$$

$$\theta = \arccos(|v - v_*|^{-1}(v - v_*, \omega)))$$

and assume that $1/2 < \beta \leq 1$ and $\Phi$ is smooth and bounded on $[0, \pi/2] \times \mathbb{R}^N$. Let $f \in L^p_2(\mathbb{R}^3) \cap L^q_1(\mathbb{R}^3)$ with $p > 6/(2\beta - 1)$ and $g \in L^p_1(\mathbb{R}^3) \cap L^q_1(\mathbb{R}^3)$. Then

$$\|Q^+(f, g)\|_{H^1(\mathbb{R}^3)} \leq C(\|f\|_{L^p_2(\mathbb{R}^3)} + \|f\|_{L^q_1(\mathbb{R}^3)}) \big(\|g\|_{L^p_1(\mathbb{R}^3)} + \|g\|_{L^q_1(\mathbb{R}^3)}\).$$

Here $L^p_k(\mathbb{R}^n)$ denotes the weighted $L^p$-space with the norm $\|f\|_{L^p_k(\mathbb{R}^n)}$:

$$\|f\|_{L^p_k(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(v)|^p (1 + |v|^2)^{\kappa/2} dv < \infty, \quad \kappa \geq 0, \quad p \geq 1. \quad (1.5)$$

The condition on $\Phi$ contains the hard sphere model ($\beta = 1, \Phi = 1$) and the inverse power potential laws (1.4): $\Phi(\theta, v') = b(\theta)(\cos \theta)^{-\beta}$ provided that $b(\theta)(\cos \theta)^{-\beta}$ ($\beta > 1/2$) is smooth on $[0, \pi/2]$. Note that the restriction $\beta > 1/2$ for (1.4) corresponds to (very) hard potentials.

In this paper we give an essentially different and direct method to study the regularity of the gain term. In fact, instead of the gain operator $Q^+$, we consider a general gain operator $G$ given by

$$G(F)(v) = \int_{\mathbb{R}^N \times S^{N-1}} B(v - v_*, \omega) F(v', v'_*) \, d\omega \, dv_*.$$

(1.6)
Of course when \( F = f \otimes g \) [i.e., \( F(v, v_\sigma) = f(v)g(v_\sigma) \)] we go back to the gain term \( Q^+(f, g) = G(f \otimes g) \). You may have just noted that such treatment on the gain terms is also considered in a recent paper [2] by Bouchut and Desvillettes [2] (see Section 5 herein for details). From the derivation of the Boltzmann equation, we see that this gain operator \( G \) comes in fact from the first equation of the Boltzmann hierarchy:

\[
\frac{\partial}{\partial t} f + v \cdot \nabla_x f = \int \int_{\mathbb{R}^N \times S^{N-1}} B(v - v_\sigma, \omega) \times [F(x, v', x_\sigma, v'_\sigma, t) - F(x, v, x_\sigma, v_\sigma, t)]_{x_\sigma = x} \, d\omega \, dv_\sigma,
\]

where \( f(x, v, t) \) is the one particle distribution function and \( F(x, v, x_\sigma, v_\sigma, t) \) is the two particles distribution function (see, for instance, [4–6] for the hard sphere model (\( N = 3 \))). This equation together with the assumption of molecular chaos [which implies that \( F(x, v, x_\sigma, v_\sigma, t) = f(x, v, g)f(x_\sigma, v_\sigma, t) \)] then lead to Eq. (B). Thus the gain operator \( G \) is more fundamental than the gain operator \( Q^+ \) in the kinetic theory of gases. It should be noted that since the previous methods above rely on the bilinear form of \( Q^+ \), they do not work on \( G \).

In Section 2 we give a general result on the Fourier transform of the gain term \( G(F) \). By this result our investigation for the regularity of \( G(F) \) is reduced to the estimation of the decay rate of a function \( (M_j(|\xi|))^{-1} \) which is independent of \( F \). In Section 3 we give some decay estimates on this function with two kinds of angular cutoff conditions. Then we prove in Section 4 the regularity of \( G(F) \) and give several corollaries for \( Q^+(f, g) \). In Section 5, we first describe the similarities and differences between our results and those by Bouchut and Desvillettes [2]. Then we present a simple method to extend a fundamental result of [2] concerning the case of separated variables [i.e., the collision kernel \( B \) is given as \( a(|z|)b(\theta) \)] to the case of some general kernels (1.3).

2. A GENERAL RESULT ON \( G(F) \wedge (\xi) \)

For the kernel \( B \) in (1.3) (which is now not necessarily nonnegative) we will use in this paper a function \( A(r) \) defined by \( A(|z|) = \int_{S^{N-1}} |B(z, \omega)| \, d\omega \), i.e.,

\[
A(r) = 2|S^{N-2}| \int_0^{\pi/2} |b(r, \theta)|(\sin \theta)^{N-2} \, d\theta,
\]

(2.1)
where \(|S^n|\) denotes the area of the unit spheres \(S^n \subset \mathbb{R}^{n+1}\) (when \(n = 0\) we define \(|S^0| = 2\). For any measurable function \(w(r) \geq 0\) in \((0, \infty)\) we define a weighted space \(L^p(R^N \times R^N, w)\) \((1 \leq p < \infty)\) by

\[ F \in L^p(R^N \times R^N, w) \iff \|F\|_{L^p(R^N \times R^N, w)}^p = \int \int_{R^N \times R^N} |F(v, v_\ast)|^p w(|v - v_\ast|) \, dv \, dv_\ast < \infty. \tag{2.2} \]

For \(p = 1, 2\), such weighted spaces are often used in this paper with different weights \(w(r)\). To unify our proofs we always assume \(N \geq 3\), but the frame used is applicable to \(N = 2\).

**Theorem 1.** Assume that \(A(r) < \infty\) for all \(r > 0\).

(i) If \(F \in L^1(R^N \times R^N, A)\), then \(G(F) \in L^1(R^N)\), \(\|G(F)\|_{L^1(R^N)} \leq \|F\|_{L^1(R^N \times R^N, A)}\) and

\[ G(F)(\xi) = \int_{R^N} K(z, \xi)(F \circ V_z)(\xi) \, dz, \quad \xi \in R^N, \tag{2.3} \]

where

\[ K(z, \xi) = \int_{S^{N-1}} B(z, \omega) \exp\left(-i(\langle z, \omega \rangle \langle \xi, \omega \rangle - \frac{1}{2}\langle z, \xi \rangle)\right) \, d\omega, \tag{2.4} \]

\[ V_z(v) = (v - z/2, v + z/2), \quad \text{i.e.,} \]

\[ (F \circ V_z)(v) = F(v - z/2, v + z/2). \]

(ii) The kernel \(K(z, \xi)\) is actually a function of \(|z|, |z| |\xi|, \text{ and } \langle z, \xi \rangle/(|z| |\xi|)\) only,

\[ K(z, \xi) = K [ h(|z|, \cdot) \left| |z| |\xi|, \frac{\langle z, \xi \rangle}{|z| |\xi|} \right|, \tag{2.5} \]

where for any function \(h\) on \([0, \pi/2]\) satisfying \(h(\theta)(\sin \theta)^{N-2} \in L^1(0, \pi/2)\),

\[ K[h](s, \tau) = a_N \int_{-1}^1 J[h](\tau, t)(1 - t^2)^{(N-3)/2} \exp\left(-i \frac{1}{2} \tau \right) \, dt, \tag{2.6} \]
\( s \in \mathbb{R}, \tau \in [-1, 1], a_N = 2^{-N}|S^{N-3}|; \)

\[
J[h](\tau, t) = \int_0^\pi (\sin \phi)^{N-3} h\left(\frac{1}{2} \arccos(\tau t + \sqrt{1 - \tau^2 (1 - t^2 \cos \phi)})\right) d\phi,
\]

(2.7)

\( \tau, t \in [-1, 1]. \) Here \( h(\theta) = h(\theta)/(\cos \theta)^{N-2}, \theta \in [0, \pi/2). \)

(iii) For any measurable function \( \rho(r) > 0 \) on \( (0, \infty) \) define \( M_\rho : [0, \infty) \to [0, \infty] \) by

\[
M_\rho(|\xi|) = \left[ \int_{\mathbb{R}^N} |K(z, \xi)|^2 \frac{dz}{\rho(|z|)} \right]^{-1}, \quad \xi \in \mathbb{R}^N.
\]

(2.8)

(As usual, if for some \( \xi \) the integral in (2.8) is 0 or \( \infty \), we define \( M_\rho(|\xi|) = \infty \) or 0, respectively.) Then for all \( F \in L^2(\mathbb{R}^N \times \mathbb{R}^N, A) \cap L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho), \)

\[
\int_{\mathbb{R}^N} M_\rho(|\xi|)|G(F)(\xi)|^2 d\xi \leq (2\pi)^N \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho)}^2.
\]

(2.9)

(iv) If \( \rho(r) > 0 \) satisfies

\[
D_\rho = |S^{N-1}| \int_0^\infty r^{N-1} |A(r)|^2 \frac{dr}{\rho(r)} < \infty,
\]

(2.10)

then the operator \( G \) is bounded from \( L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho) \) into \( L^2(\mathbb{R}^N), \) and (2.9) still holds for all \( F \in L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho). \) Moreover we have

\[
1/D_\rho \leq M_\rho(|\xi|) \to \infty \quad \text{as} \quad |\xi| \to \infty.
\]

(2.11)

To prove Theorem 1, we need two lemmas.

**Lemma 1.** Let \( g \in L^1(S^{N-1}) \) and \( \sigma \in S^{N-1}. \) Then

\[
\int_{S^{N-1}} g(\omega) d\omega = \int_0^\pi (\sin \theta)^{N-2} \int_{S^{N-2}(\sigma)} g(\cos \theta \sigma + \sin \theta \omega) d\omega \, d\theta,
\]

where \( S^{N-2}(\sigma) = \{\omega \in S^{N-1} | \omega \perp \sigma\} \) and \( d\omega \) denotes the Lebesgue sphere measure on \( S^{N-2}(\sigma). \)

**Proof:** Use coordinate rotation and the Fubini theorem. \( \square \)
Lemma 2. Assume that $h(\theta)(\sin \theta)^{N-2} \in L^2(0, \pi/2)$. Let $s \in \mathbb{R}$ and $\sigma, \zeta \in \mathbb{S}^{N-1}$. Then

$$K[h](s, \langle \sigma, \zeta \rangle)$$

$$= \int_{\mathbb{S}^{N-1}} h(\arccos(\langle \sigma, \omega \rangle))$$

$$\times \exp \left( -i s \left( \langle \sigma, \omega \rangle \langle \zeta, \omega \rangle - \frac{1}{2} \langle \sigma, \zeta \rangle \right) \right) d\omega,$$  \hspace{1cm} (2.12)

$$K[h](0, \tau) = 2|\mathbb{S}^{N-2}| \int_0^{\pi/2} h(\theta)(\sin \theta)^{N-2} d\theta \quad \forall \tau \in [-1, 1],$$  \hspace{1cm} (2.13)

where $K[h](\cdot, \cdot)$ is defined in (2.6) and (2.7).

Proof. Let $I(s, \sigma, \zeta)$ be the right-hand side of (2.12) and denote $H(t) = h(\arccos t)$ and $E'(t) = \exp(-i t^2/2)$. Then by Lemma 1 we have

$$I(s, \sigma, \zeta) = 2 \int_{\mathbb{S}^{N-1}} H(\langle \sigma, \omega \rangle) 1_{\langle \sigma, \omega \rangle \geq 0} E'(\langle \zeta, \sigma, \omega \rangle \omega - \sigma) \, d\omega$$

$$= 2 \int_0^{\pi/2} (\sin \theta)^{N-2} H(\cos \theta)$$

$$\times \int_{\mathbb{S}^{N-1}(\sigma)} E'(\langle \zeta, \cos \theta \sigma + \sin \theta \omega \rangle \omega \sigma) \, d\omega \, d\theta$$

$$= 2^{2-N} \int_0^{\pi} (\sin \theta)^{N-2} \tilde{H}(\cos \theta)$$

$$\times \int_{\mathbb{S}^{N-1}(\sigma)} E'(\langle \zeta, \cos \theta \sigma + \sin \theta \omega \rangle \omega \sigma) \, d\omega \, d\theta$$

$$= 2^{2-N} \int_{\mathbb{S}^{N-1}} \tilde{H}(\langle \sigma, \omega \rangle) E'(\langle \zeta, \omega \rangle) \, d\omega,$$

where $\tilde{H}(t) = H(\sqrt{(1 + t)/2})/(\sqrt{(1 + t)/2})^{N-2}, \ t \in (-1, 1)$, using Lemma 1 we obtain

$$\tilde{H}(\langle \sigma, \omega \rangle) \cos \theta + \langle \sigma, \omega \rangle \sin \theta) \, d\omega \, d\theta.$$  \hspace{1cm} (2.14)

Since $|\sigma - \langle \sigma, \zeta \rangle \omega| = \sqrt{1 - \langle \sigma, \zeta \rangle^2}$ we have, for some $u \in \mathbb{S}^{N-1},$

$$\langle \sigma, \omega \rangle = \langle \sigma - \langle \sigma, \zeta \rangle \omega \rangle$$

$$= \sqrt{1 - \langle \sigma, \zeta \rangle^2} \langle u, \omega \rangle \quad \forall \omega \in \mathbb{S}^{N-2}(\zeta).$$
Also, it is easily checked that $\tilde{H}(t) = h(\tfrac{1}{2}\arccos t)$, $t \in (-1,1]$. Thus the inner integral in (2.14) is equal to [by (2.7)]

$$\left| S^{N-3} \int_0^\pi (\sin \phi)^{N-3} \tilde{H} \left( \langle \sigma, \zeta \rangle \cos \theta + \sqrt{1 - \langle \sigma, \zeta \rangle^2 \sin \theta \cos \phi} \right) d\phi \right|$$

$$= |S^{N-3}| J[h](\langle \sigma, \zeta \rangle, \cos \theta).$$

This together with (2.14) yield (2.12). Equality (2.13) is obvious. □

Proof of Theorem 1. First of all we know from the collision relation (1.2) that for any fixed $\omega \in S^{N-1}$ the map $(v, v_\ast) \mapsto (v', v'_\ast)$ is an orthogonal linear transform and satisfies

$$|v' - v'_\ast| = |v - v_\ast|, \quad |\langle v' - v'_\ast, \omega \rangle| = |\langle v - v_\ast, \omega \rangle|.$$ 

Since $B(v - v_\ast, \omega)$ depends only on $|v - v_\ast|$ and $|\langle v - v_\ast, \omega \rangle|$, part (i) follows from the Fubini theorem and simple changes of variables. Part (ii) is a consequence of Lemma 2 by letting $h(\theta) = b(|z|, \theta)$ (for any fixed $z \neq 0$, $\sigma = \xi/|\xi|$, and $s = \tfrac{1}{2}z||\xi|$. Part (iii) [i.e. (2.9)] is easily derived by first using the Cauchy–Schwarz inequality to obtain

$$\left| G(F)^\wedge (\xi) \right|^2 \leq \int_{R^N} |K(z, \xi)|^2 \frac{dz}{\rho(|z|)} \int_{R^N} \rho(|z|)|F \circ V_z|^\wedge (\xi)|^2 dz,$$

which implies (recall that $\infty \cdot 0 = 0$)

$$M_\rho(|\xi|)G(F)^\wedge (\xi)|^2 \leq \int_{R^N} \rho(|z|)|F \circ V_z|^\wedge (\xi)|^2 dz,$$

and then using the Fubini theorem and the Plancherel theorem.

Now we prove part (iv). Consider the positive operator $G^+$ defined by the substitution of $|B|$ for $B$ in (1.6). Let $F \in L^2(R^N \times R^N, \rho).$ For any nonnegative $\psi \in L^2(R^N)$ we have, by the Fubini–Tonelli theorem and the properties of $(v, v_\ast)$ and $(v', v'_\ast)$ mentioned above,

$$\int_{R^N} G^+ (|F|) (v) \psi(v) dv$$

$$= \int_{R^N \times R^N} |F(v, v_\ast)| \left( \int_{S^{N-1}} |B(v - v_\ast, \omega)| \psi(v') d\omega \right) dv dv_\ast$$

$$\leq \|F\|_{L^2(R^N \times R^N, \rho)} \left[ \int_{R^N \times R^N} \frac{1}{\rho(|v - v_\ast|)} \right.$$ 

$$\times \left( \int_{S^{N-1}} |B(v - v_\ast, \omega)| \psi(v') d\omega \right)^2 dv dv_\ast \right]^{1/2}. \quad (2.15)$$
Since \( A(z) = \int_{S^{N-1}} |B(z, \omega)| d\omega \) and \( D_\rho = \int_{R^N} \|A(z)\|^2 / \rho(|z|) \, dz \), the last integral in (2.15) is less than or equal to (by Cauchy–Schwarz inequality)

\[
\int \int_{R^N \times R^N} \frac{A(|v - v_*|)}{\rho(|v - v_*|)} |B(v - v_*, \omega)|[\psi(v')]^2 \, d\omega \, dv \, dv_*
\]

\[
= \int \int_{R^N \times R^N} A(|v - v_*|) |B(v - v_*, \omega)| [\psi(v)]^2 \, dv \, dv_*
\]

\[
= \int \int_{R^N \times R^N} \left[ A(|v - v_*|) \right]^2 \rho(|v - v_*|) [\psi(v)]^2 \, dv \, dv_* = D_\rho \|\psi\|^2_{L^2(R^N)}.
\]

Thus \( G^+(\{|F|\}) \) and therefore \( G(F) \) belong to \( L^2(R^N) \) and

\[
\|G(F)\|_{L^2(R^N)} \leq \sqrt{D_\rho} \|F\|_{L^2(R^N \times R^N, \rho)}.
\]

Next, let \( F_n(v, u_*^j) = F(v, u_*^j) \mathbb{1}_{|v| + |u_*^j| \leq n} \). Then it is easily shown that \( F_n \in L^2(R^N \times R^N, A) \cap L^2(R^N \times R^N, \rho) \) and so (2.9) holds for all \( F_n \).

Denote, for each \( k \geq 1 \), \( M_{\rho, k}(|\xi|) = \min(M_\rho(|\xi|), k) \). Then, since \( G(F) \in L^2(R^N) \) and \( |F_n| \leq |F| \),

\[
\left[ \int_{R^N} M_{\rho, k}(|\xi|) |G(F)(\xi)|^2 d\xi \right]^{1/2} \leq \left[ \int_{R^N} M_{\rho, k}(|\xi|) |G(F - F_n)(\xi)|^2 d\xi \right]^{1/2}
\]

\[
+ (2\pi)^{N/2} \|F_n\|_{L^2(R^N \times R^N, \rho)}
\]

\[
\leq \sqrt{k} (2\pi)^{N/2} \sqrt{D_\rho} \|F - F_n\|_{L^2(R^N \times R^N, \rho)} + (2\pi)^{N/2} \|F\|_{L^2(R^N \times R^N, \rho)}.
\]

Therefore, first letting \( n \to \infty \) and then letting \( k \to \infty \) we obtain (2.9) by Fatou’s lemma. Finally we prove (2.11). This is equivalent to showing that

\[
[M_{\rho}(|\xi|)]^{-1} = \int_{R^N} |K(z, \xi)|^2 \frac{dz}{\rho(|z|)} \to 0 \quad \text{as} \ |\xi| \to \infty. \quad (2.16)
\]

Let \( \mathbf{1} = (1, 0, \ldots, 0) \in S^{N-1} \). By representations (2.5)–(2.7), Lemma 2, and (2.1) we have

\[
\int_{R^N} |K(z, \xi)|^2 \frac{dz}{\rho(|z|)} = \int_{S^{N-1}} d\sigma \int_0^\infty r^{N-1} |K(b(r, \sigma))|(r|\xi|, \sigma, \mathbf{1})^2 \frac{dr}{\rho(r)}
\]

(2.17)
and

\[ |K[b(r, \cdot)](r|\xi|, \langle \sigma, 1 \rangle)| \]

\[ \leq a_N \int_{-1}^{1} J[b(r, \cdot)](\langle \sigma, 1 \rangle, r)(1 - r^2)^{(N-3)/2} dt \]

\[ = K[b(r, \cdot)](0, \langle \sigma, 1 \rangle) = 2\pi^{N-2} \int_{0}^{\pi/2} b(r, \theta)(\sin \theta)^{N-2} d\theta \]

\[ = A(r) < \infty \quad \forall r > 0. \quad (2.18) \]

(2.18) together with the integral representation (2.6) and the Riemann–
Lebesgue lemma imply that for any \( r > 0 \) and any \( \sigma \in S^{N-1}, \)

\[ K[b(r, \cdot)](r|\xi|, \langle \sigma, 1 \rangle) \to 0 \quad \text{as} \quad |\xi| \to \infty. \quad (2.19) \]

In addition, (2.18) also shows that the integrand in the right-hand side of
(2.17) is dominated by the function \( r^{N-1}[A(r)]^2/\rho(r) \in L^1(0, \infty). \) Therefore (2.14) follows from (2.17), (2.19), and the Lebesgue dominated convergence theorem.

Theorem 1 shows that the regularity of the gain term \( G(F) \) is deter-
mined by the decay rate of the function in (2.16). An immediate application of this theorem is for the hard sphere model \( b(z, \theta) = |z| \cos \theta \) with \( N = 3. \) In this case we have \( J[b(|z|, \cdot)](r, t) = \pi |z| \) and so by (2.5) and
(2.6) we obtain

\[ K(z, \xi) = 4\pi \frac{\sin(\frac{\pi}{2}|z| |\xi|)}{|\xi|}. \quad (2.20) \]

If we choose \( \rho(r) = (1 + r^2)^k \) with \( k > 5/2, \) then, since \( A(r) = 2\pi r, \) the condition (2.10) for \( N = 3 \) is satisfied. Thus by (2.20) we get for some \( C = C_k < \infty, \)

\[ \int_{\mathbb{R}^3} |K(z, \xi)|^2 \frac{dz}{\rho(|z|)} \leq C(1 + |\xi|^2)^{-1}, \quad \text{i.e.,} \]

\[ 1 + |\xi|^2 \leq CM_{\rho}(|\xi|), \quad \xi \in \mathbb{R}^3. \]

Therefore if \( F \in L^2(\mathbb{R}^3 \times \mathbb{R}^3, \rho), \) then \( \|G(F)\|_{L^1(\mathbb{R}^3)} \leq C\|F\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3, \rho)}. \)

We end this section with two facts (from Theorem 1) that will be used later.

(1) Because of the condition \( A(r) < \infty \quad (\forall r > 0) \), the function
\( K(z, \xi) \) in (2.4) is well defined and so the integrals in (2.17) make sense and the equality (2.17) holds for all positive measurable functions \( \rho(r). \)
(2) If \( F \in L^1(\mathbb{R}^N \times \mathbb{R}^N, A) \cap L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho) \), then the inequalities
\[
|G(F)(\xi)| \leq \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, A)} \quad \text{and} \quad \|G(F)(\xi)\|^2 \leq C_N(\|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, A)}^2 + \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho)})^2.
\]

\[
\int_{\mathbb{R}^N} \left(1_{|\xi| \leq 1} + M_\rho(|\xi|)1_{|\xi| > 1}\right)|G(F)(\xi)|^2 d\xi
\leq C_N(\|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, A)} + \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho)})^2. \tag{2.21}
\]

3. DECAY ESTIMATES FOR \([M_\rho(|\xi|)]^{-1}\)

Under some angular cutoff conditions on \( B(z, \omega) \) we will give decay estimates for the function in (2.16). Throughout this section the finite constants \( C \) depend only on the kernel \( B \), dimension \( N \), and the given weight \( \rho \); the constants \( C_k \) depend only on \( k \).

**Proposition 1.** Let \( B(z, \omega) = a(|z|)b(\theta) \), where \( a(r) \) and \( b(\theta) \) satisfy that for some positive function \( \rho(r) \) on \((0, \infty)\) and some constant \( 1 < p < 2 \),
\[
\int_0^\infty \left( \frac{r^{N-1}|a(r)|}{\rho(r)} \right)^{p/(2-p)} dr < \infty,
\]
\[
\int_0^{\pi/2} \frac{|b(\theta)|^p}{(\cos \theta)^{(N-2k)(p-1)}} (\sin \theta)^{N-2} d\theta < \infty.
\]

Then

\[
\int_{\mathbb{R}^N} |K(z, \xi)|^2 \frac{dz}{\rho(|z|)} \leq C|\xi|^{(2/p) - 2}, \quad \xi \in \mathbb{R}^N \setminus \{0\}, \tag{3.2}
\]

with
\[
C = C_N \left( \int_0^\infty \left( \frac{r^{N-1}|a(r)|}{\rho(r)} \right)^{p/(2-p)} dr \right)^{(2-p)/p}
\times \left[ \int_0^{\pi/2} \frac{|b(\theta)|^p}{(\cos \theta)^{(N-2k)(p-1)}} (\sin \theta)^{N-2} d\theta \right]^{2/p}.
\]

**Proof.** First we note that the second condition in (3.1) implies that \( b(\theta)(\sin \theta)^{N-2} \in L^2(0, \pi/2) \) so that \( K(z, \xi) \) makes sense. Let \( q = p/(p-1) > 2 \) and \( p_1 = p/(2-p) \). Then
\[
(1/p_1) + (2/q) = 1 \quad \text{and} \quad (1/p) +}
Using (2.17) [with \( b(r, \theta) = a(r)b(\theta) \)] and Hölder inequality we have

\[
\int_{\mathbb{R}^N} |K(z, \xi)|^2 \frac{dz}{\rho(|z|)} = \int_{S^{N-1}} d\sigma \int_0^\infty r^{N-1} |a(r)|^2 \frac{|K[b](r|\xi|, \langle \sigma, \mathbf{1} \rangle)|^2}{\rho(r)} dr \\
\leq \|\Phi\|_{L^p(0, \infty)} \int_{S^{N-1}} d\sigma \left( \int_0^\infty |K[b](r|\xi|, \langle \sigma, \mathbf{1} \rangle)|^q dr \right)^{2/q} \\
= \|\Phi\|_{L^p(0, \infty)} \int_{S^{N-1}} d\sigma \left( \int_0^\infty |K[b](r, \langle \sigma, \mathbf{1} \rangle)|^q dr \right)^{2/q} |\xi|^{-2/q},
\]

where \( \Phi(r) = (r^{N-1}|a(r)|^2)/\rho(r) \in L^p(0, \infty) \) by assumption. Now, if we define

\[
J[b](\tau, t)(1 - t^2)^{(N-3)/2} = 0 \quad \text{for all } t \in \mathbb{R} \setminus [-1, 1],
\]

then by representation (2.6) the function \( r \mapsto K[b](2r, \langle \sigma, \mathbf{1} \rangle) \) is the Fourier transform of the function \( t \mapsto a_N J[b](\langle \sigma, \mathbf{1} \rangle, t)(1 - t^2)^{(N-3)/2} \). Therefore, applying the Hausdorff–Young inequality (see, e.g., [11]),

\[
\|f \|^q_{L^p(\mathbb{R})} \leq (2\pi)^{1/q} \|f\|_{L^p(\mathbb{R})}, \quad (1/p) + (1/q) = 1, \quad 1 < p < 2,
\]

we obtain

\[
\left( \int_0^\infty |K[b](r, \langle \sigma, \mathbf{1} \rangle)|^q dr \right)^{1/q} \leq 2^{1/q}(2\pi)^{1/q} a_N \left( \int_{-1}^1 |J[b](\langle \sigma, \mathbf{1} \rangle, t)|^p (1 - t^2)^{(N-3)p/2} dt \right)^{1/p}.
\]

Next, if we set \( h(\theta) = |b(\theta)|^p / (\cos \theta)^{(N-2)(p-1)} \), then \( |b(\theta)|/(\cos \theta)^{(N-2)} \) \( = h(\theta)/(\cos \theta)^{(N-2)} \) and by definition of \( J \) [see (2.7)] and Hölder inequality we have

\[
|J[b](\langle \sigma, \mathbf{1} \rangle, t)|^p \leq \left( \int_0^\pi (\sin \phi)^{N-3} d\phi \right)^{p/q} J[h](\langle \sigma, \mathbf{1} \rangle, t).
\]
This together with (2.6) and Lemma 2 [(2.13)] imply that
\[
\int_{-1}^{1} |J[b]((σ, 1), t)|^p (1 - t^2)^{-(N-3)p/2} dt \\
\leq π \int_{-1}^{1} J[h]((σ, 1), t)(1 - t^2)^{(N-3)/2} dt \\
= \frac{π}{a_N} K[h](0, ⟨σ, 1⟩) \\
= C_N \int_0^{π/2} h(θ)(sin θ)^{N-2} dθ < ∞.
\]
(3.5)

Therefore (3.2) follows from (3.3)–(3.5).

In the following, we denote by \([y]\) the least integer not less than \(y\).

**Proposition 2.** Let \(m = \lfloor (N - 1)/2 \rfloor\). Assume that \(B(z, w) = b(|z|, θ)(cos θ)^{N-2}\), where the function \(b(r, θ)\) for any fixed \(r > 0\) is in \(C^m[0, π/2]\) and satisfies
\[
\tilde{A}(r) \equiv \max_{0 ≤ k ≤ m} \max_{θ ∈ [0, π/2]} \left| \frac{∂^k}{∂θ^k} \tilde{b}(r, θ) \right| < ∞ \quad ∀r > 0.
\]
(3.6)

Then for any measurable function \(ρ(r) > 0\) on \((0, ∞)\) and for all \(ξ ∈ \mathbb{R}^N\),
\[
\int_{\mathbb{R}^N} |K(z, ξ)|^2 \frac{dz}{ρ(|z|)} \\
≤ C_N (1 + |ξ|^2)^{-(N-1)/2} \int_0^{∞} (1 + r^2)^{(N-1)/2} [\tilde{A}(r)]^2 \frac{dr}{ρ(r)}
\]
(3.7)
and
\[
\int_{\mathbb{R}^N} |K(z, ξ)|^2 \frac{dz}{ρ(|z|)} \leq C_N |ξ|^{-N+1} \int_0^{∞} [\tilde{A}(r)]^2 \frac{dr}{ρ(r)}.
\]
(3.8)

To prove Proposition 2, we first prove three lemmas. If we only considered the case \(N = 3\), these lemmas would not need to be introduced; Proposition 2 can be proven directly.

**Lemma 3.** Let \(k\) be a positive integer, \(ε ∈ [0, 1]\), and let \(φ ∈ C^k[0, π]\). Then
\[
\left| \frac{d^k}{dθ^k} [φ(\arccos(ε cos θ))] \right| ≤ C_ε \|φ\|_{k, [0, π]} \left( \frac{1}{\sqrt{1 - ε^2}} \right)^{k-1}, \quad θ ∈ \mathbb{R},
\]
(3.9)
where
\[
\|\varphi\|_{k, [a, b]} = \max_{0 \leq j \leq k} \max_{\theta \in [a, b]} \left| \frac{d^j}{d\theta^j} \varphi(\theta) \right|.
\]

Proof. Denote \(\|\varphi\|_k = \|\varphi\|_{k, [0, \pi]}\). Let
\[
\lambda = e/\sqrt{1 - e^2}, \quad R(t) = t/\sqrt{1 + t^2},
\]
and \(\varphi(\theta) = (d/d\theta)\varphi(\theta)\). Then
\[
\frac{d}{d\theta} [\varphi(\arccos(e \cos \theta))] = \varphi_1(\arccos(e \cos \theta)) R(\lambda \sin \theta), \quad (3.10)
\]
and so (3.9) holds for \(k = 1\). Suppose that (3.9) holds for \(k \leq n\). Let \(\varphi \in C^{n+1}[0, \pi]\). By (3.10) and the Leibniz rule we have
\[
\frac{d^{n+1}}{d\theta^{n+1}} [\varphi(\arccos(e \cos \theta))] = \sum_{k=0}^{n} C_{n,k} [\varphi_1(\arccos(e \cos \theta))]^{(k)} [R(\lambda \sin \theta)]^{(n-k)}. \quad (3.11)
\]
Since \(\varphi_1 \in C^n[0, \pi]\), we have, by inductive hypothesis,
\[
\left| [\varphi_1(\arccos(e \cos \theta))]^{(k)} \right| \leq C_k \|\varphi_1\|_k \left( \frac{1}{\sqrt{1 - e^2}} \right)^{k-1}, \quad k = 1, \ldots, n. \quad (3.12)
\]
On the other hand, it is easily shown that for any \(j \geq 1\), \(R^{(j)}(t) = (1 + t^2)^{-1/2} P_{j-1}(t)\), where \(P_{j-1}\) is a polynomial of degree \(j - 1\). Thus \(R^{(j)}(t)\) is bounded on \(R\) and so one easily obtains that
\[
\left| [R(\lambda \sin \theta)]^{(j)} \right| \leq C_j \left( \frac{1}{\sqrt{1 - e^2}} \right)^j, \quad \theta \in R, j \geq 0. \quad (3.13)
\]
The estimates (3.12) and (3.13) then imply that (3.11) is bounded by
\[
C_n \|\varphi_1\|_0 \left( \frac{1}{\sqrt{1 - e^2}} \right)^n + C_n \|\varphi_1\|_n \left( \frac{1}{\sqrt{1 - e^2}} \right)^{n-1}
\]
\[
\leq C_{n+1} \|\varphi\|_{n+1} \left( \frac{1}{\sqrt{1 - e^2}} \right)^n.
\]
Lemma 4. Let $k \geq 0$ be an integer, $m = [(k + 1)/2]$, and $\varphi \in C^m[0, \pi]$. Define

$$I_k[\varphi](s) = \int_0^\pi e^{is \cos \theta} (\sin \theta)^k \varphi(\theta) \, d\theta, \quad s \in \mathbb{R}.$$ 

Then

$$|I_k[\varphi](s)| \leq C_k \|\varphi\|_{m, [0, \pi]} |1 + s^2|^{-(k+1)/4}, \quad s \in \mathbb{R}.$$

Proof. Denote $\|\varphi\|_m = \|\varphi\|_{m, [0, \pi]}$. It suffices to show that

$$|I_k[\varphi](s)| \leq C_k \|\varphi\|_m |s|^{-(k+1)/2}, \quad |s| \geq 1. \quad (3.14)$$

For $k = 0$, let $\delta = |s|^{-1/2} \leq 1$. We have

$$|I_0[\varphi](s)| \leq C_0 \|\varphi\|_0 \delta + \frac{\delta}{\sin \delta} \int_\delta^{\pi/2} e^{is \cos \varphi(\theta)} \, d\theta.$$

Write $e^{is \cos \varphi(\theta)} = e^{is \cos \varphi(\theta)} e^{is \sin \varphi(\theta)/\sin \theta}$. Then integration by parts and the inequality $\sin \delta \geq (2/\pi)\delta$ imply

$$\left| \int_\delta^{\pi/2} e^{is \cos \varphi(\theta)} \, d\theta \right| \leq C_0 \|\varphi\|_0 \frac{1}{|s| \delta} + C_0 \|\varphi\|_1 \frac{1}{|s| \delta} \int_\delta^{\pi/2} \frac{d\theta}{(\sin \theta)^2} \leq C_0 \|\varphi\|_1 \frac{1}{|s| \delta}.$$

Thus $|I_0[\varphi](s)| \leq C_0 \|\varphi\|_1 |s|^{-1/2}$. For $k = 1, (3.14)$ obviously holds. Suppose that (3.14) holds for $k \leq n$ ($n \geq 1$). Let $\varphi \in C^{([n+2)/2]}[0, \pi]$. Then integration by parts gives

$$I_{n+1}[\varphi](s) = \frac{1}{1s} I_n[\psi](s)$$

where $\psi(\theta) = n \cos \theta \varphi(\theta) + \sin \theta \varphi' (\theta)$. Since $[n/2] + 1 = [(n + 2)/2]$, it follows that

$$|I_{n+1}[\varphi](s)| \leq \frac{1}{|s|} C_{n-1} \|\psi\|_{[n/2]} |s|^{-n/2} \leq C_{n+1} \|\varphi\|_{[(n+2)/2]} |s|^{-(n+2)/2}.$$

Lemma 5. Let $b(\theta) = \tilde{b}(\theta) (\cos \theta)^{N-2}$, $\tilde{b} \in C^m[0, \pi/2]$, and $m = [(N-1)/2]$ ($N \geq 3$). Then for all $s \in \mathbb{R}$, $\tau \in [-1, 1],

$$|K[b](s, \tau)| \leq C_N \|\tilde{b}\|_{m, [0, \pi/2]} \left( \frac{1}{\sqrt{1 - \tau^2}} \right)^{m-1} (1 + s^2)^{-(N-1)/4}. \quad (3.15)$$
Proof. By definition of $K[\cdot]$ [(2.6) and (2.7)] we have

$$K[b](s, \tau) = a_N \int_{0}^{\pi} \exp \left( -\frac{1}{2} s \cos \theta \right) (\sin \theta)^{N-2} J[b](\tau, \cos \theta) \, d\theta,$$

$$J[b](\tau, \cos \theta) = \int_{0}^{\pi} (\sin \phi)^{N-3} \tilde{b} \left( \frac{1}{2} \arccos(e \cos(\theta - \alpha)) \right) \, d\phi,$$

where $e = \sqrt{\tau^2 + (1 - \tau^2) \cos^2 \phi}$ and $\alpha = \arg(\tau, \sqrt{1 - \tau^2} \cos \phi)$. We can suppose that $|\tau| < 1$. For any fixed $\phi \in (0, \pi)$, since $0 \leq e < 1$, the function $\theta \mapsto b \left( \frac{1}{2} \arccos(e \cos(\theta - \alpha)) \right)$ is of $C^m$ on $\mathbb{R}$ and by Lemma 3 we have for all $k = 0, 1, \ldots, m$ ($= [(N - 1)/2] \leq N - 2$) and all $\theta \in \mathbb{R}$,

$$\left| \frac{\partial^k}{\partial \theta^k} \left[ b \left( \frac{1}{2} \arccos(e \cos(\theta - \alpha)) \right) \right] (\sin \phi)^{N-3} \right| \leq C_k \| \tilde{b} \|_{k, [0, \pi/2]} \left( \frac{1}{\sqrt{1 - e^2}} \right)^{k-1} (\sin \phi)^{N-3},$$

$$\leq C_N \| \tilde{b} \|_{m, [0, \pi/2]} \left( \frac{1}{\sqrt{1 - \tau^2}} \right)^{m-1}.$$

Thus interchanging derivatives and integrals imply that $J[b](\tau, \cos \theta) \in C^m[0, \pi]$ in $\theta$ and

$$\| J[b](\tau, \cos \cdot) \|_{m, [0, \pi]} \leq C_N \| \tilde{b} \|_{m, [0, \pi/2]} \left( \frac{1}{\sqrt{1 - \tau^2}} \right)^{m-1}.$$

Therefore (3.15) follows from Lemma 4 with $k = N - 2$ and $\varphi(\theta) = J[b](\tau, \cos \theta)$. $\blacksquare$

Proof of Proposition 2. By Lemma 5 we have for any $r > 0$ and any $\sigma \in S^{N-1}$,

$$| K[b(\cdot, \cdot)](r, \xi, \langle \sigma, 1 \rangle) |^2 \leq C_N \left( \| \tilde{b}(r, \cdot) \|_{m, [0, \pi/2]} \right)^2 \left( \frac{1}{\sqrt{1 - \langle \sigma, 1 \rangle^2}} \right)^{2m-2} (1 + r^2 |\xi|^2)^{-(N-1)/2}.$$
Since \( \tilde{A}(r) = \|\tilde{b}(r, \cdot)\|_{m, [0, \pi/2]} \) and \( 2m \leq N \), it follows from (2.17) that

\[
\int_{\mathbb{R}^N} |K(z, \xi)|^2 \frac{dz}{\rho(|z|)} \\
\leq C_N \int_0^\pi (\sin \theta)^{N-2} \left( \frac{1}{\sin \theta} \right)^{2m-2} d\theta \\
\times \int_0^\infty r^{N-1} (1 + r^2|\xi|^2)^{-(N-1)/2} \left[ \tilde{A}(r) \right]^2 \frac{dr}{\rho(r)} \\
= C_N \int_0^\infty \left( \frac{r^2}{1 + r^2|\xi|^2} \right)^{(N-1)/2} \left[ \tilde{A}(r) \right]^2 \frac{dr}{\rho(r)},
\]

which implies (3.7) and (3.8) since \( r^2/(1 + r^2|\xi|^2) \leq (1 + r^2)/(1 + |\xi|^2) \).

\[\blacksquare\]

4. Regularity of \( G(F) \) and \( Q^+(f, g) \)

In this section the regularizing properties of \( G(F) \) [therefore \( Q^+(f, g) \)] are given in \( H^s(\mathbb{R}^N) \) for weak angular cutoff condition (with small \( s > 0 \), using Proposition 1) and for smooth angular cutoff condition (with \( s = (N-1)/2 \), using Proposition 2), respectively. The weighted \( L^p \)-space \( L^p_{\rho} (\mathbb{R}^N \times \mathbb{R}^N) \) (\( 1 \leq p < \infty \)) used below is defined according to (1.5), i.e.,

\[
\|F\|_{L^p_{\rho} (\mathbb{R}^N \times \mathbb{R}^N)} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |F(v, v_*)|^p (1 + |v|^2 + |v_*|^2)^{\kappa p/2} dv dv_* < \infty.
\]

For such spaces, the following elementary inequalities are often used:

\[
2^{-\kappa} (1 + |v - v_*|^2)^\kappa \\
\leq (1 + |v|^2 + |v_*|^2)^\kappa \\
\leq (1 + |v|^2)^\kappa (1 + |v_*|^2)^\kappa, \quad v, v_* \in \mathbb{R}^N, \kappa \geq 0. \quad (4.1)
\]

As in the above sections, all positive constants \( C \) that appear in this section depend only on the kernels \( B \), dimension \( N \), and the given weights.
Theorem 2. Let $B(z, \omega) = a(|z|)b(\theta)$, where $a(r)$ and $b(\theta)$ satisfy

\[
|a(r)| \leq C_\alpha (r^{-\alpha} + r^\beta), \quad r \in (0, \infty);
\]
\[
|b(\theta)| \leq (\cos \theta)^{-\gamma}, \quad \theta \in [0, \pi/2),
\]
for some constants $C_\alpha < \infty$, $0 \leq \alpha < N/2$, $\beta \geq 0$, and $0 < \gamma < 1$. Let $\kappa > (N/2) + \beta$. Then for any $s$ satisfying

\[
0 < s < \min\{(1 - \gamma)/(N - 1), (N/2) - \alpha\},
\]
the gain operator $G$ is bounded from $L^2_\alpha(\mathbb{R}^N \times \mathbb{R}^N)$ into $H^s(\mathbb{R}^N)$:

\[
\|G(F)\|_{H^s(\mathbb{R}^N)} \leq C\|F\|_{L^2_\alpha(\mathbb{R}^N \times \mathbb{R}^N)},
\]

(4.3)

Consequently, for all $f, g \in L^2_\alpha(\mathbb{R}^N)$,

\[
\|Q^*(f, g)\|_{H^s(\mathbb{R}^N)} \leq C\|f\|_{L^2_\alpha(\mathbb{R}^N)}\|g\|_{L^2_\alpha(\mathbb{R}^N)}.
\]

(4.4)

Remark. For the inverse $k$th power forces of interacting particles, the function $b(\theta)$ has a nonintegrable singularity at $\theta = \pi/2$ (see [4, 12]):

\[
b(\theta) = O(\cos \theta)^{-((k+1)/(k-1)}), \quad \theta \to \pi/2 \quad (3 < k < \infty).
\]

The angular cutoff condition in (4.2) is referred to this model with integrable singularity.

Proof of Theorem 2. Let $\rho(r) = (1 + r^2)^\kappa$ and $p = 1/(1 - s)$. Then $1 < p < 2$, $1 - (1/p) = s < (N/2) - \alpha$, and $\epsilon := (N - 2)(p - 1) + \gamma p < 1$ so that we have

\[
\int_0^\infty \left( \frac{r^{N-1}|a(r)|^2}{\rho(r)} \right)^{p/(2-p)} dr \leq C \int_0^\infty \left( \frac{r^{N-1}(r^{-2\alpha} + r^{2\beta})}{(1 + r^2)^\alpha} \right)^{p/(2-p)} dr < \infty,
\]

\[
\int_0^{\pi/2} \frac{|b(\theta)|^p}{(\cos \theta)^{(N-2)(p-1)}} (\sin \theta)^{N-2} d\theta
\]

\[
\leq \int_0^{\pi/2} (\cos \theta)^{-\epsilon} (\sin \theta)^{N-2} d\theta < \infty.
\]

Thus the conditions in Proposition 1 (3.1) hold. Moreover, since \([3.1]\) $b(\theta)(\sin \theta)^{N-2} \in L^1(0, \pi/2)$, it follows that $A(r) \leq C(r^{-\alpha} + r^\beta)$ and so the function $\int_\mathbb{R}^N [K(z, \xi)]^2/\rho(|\xi|) d\xi$ is bounded by

\[
|S^{N-1}| \int_0^\infty \frac{r^{N-1}[A(r)]^2}{\rho(r)} dr \leq C \int_0^\infty \frac{r^{N-1}(r^{-2\alpha} + r^{2\beta})}{(1 + r^2)^\alpha} dr < \infty.
\]
This together with Proposition 1 imply
\[ \int_{\mathbb{R}^N} |K(z, \xi)|^2 \frac{dz}{\rho(|z|)} \leq C \left(1 + |\xi|^2\right)^{-s}, \quad \text{i.e.,} \quad \left(1 + |\xi|^2\right)^s \leq CM_{p}(|\xi|). \]

Therefore (4.3) and (4.4) follow from part (iv) of Theorem 1 and the inequality [by (4.1)]
\[ \|f \otimes g\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho)} \leq \|f\|_{L^2(\mathbb{R}^N)} \|g\|_{L^2(\mathbb{R}^N)} \left[\rho(r) = (1 + r^2)^s\right]. \quad (4.5) \]

For general collision kernel (1.3) with smooth angular cutoff we have the following theorem.

**Theorem 3.** Assume that \(B(z, \omega) = b(|z|, \theta)\) with \(b(r, \theta) = \tilde{b}(r, \theta)(\cos \theta)^{N-2}\) satisfying the angular cutoff condition in Proposition 2. Let \(A(r)\) and \(\tilde{A}(r)\) be defined in (2.1) and (3.6), respectively. Then we have

(i) If \(\rho(r) > 0\) satisfies
\[ \int_0^\infty \left[\tilde{A}(r)\right]^2 \frac{dr}{\rho(r)} < \infty, \quad (4.6) \]
then for all \(F \in L^2(\mathbb{R}^N \times \mathbb{R}^N, A) \cap L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho),\)
\[ \|G(F)\|_{H^{(N-1)/2}(\mathbb{R}^N)} \leq C \left(\|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, A)} + \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho)}\right). \quad (4.7) \]

(ii) If \(\rho(r) > 0\) satisfies
\[ \int_0^\infty (1 + r^2)^{(N-1)/2} \left[\tilde{A}(r)\right]^2 \frac{dr}{\rho(r)} < \infty, \quad (4.8) \]
then for all \(F \in L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho),\)
\[ \|G(F)\|_{H^{(N-1)/2}(\mathbb{R}^N)} \leq C \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, \rho)}. \quad (4.9) \]

**Proof.** (i) By condition (4.6) and Proposition 2 [(3.8)] we have
\[ (1 + |\xi|^2)^{(N-1)/2} \leq C \left(1_{|\xi| \leq 1} + M_{p}(|\xi|)1_{|\xi| > 1}\right). \]
Thus (4.7) follows from (2.21).

(ii) The condition (4.8) and Proposition 2 [(3.7)] imply the condition (2.10) and
\[ (1 + |\xi|^2)^{(N-1)/2} \leq CM_{p}(|\xi|). \]
Therefore (4.9) follows from part (iv) of Theorem 1. \(\blacksquare\)
**Corollary 1 to Theorem 3** ($\rho = 1$). If the kernel $B$ satisfies the condition in Theorem 1, then $G$ is bounded from $L^2(\mathbb{R}^N \times \mathbb{R}^N)$ into $H^{(N-1)/2}(\mathbb{R}^N)$:

$$
\|G(F)\|_{H^{(N-1)/2}(\mathbb{R}^N)} \leq C\|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N)}.
$$

Especially for all $f, g \in L^2(\mathbb{R}^N)$,

$$
\|Q^+(f, g)\|_{H^{(N-1)/2}(\mathbb{R}^N)} \leq C\|f\|_{L^2(\mathbb{R}^N)}\|g\|_{L^2(\mathbb{R}^N)}.
$$

**Remark.** In this corollary, the regularity of $Q^+(f, g)$ involves only the $L^2(\mathbb{R}^N)$ norm for $f$ and $g$. In practice, the most important case is that $f = g$. In this case, Theorem 1 requires that $f \in L^2(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$.

**Corollary 2 to Theorem 3.** Let $B(z, \omega)$ be given in Theorem 3. Suppose further that the function $A(r)$ satisfies

$$
\tilde{A}(r) \leq C \left( r^{-\alpha} + r^\beta \right), \quad r > 0, 0 \leq \alpha < 0, \beta \geq 0, \quad (4.10)
$$

with constants $C_\alpha$, $\alpha$, and $\beta$. Then we have

(i) If $\kappa > (N/2) + \beta$, then for all $f, g \in L^2_\kappa(\mathbb{R}^N)$,

$$
\|Q^+(f, g)\|_{H^{N-1/2}(\mathbb{R}^N)} \leq C\|f\|_{L^2(\mathbb{R}^N)}\|g\|_{L^2(\mathbb{R}^N)}. \quad (4.11)
$$

(ii) If $\kappa > (1/2) + \beta$, then for all $f, g \in L^2_\kappa(\mathbb{R}^N) \cap L^2_\lambda(\mathbb{R}^N)$,

$$
\|Q^+(f, g)\|_{H^{(N-1)/2}(\mathbb{R}^N)} \leq C\left(\|f\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^2_\kappa(\mathbb{R}^N)}\right)\left(\|g\|_{L^2(\mathbb{R}^N)} + \|g\|_{L^2_\lambda(\mathbb{R}^N)}\right). \quad (4.12)
$$

**Proof.** Choose $\rho(r) = (1 + r^2)^\kappa$.

(i) Let $\kappa > (N/2) + \beta$. By condition (4.10) we have

$$
\int_0^1 (1 + r^2)^{(N-1)/2}\left[ \tilde{A}(r) \right]^2 \frac{dr}{\rho(r)} \leq C\int_0^1 r^{-2\alpha} \, dr + C\int_1^\infty \left( \frac{1}{1 + r^2} \right)^{\kappa - (N - 1)/2 - \beta} \, dr < \infty.
$$

Thus (4.11) follows from part (ii) of Theorem 3 and the inequality (4.5).

(ii) Let $\kappa > (1/2) + \beta$. Then

$$
\int_0^\infty \left[ \tilde{A}(r) \right]^2 \frac{dr}{\rho(r)} \leq C\int_0^1 r^{-2\alpha} \, dr + C\int_1^\infty \left( \frac{1}{1 + r^2} \right)^{\kappa - \beta} \, dr < \infty.
$$
REGULARITY OF THE GAIN TERM

Next, by definition of $A(r)$ [2.1] and $\tilde{A}(r)$ [(3.6)] and the condition (4.10) we have $A(r) \leq C_{\Omega}A(r) \leq C(r^{-\alpha} + r^\beta)$. This gives

$$\|f \otimes g\|_{L^1(\mathbb{R}^N \times \mathbb{R}^N, A)} \leq C \int_{\mathbb{R}^N} |f(v)| \left[ \int_{\mathbb{R}^N} \left( |g(v_*)| (|v - v_*|^{-\alpha} + |v - v_*|^\beta) \right) dv_* \right] dv$$

which, together with (4.13) and (4.5), yields (4.12).

5. AN EXTENSION OF A NEW RESULT

In this section, we first compare our main results with those of Bouchut and Desvillettes [2]. Then we extend a new fundamental result of [2] concerning the case of separated variable kernels to a general case. Let us first introduce another representation of the gain operator $G$ used in [2]:

$$G(F)(v) = \int \int_{\mathbb{R}^N \times S^{N-1}} \tilde{B}(|v - v_*|, |v - v_*|^{-1}(v - v_*), \omega)$$

where the new kernel $\tilde{B}(r, t)$ has the following relation with the original kernel $B(z, \omega) = b(|z|, \omega)$:

$$\tilde{B}(r, \cos 2\theta) = 2^{2-N} \tilde{B}(r, \theta) \quad \equiv 2^{2-N} \frac{b(r, \theta)}{(\cos \theta)^{N-2}},$$

$$r > 0, \theta \in [0, \pi/2].$$

For the case $N = 3$, it is proved in [2] that if $\tilde{B}$ is continuous on $(0, \infty) \times [-1, 1]$, admitting a continuous derivative in the second variable, and satisfies for some constant $K$,

$$|\tilde{B}(r, t)| + \left| \frac{\partial}{\partial t} \tilde{B}(r, t) \right| \leq K(1 + r) \quad \text{on } (0, \infty) \times [-1, 1],$$
then for any $\kappa > 3/2$, there exists a constant $C = C_{\kappa, \kappa}$ such that for all $f \in L^2_\delta(\mathbb{R}^3) \cap L^2_\delta(\mathbb{R}^3)$,

$$\|Q^+(f, f)\|_{H^\kappa(\mathbb{R}^3)} \leq C\left(\|f\|_{L^2_\delta(\mathbb{R}^3)}^2 + \|f\|_{L^2_\delta(\mathbb{R}^3)}^2\right).$$

The method of the proof in [2] is similar to ours. The main difference is that in our results (Theorem 3 and its Corollary 2) the smoothness condition is added on the angular variable $\theta$ for $b(r, \theta)$ rather than on the variable $r$ for $B(r, \theta)$; in fact, the angular form of kernels makes our proofs relatively easy for general dimension $N \geq 3$ (Lemmas 3–5).

In previous sections we have shown that due to the specific structure of the gain operator $G$ our investigation for regularizing properties can be simply reduced to the estimate of decay rate of a fixed function $\xi \rightarrow [M_p(|\xi|)]^{-1} = \int_{|\xi|} \frac{|K(z, \xi)|^2}{\rho(|z|)} dz$. This is of course at least feasible for the case of separated variable kernels $B(z, \omega) = a(|z|)b(\theta)$ with weak angular cutoff (Theorem 2) and for the general case with smooth angular cutoff (Theorem 3). However, for the first case with a “strong” angular cutoff, Bouchut and Desvillettes [2] gave the following important observation by making full use of Fourier transform of $F$:

Suppose first that $B(z, \omega) = b(\theta)$, i.e., $a(r) = 1$. Let the function $\tilde{b}(r)$ be defined by

$$\tilde{b}(\cos 2\theta) = 2^{2-N} \frac{b(\theta)}{(\cos \theta)^{N-2}}, \quad \theta \in (0, \pi/2).$$

Then the gain term $G(F)$ and its Fourier transform have a similar structure,

$$G(F)(v) = \int_{\mathbb{R}^N} \tilde{b}(v - v_*)^{-1}(v - v_*, \omega)$$

$$\times F\left(\frac{v + v_*}{2} - \frac{|v - v_*|}{2} \omega, \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \omega\right) d\omega dv_*,$$

$$G(F)(\xi) = \int_{\mathbb{S}^{N-1}} \tilde{b}(\xi |^{-1}(\xi, \omega) F\left(\frac{\xi - |\xi| \omega}{2}, \frac{\xi + |\xi| \omega}{2}\right) d\omega,$$

where $F^\wedge$ is the Fourier transform of $F$ in both variables. Assume that $\tilde{b}$ satisfy $|\tilde{b}(r)|^2(1 - r^2)^{(N-3)/2} \in L^1(-1, 1)$, i.e., $|b(\theta)|^2(\tan \theta)^{N-2} \in L^1(-1, 1)$. Then for any $k > 3/2$, there exists a constant $C = C_{\kappa, \kappa}$ such that for all $f \in L^2_\delta(\mathbb{R}^3) \cap L^2_\delta(\mathbb{R}^3)$,

$$\|Q^+(f, f)\|_{H^\kappa(\mathbb{R}^3)} \leq C\left(\|f\|_{L^2_\delta(\mathbb{R}^3)}^2 + \|f\|_{L^2_\delta(\mathbb{R}^3)}^2\right).$$

The method of the proof in [2] is similar to ours. The main difference is that in our results (Theorem 3 and its Corollary 2) the smoothness condition is added on the angular variable $\theta$ for $b(r, \theta)$ rather than on the variable $r$ for $B(r, \theta)$; in fact, the angular form of kernels makes our proofs relatively easy for general dimension $N \geq 3$ (Lemmas 3–5).

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$$\times F\left(\frac{v + v_*}{2} - \frac{|v - v_*|}{2} \omega, \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \omega\right) d\omega dv_*,$$

$$G(F)(\xi) = \int_{\mathbb{S}^{N-1}} \tilde{b}(\xi |^{-1}(\xi, \omega) F\left(\frac{\xi - |\xi| \omega}{2}, \frac{\xi + |\xi| \omega}{2}\right) d\omega,$$

where $F^\wedge$ is the Fourier transform of $F$ in both variables. Assume that $\tilde{b}$ satisfy $|\tilde{b}(r)|^2(1 - r^2)^{(N-3)/2} \in L^1(-1, 1)$, i.e., $|b(\theta)|^2(\tan \theta)^{N-2} \in L^1(-1, 1)$.
Let $F$ be in (for instance) $C_\infty^0(\mathbb{R}^N \times \mathbb{R}^N)$. Then

$$
\left| F^{\wedge}\left(\frac{\xi - |\xi|\omega}{2}, \frac{\xi + |\xi|\omega}{2}\right) \right|^2
= - \int_{r = |\xi|}^\infty \frac{\partial}{\partial r} \left| F^{\wedge}\left(\frac{\xi - r\omega}{2}, \frac{\xi + r\omega}{2}\right) \right|^2 \, dr
$$

and so using the Cauchy–Schwarz inequality and elementary properties of Fourier transform, it is obtained in [2] that

$$
\int_{\mathbb{R}^N} |\xi|^{N-1} |G(F)^{\wedge}(\xi)|^2 \, d\xi \leq C_{N, b} \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N)} \|U - u_* \|_{L^2(\mathbb{R}^N \times \mathbb{R}^N)},
$$

where $C_{N, b} = C_N \int_0^{\pi/2} |b(\theta)|^2 (\tan \theta)^{N-2} \, d\theta$. For the case $a(r) \neq$ constant, $F(v, u_n)$ will be replaced by $a(u - u_n) b F(v, u_n)$. This yields the following result of Bouchut and Desvillettes [2, Proposition 2.2] which can be stated in terms of our notation (2.2) for weighted spaces $L^p(\mathbb{R}^N \times \mathbb{R}^N, w)$:

**Theorem BD.** Assume the kernel $B(z, \omega) = a(|z|) b(\theta)$ satisfies

$$
C_b^2 = \int_0^{\pi/2} |b(\theta)|^2 (\tan \theta)^{N-2} \, d\theta < \infty.
$$

Let $w_0(r) = |a(r)|^2$, $w_2(r) = |a(r)|^2$, and $w_2(r) = r^2 |a(r)|^2$. Then for all $F \in L^2(\mathbb{R}^N \times \mathbb{R}^N, w_0) \cap L^2(\mathbb{R}^N \times \mathbb{R}^N, w_2) \cap L^2(\mathbb{R}^N \times \mathbb{R}^N, w_2)$,

$$
\|G(F)\|_{\dot{H}^{N-1/2}(\mathbb{R}^N)} \leq C_N C_b \left[ \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_1)} \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_2)} \right]^{1/2},
$$

where

$$
\|G(F)\|_{\dot{H}^{N-1/2}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |\xi|^{N-1} |G(F)^{\wedge}(\xi)|^2 \, d\xi \right)^{1/2}.
$$

It should be noted that in the original result [2, Proposition 2.2] there is no restriction $F \in L^2(\mathbb{R}^N \times \mathbb{R}^N, w_0)$ [which insures that $G(F) \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$ so that $G(F)^{\wedge}(\xi)$ makes sense]. Without this restriction, the left-hand side of (5.2) should be understood as the limit of $\|G(F_n)\|_{\dot{H}^{N-1/2}(\mathbb{R}^N)}$ of any sequence $F_n$ satisfying the condition in Theorem BD and $\|F - F_n\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_1)} \rightarrow 0 (n \rightarrow \infty)$.

Now we extend Theorem BD to a general kernel.
Theorem 4. Assume \( B(z, \omega) = b(|z|, \theta) \) satisfy
\[
\int_0^{\pi/2} |b(1, \theta)|^2 (\tan \theta)^{N-2} \, d\theta < \infty
\]
and there exists a function \( a: (0, \infty) \to \mathbb{R} \setminus \{0\} \) such that for any \( \theta \in (0, \pi/2) \), the function \( r \to b(r, \theta)/a(r) \) is absolutely continuous on every closed subinterval of \((0, \infty)\) and
\[
\int_0^\infty \left( \int_0^{\pi/2} \left| \frac{\partial}{\partial r} \left( \frac{b(r, \theta)}{a(r)} \right) \right|^2 (\tan \theta)^{N-2} \, d\theta \right)^{1/2} \, dr < \infty.
\]
Let \( w_0, w_1 \), and \( w_2 \) be defined in Theorem BD. Then for all \( F \in L^2(\mathbb{R}^N \times \mathbb{R}^N, w_0) \cap L^2(\mathbb{R}^N \times \mathbb{R}^N, w_1) \cap L^2(\mathbb{R}^N \times \mathbb{R}^N, w_2) \),
\[
\|G(F)\|_{H^0(\mathbb{R}^N)} \leq C_{a,b} \left[ \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_0)} \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_1)} \right]^{1/2},
\]
(5.3)
\[
\|G(F)\|_{H^0(\mathbb{R}^N)} \leq C_{a,b} \left[ \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_0)} + \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_1)} \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_2)} \right]^{1/2},
\]
(5.4)
where
\[
C_{a,b} = \left[ \int_0^{\pi/2} \frac{b(1, \theta)}{a(1)}^2 (\tan \theta)^{N-2} \, d\theta \right]^{1/2} + \int_0^\infty \left( \int_0^{\pi/2} \left| \frac{\partial}{\partial r} \left( \frac{b(r, \theta)}{a(r)} \right) \right|^2 (\tan \theta)^{N-2} \, d\theta \right)^{1/2} \, dr.
\]
Remarks. (1) If \( b(r, \theta) = a(r)b(\theta) \), we go back to Theorem BD. (2) As shown in (2) for Theorem BD, some special cases of Theorem 4 are also given easily. For instance, if in Theorem 4, \( a(r) = (1 + r^2)^{\kappa/2} \) (\( \kappa \geq 0 \)) and \( f, g \in L^2(\mathbb{R}^N) \cap L^1_{\kappa+}(\mathbb{R}^N) \), then
\[
\|Q^+ (f, g)\|_{H^0(\mathbb{R}^N)} \leq C(\|f\|_{L^2(\mathbb{R}^N)} \|g\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^1_{\kappa}(\mathbb{R}^N)} \|g\|_{L^1_{\kappa}(\mathbb{R}^N)}).
\]
Proof of Theorem 4. Define
\[
b_s(\theta) = \frac{b(1, \theta)}{a(1)}, \quad b_\gamma(\theta) = \frac{\partial}{\partial s} \left( \frac{b(s, \theta)}{a(s)} \right),
\]
\[
a_\gamma(r) = a(r) \text{sign}(r - 1)1_{1 \leq s \leq 1 \vee r},
\]
where \( y_1 \wedge y_2 = \min(y_1, y_2) \) and \( y_1 \vee y_2 = \max(y_1, y_2) \). Then the kernel \( b(r, \theta) \) is written as a summation of separated variable kernels:

\[
b(r, \theta) = a(r) b^\varphi(\theta) + \int_{s=0}^{\infty} a_s(r) b_s(\theta) \, ds, \quad r \in (0, \infty), \theta \in (0, \pi/2).
\]

Accordingly, let

\[
B^\varphi(z, \omega) = a(|z|) b^\varphi(\theta), \quad B_s(z, \omega) = a_s(|z|) b_s(\theta)
\]

and let \( G^\varphi, G_s \) be the gain operators corresponding to the kernels \( B^\varphi(z, \omega) \) and \( B_s(z, \omega) \), respectively. Then

\[
G(F)(\nu) = G^\varphi(F)(\nu) + \int_{s=0}^{\infty} G_s(F)(\nu) \, ds, \quad \text{(5.5)}
\]

and

\[
G(F) (\xi) = G^\varphi(F) (\xi) + \int_{s=0}^{\infty} G_s(F) (\xi) \, ds. \quad \text{(5.6)}
\]

The two identities (5.5) and (5.6) can be easily shown to hold rigorously. In fact, if we denote by \( G^{\varphi,+} \) and \( G_s^+ \) the positive operators corresponding to the kernels \( |B^\varphi(z, \omega)| \) and \( |B_s(z, \omega)| \), respectively, then, by the Cauchy–Schwarz inequality [note that \( |a_s(r)| \leq |a(r)| = w_0(r) \)] we have

\[
\int_{\mathbb{R}^N} G^{\varphi,+} (|F|) (\nu) \, d\nu + \int_{0}^{\infty} ds \int_{\mathbb{R}^N} G^+_s (|F|) (\nu) \, d\nu
\]

\[
\leq 2|\mathbb{S}|^{N-2} \left( \int_{0}^{\pi/2} \left| b^\varphi(\theta) (\sin \theta)^{N-2} \, d\theta + \int_{0}^{\infty} ds \cdot \int_{0}^{\pi/2} \left| b_s(\theta) (\sin \theta)^{N-2} \, d\theta \right| \right) ||F||_{L_h^1(\mathbb{R}^n \times B^\varphi, w_0)}
\]

\[
\leq 2|\mathbb{S}|^{N-2} |C_{a,b} \left( \int_{0}^{\pi/2} (\cos \theta \sin \theta)^{N-2} \, d\theta \right)^{1/2} ||F||_{L_h^1(\mathbb{R}^n \times B^\varphi, w_0)} < \infty. \quad \text{(5.7)}
\]

Now, using the Minkowski inequality to (5.6) we obtain

\[
||G(F)||_{L_2^{(N-1)/2}(\mathbb{R}^n)} \leq ||G^\varphi(F)||_{L_2^{(N-1)/2}(\mathbb{R}^n)} + \int_{s=0}^{\infty} ||G_s(F)||_{L_2^{(N-1)/2}(\mathbb{R}^n)} \, ds. \quad \text{(5.8)}
\]
Furthermore, applying Theorem BD (5.2) and (5.1) we have
\[
\|G^s(F)\|_{L^{(N-1)/2}(\mathbb{R}^N)} \leq C_N \left[ \int_0^{\pi/2} |b^s(\theta)|^2 (\tan \theta)^{N-2} d\theta \right]^{1/2} \\
\times \left[ \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_j)} \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_j)} \right]^{1/2},
\]
and, for all \( s > 0 \),
\[
\|G_s(F)\|_{L^{(N-1)/2}(\mathbb{R}^N)} \leq C_N \left[ \int_0^{\pi/2} |b_s(\theta)|^2 (\tan \theta)^{N-2} d\theta \right]^{1/2} \\
\times \left[ \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_j)} \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_j)} \right]^{1/2}.
\]
Combining these with (5.8) yields (5.3). Finally, since by (5.7),
\[
|G(F)^s(\xi)| \leq \|G(F)\|_{L^2(\mathbb{R}^N)} \leq C_N C_{\alpha, \beta} \|F\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N, w_j)},
\]
(5.4) follows from (5.3).

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REFERENCES