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On calculus of local fractional derivatives

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Abstract

Local fractional derivative (LFD) operators have been introduced in the recent literature (Chaos 6 (1996) 505–513). Being local in nature these derivatives have proven useful in studying fractional differentiability properties of highly irregular and nowhere differentiable functions. In the present paper we prove Leibniz rule, chain rule for LFD operators. Generalization of directional LFD and multivariable fractional Taylor series to higher orders have been presented. © 2002 Elsevier Science (USA). All rights reserved.

Keywords: Riemann–Liouville fractional derivatives/integrals; Local fractional derivatives; Local fractional Taylor series

1. Introduction

Fractional calculus [1,2] developed since 17th century thorough the pioneering works of Leibniz, Euler, Lagrange, Abel, Liouville and many others deals with generalization of differentiation and integration to fractional order. In recent years the term "fractional calculus" refers to integration and differentiation to an arbitrary order. Complex analytic version of fractional differentiation/integration has been discussed by Srivastava and Owa [3]. Interestingly these derivatives/integrals are not mere mathematical curiosities but have applications in visco-elasticity, feedback amplifiers, electrical circuits, electro-analytical chemistry, fractional

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multipoles, neuron modelling and related areas in physics, chemistry, and biological sciences [2]. It is well known that the fractional derivatives/integrals have been defined in a variety of ways as [1,2] given by Riemann, Liouville, Grűnwald, Weyl and others. These definitions, however, are non-local in nature, which makes them unsuitable for investigating properties related to local scaling or fractional differentiability [4]. Kolwankar and Gangal [4,5] have proposed local fractional derivative (LFD) operator through renormalization of Riemann–Liouville definition. LFD follows as a natural generalization of the usual derivatives to fractional order conserving the local nature of the derivatives in contrast to traditional definitions of fractional derivatives and used further to explore local scaling properties of highly irregular and nowhere differentiable Weierstrass functions [4]. LFD operators engender a new kind of differential equations, referred as local fractional differential equations (LFDE) different from the conventional fractional differential equations. The fractional analog [6] of the Fokker–Planck equation [7] involving LFDs has been used in modelling phenomena involving fractal time. LFDs therefore provide a much needed tool for calculus of fractal space–time.

As a pursuit of these we herein investigate the formal properties of LFD operators. In the present work we prove Leibniz rule for a product of functions and subsequently derive chain rule for evaluating LFD of composite function. Generalizations of directional LFD and fractional multivariable Taylor series to higher orders have also been presented.

The paper has been organised as follows. In Section 2 we give basic definitions in Riemann–Liouville fractional calculus and LFD operator. Leibniz rule and chain rule for LFD have been derived in Section 3 and Section 4. Extensions of directional LFDs and local fractional Taylor series to higher orders have been presented in Sections 5 and 6.

2. Basic definitions and preliminaries

2.1. Riemann–Liouville fractional calculus

Definitions of Riemann–Liouville fractional derivative/integral and their properties are given below.

Riemann–Liouville fractional derivative of a real function f is given for $x > a$ as [1,2]

$$\frac{d^\alpha f(x)}{d(x-a)^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\, dt, \quad n-1 \leqslant \alpha < n, \tag{1}$$

where $n \in \mathbb{N}$. If further $f(x) \in C^n(\mathbb{R})$, repeated integration by parts leads to

$$\frac{d^\alpha f(x)}{d(x-a)^\alpha}=\sum_{k=0}^{n-1}\frac{f^{(k)}(a)(x-a)^{-\alpha+k}}{\Gamma(k+1-\alpha)}+\frac{1}{\Gamma(n-\alpha)}\int_a^x\frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\,dt. \tag{2}$$

Riemann–Liouville fractional integral of a real function f is given by

$$\frac{d^\alpha f(x)}{d(x-a)^\alpha}=\frac{1}{\Gamma(-\alpha)}\int_a^x\frac{f(t)}{(x-t)^{\alpha+1}}\,dt,\quad \alpha<0,\ x>a. \tag{3}$$

Note that [1,2] if $f(x)=(x-a)^\beta$, $\beta>-1$, $x>a$, then

$$\frac{d^\alpha f(x)}{d(x-a)^\alpha}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}. \tag{4}$$

From (3) and (4) it follows that

$$\frac{d^\alpha 1}{d(x-a)^\alpha}=\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)},\quad x>a, \tag{5}$$

where α is any real number. Composition of the Riemann–Liouville fractional derivative with integer-order derivatives for $f\in C^n$, $\alpha>0$, $n\in\mathbb{N}$ [2] gives

$$\frac{d^{\alpha+n} f(x)}{d(x-a)^{\alpha+n}}=\frac{d^\alpha f^{(n)}(x)}{d(x-a)^\alpha}+\sum_{j=0}^{n-1}\frac{f^{(j)}(a)(x-a)^{j-\alpha-n}}{\Gamma(1+j-\alpha-n)},\quad x>a. \tag{6}$$

If the fractional derivative $d^\alpha f(x)/d(x-a)^\alpha$ of a function $f(x)$ is integrable, then [2]

$$\frac{d^{-\alpha}\left(\frac{d^\alpha f(x)}{d(x-a)^\alpha}\right)}{d(x-a)^{-\alpha}}=f(x)-\sum_{j=1}^{n}\left[\frac{d^{\alpha-j} f(x)}{d(x-a)^{\alpha-j}}\right]_{x=a}\frac{(x-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}, \tag{7}$$

where $n-1\leqslant\alpha<n$, $x>a$.

Leibniz rule for fractional differentiation is given below [2].

If $f(x)$ is continuous in $[a,b]$ and $\varphi(x)\in C^{n+1}[a,b]$, then the fractional derivative of the product $\varphi(x)f(x)$ is given by

$$\frac{d^\alpha(\varphi(x)f(x))}{d(x-a)^\alpha}=\sum_{k=0}^{n}\binom{\alpha}{k}\varphi^{(k)}(x)\frac{d^{\alpha-k}f(x)}{d(x-a)^{\alpha-k}}-R_n^\alpha(x),$$
$$0<\alpha\leqslant n-1, \tag{8}$$

where

$$R_n^\alpha(x)=\frac{1}{n!\Gamma(-\alpha)}\int_a^x(x-u)^{-\alpha-1}f(u)\,du\int_u^x\varphi^{(n+1)}(r)(u-r)^n\,dr \tag{9}$$

and $\binom{\alpha}{k}$ is the generalized binomial coefficient $(=\Gamma(\alpha+1)(k!\Gamma(\alpha-k+1))^{-1})$.

The fractional derivative of the composite analytic function $\varphi(x) = f(h(x))$ [2] turns out to be

$$\frac{d^\alpha \varphi(x)}{d(x-a)^\alpha} = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}\varphi(x) + \sum_{k=1}^{\infty} \binom{\alpha}{k} \frac{k!(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{m=1}^{k} f^{(m)}\bigl(h(x)\bigr) \times \sum \prod_{r=1}^{k} \frac{1}{a_r!}\left(\frac{h^{(r)}(x)}{r!}\right)^{a_r}, \tag{10}$$

where the sum $\sum$ extends over all combinations of non-negative integral values of $a_1, a_2, \ldots, a_k$ such that $\sum_{r=1}^{k} r a_r = n$ and $\sum_{r=1}^{k} a_r = m$.

2.2. Local fractional derivative

Kolwankar and Gangal [4] have defined LFD as follows. If for a function $f:[0,1] \to \mathbb{R}$, the limit

$$\mathbb{D}_{\pm}^{\alpha} f(x) = \lim_{y \to x\pm} \frac{d^\alpha (f(y) - f(x))}{d(\pm(y-x))^\alpha}, \quad 0 < \alpha < 1, \tag{11}$$

exists and is finite, then f is said to have right (left) LFD of order α at $y = x$. If for a function $f:[0,1] \to \mathbb{R}$, the limit

$$\mathbb{D}_{\pm}^{\alpha} f(x) = \lim_{y \to x\pm} \frac{d^\alpha \bigl(f(y) - \sum_{k=0}^{n} (f^{(k)}(x)/\Gamma(k+1))(y-x)^k\bigr)}{d(\pm(y-x))^\alpha} \tag{12}$$

exists and is finite, where n is the largest integer for which nth-order derivative of $f(y)$ at x exist and is finite, then $\mathbb{D}_{\pm}^{\alpha} f(x)$ are called as the right (left) LFD of order α $(n < \alpha < n+1)$, at $y = x$.

The definition of right (left) directional local fractional derivative has been given as follows [8].

Consider a function $f:\mathbb{R}^n \to \mathbb{R}$. The right (left) directional LFD of f at $X \in \mathbb{R}^n$ of order α in direction $V \in \mathbb{R}^n$ is given by

$$\bigl(\mathbb{D}_{V}^{\alpha}\bigr)_{\pm} f(X) = \lim_{t \to 0^+} \frac{d^\alpha (f(X \pm tV) - f(X))}{dt^\alpha}, \quad 0 < \alpha < 1, \tag{13}$$

provided the limit on the R.H.S. exists.

3. Leibniz rule for LFD

In order to prove Leibniz rule for LFD we prove the following lemma.

Lemma 3.1. *Let $f:[a,b] \to \mathbb{R}$ be a function of class C^{r+1}. Then $\mathbb{D}_{\pm}^{\alpha} f(x) = 0$, $x \in (a,b)$ and $\mathbb{D}_{+}^{\alpha} f(a) = \mathbb{D}_{-}^{\alpha} f(b) = 0$, for $m < \alpha < m+1 \leqslant r+1$, $m \in \mathbb{N} \cup \{0\}$.*

Proof. For $m < \alpha < m + 1$,

$$\mathbb{D}_+^\alpha f(x) = \lim_{\delta \to 0^+} \frac{d^\alpha \big(f(x+\delta) - \sum_{n=0}^m \frac{f^{(n)}(x)}{\Gamma(n+1)} \delta^n \big)}{d\delta^\alpha}.$$

For a given x define

$$g(\delta) = f(x+\delta) - \sum_{n=0}^m \frac{f^{(n)}(x)}{\Gamma(n+1)} \delta^n.$$

Then in view of (2) we get

$$\mathbb{D}_+^\alpha f(x) = \lim_{\delta \to 0^+} \left[\sum_{k=0}^m \frac{g^{(k)}(0)\delta^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \int_0^\delta \frac{(\delta-u)^{m-\alpha} g^{(m+1)}(u)\, du}{\Gamma(-\alpha+m+1)} \right]. \tag{14}$$

Since

$$g^{(k)}(\delta) = f^{(k)}(x+\delta) - \sum_{n=k}^m \frac{f^{(n)}(x)}{\Gamma(n+1)} n(n-1)\dots(n-k+1)\delta^{n-k},$$

$$g^{(k)}(0) = f^{(k)}(x) - f^{(k)}(x) = 0, \tag{15}$$

Eqs. (14) and (15) lead to

$$\mathbb{D}_+^\alpha f(x) = \lim_{\delta \to 0^+} \frac{1}{\Gamma(-\alpha+m+1)} \int_0^\delta (\delta-u)^{m-\alpha} g^{(m+1)}(u)\, du = 0,$$

as $g^{(m+1)}(u)$ is continuous and $\lim_{\delta \to 0^+} \int_0^\delta (\delta-u)^{m-\alpha}\, du = 0$. Similarly we can show that $\mathbb{D}_-^\alpha f(x) = 0$, $x \in (a,b)$ and $\mathbb{D}_+^\alpha f(a) = \mathbb{D}_-^\alpha f(b) = 0$. □

Now we proceed to prove Leibniz rule for LFD of order α, $0 < \alpha < 1$, for product of two functions.

Theorem 3.1. *Let $f(x)$ be continuous on $[a,b]$ and $\mathbb{D}_+^\alpha f(a)$, $\mathbb{D}_-^\alpha f(b)$ and $\mathbb{D}_\pm^\alpha f(x)$ exist for every $x \in (a,b)$. If further $\varphi(x) \in C^3[a,b]$, then for $0 < \alpha < 1$*

$$\mathbb{D}_+^\alpha \big((\varphi f)(a)\big) = \varphi(a)\mathbb{D}_+^\alpha f(a), \qquad \mathbb{D}_-^\alpha \big((\varphi f)(b)\big) = \varphi(b)\mathbb{D}_+^\alpha f(b),$$

$$\mathbb{D}_\pm^\alpha \big((\varphi f)(x)\big) = \varphi(x)\mathbb{D}_\pm^\alpha f(x), \quad x \in (a,b). \tag{16}$$

Proof.

$$\mathbb{D}_+^\alpha \big((\varphi f)(x)\big) = \lim_{\delta \to 0^+} \frac{d^\alpha ((f\varphi(x+\delta)) - (f\varphi)(x))}{d\delta^\alpha}$$

$$= \varphi(x)\mathbb{D}_{+}^{\alpha} f(x) + \lim_{\delta \to 0^+} \frac{d^\alpha (f(x+\delta)[\varphi(x+\delta) - \varphi(x)])}{d\delta^\alpha}. \tag{17}$$

From Leibniz rule for ordinary fractional derivative (cf. Eq. (8)) we get

$$\frac{d^\alpha (f(x+\delta)[\varphi(x+\delta) - \varphi(x)])}{d\delta^\alpha} = \sum_{j=0}^{2} \binom{\alpha}{j} \bigl[\varphi(x+\delta) - \varphi(x)\bigr]^{(j)} \frac{d^{\alpha-j} f(x+\delta)}{d\delta^{\alpha-j}} - R_2^\alpha(\delta), \tag{18}$$

where

$$R_2^\alpha(\delta) = \frac{1}{2\Gamma(-\alpha)} \int_0^\delta (\delta-u)^{-\alpha-1} f(x+u)\, du \int_u^\delta \varphi^{(3)}(x+r)(u-r)^2\, dr.$$

Note that

$$\begin{aligned} &\lim_{\delta \to 0^+} \frac{d^\alpha f(x+\delta)}{d\delta^\alpha} \bigl(\varphi(x+\delta) - \varphi(x)\bigr) \\ &\quad = \frac{f(x)}{\Gamma(1-\alpha)} \lim_{\delta \to 0^+} \frac{\varphi(x+\delta) - \varphi(x)}{\delta^\alpha} \\ &\quad = \frac{f(x)}{\Gamma(1-\alpha)} \frac{d\varphi(\delta)}{d\delta} \lim_{\delta \to 0^+} \delta^{1-\alpha} = 0. \end{aligned} \tag{19}$$

Observe for $j = 1, 2$

$$\lim_{\delta \to 0^+} \frac{d^{\alpha-j} f(x+\delta)}{d\delta^{\alpha-j}} = \frac{1}{\Gamma(j-\alpha)} \lim_{\delta \to 0^+} \int_0^\delta \frac{f(x+u)}{(\delta-u)^{\alpha-j+1}}\, du = 0, \tag{20}$$

as f is bounded and $\lim_{\delta \to 0^+} \int_0^\delta (\delta-u)^{j-\alpha-1}\, du = 0$.

On substituting $r = u + h(\delta - u)$, it follows

$$\begin{aligned} \lim_{\delta \to 0^+} R_2^\alpha(\delta) = \frac{1}{2\Gamma(-\alpha)} \lim_{\delta \to 0^+} \int_0^\delta (\delta-u)^{2-\alpha} f(x+u)\, du \\ \times \int_0^1 h^2 \varphi^{(3)}\bigl(x+u+h(\delta-u)\bigr)\, dh = 0, \end{aligned} \tag{21}$$

as $\int_{0^+}^1 h^2\varphi^{(3)}(x+u+h(\delta-u))\, dh$ is finite, f is bounded and $\lim_{\delta \to 0^+} \int_0^\delta (\delta - u)^{2-\alpha}\, du = 0$.

Thus, in view of Eqs. (19)–(21),

$$\lim_{\delta\to 0^+} \frac{d^\alpha(f(x+\delta)[\varphi(x+\delta)-\varphi(x)])}{d\delta^\alpha} = 0. \tag{22}$$

Hence from Eqs. (17) and (22) we get $\mathbb{D}_+^\alpha(\varphi(x)f(x)) = \varphi(x)\mathbb{D}_+^\alpha f(x)$, where $0<\alpha<1$.

A similar proof can be given for $\mathbb{D}_-^\alpha f$. Further note that $\mathbb{D}_+^\alpha((\varphi f))(a) = \varphi(a)\mathbb{D}_+^\alpha f(a)$ and $\mathbb{D}_-^\alpha((\varphi f))(b) = \varphi(b)\mathbb{D}_-^\alpha f(b)$ whenever $\mathbb{D}_+^\alpha f(a)$ and $\mathbb{D}_-^\alpha f(b)$ exist. □

We further extend the Leibniz rule to the case where $n<\alpha<n+1$, using Theorem 3.1.

Theorem 3.2. *Let $f(x)\in C^r[a,b]$ and $\varphi(x)\in C^{r+3}[a,b]$. If $\mathbb{D}^{\alpha-n}f^{(n)}(x)$ exists, where $n<\alpha<n+1\leqslant r+1$, $x\in(a,b)$, $n\in\mathbb{N}\cup\{0\}$, then*

$$\mathbb{D}_\pm^\alpha\big(\varphi(x)f(x)\big) = \varphi(x)\mathbb{D}_\pm^{\alpha-n}f^{(n)}(x). \tag{23}$$

Proof. For fixed $x\in(a,b)$

$$\mathbb{D}_+^\alpha\big((\varphi f)(x)\big) = \lim_{\delta\to 0^+}\frac{d^\alpha g(\delta)}{d\delta^\alpha},$$

where

$$g(\delta) = (f\varphi)(x+\delta) - \sum_{k=0}^{n}\Big(\big[(f\varphi)(x)\big]^{(k)}\big/\Gamma(k+1)\Big)\delta^k.$$

Using (6),

$$\frac{d^\alpha g(\delta)}{d\delta^\alpha} = \frac{d^{\alpha-n}}{d\delta^{\alpha-n}}\left[\frac{d^n g(\delta)}{d\delta^n} + \sum_{j=0}^{n-1}\frac{\delta^{j-\alpha-n}}{\Gamma(1+j-\alpha-n)}\frac{d^j g(0)}{d\delta^j}\right]. \tag{24}$$

Note that

$$\begin{aligned}\frac{d^n g(\delta)}{d\delta^n} &= \sum_{k=0}^{n}\binom{n}{k}\varphi^{(k)}(x+\delta)f^{(n-k)}(x+\delta) - \big(\varphi(x)f(x)\big)^{(n)}\\ &= \sum_{k=0}^{n}\binom{n}{k}\big\{f^{(n-k)}(x+\delta)\big[\varphi^{(k)}(x+\delta)-\varphi^{(k)}(x)\big]\big\}\\ &\quad+\sum_{k=0}^{n}\binom{n}{k}\big\{\varphi^{(k)}(x)\big[f^{(n-k)}(x+\delta)-f^{(n-k)}(x)\big]\big\}.\end{aligned} \tag{25}$$

Denote $\alpha-n=\beta$. From Eq. (22) we get

$$\lim_{\delta\to 0^+}\frac{d^\beta(f^{(n-k)}(x+\delta)[\varphi^{(k)}(x+\delta)-\varphi^{(k)}(x)])}{d\delta^\beta} = 0, \quad k=0,1,\ldots,n. \tag{26}$$

Equation (26), Theorem 3.1 and the definition of LFD together imply

$$\lim_{\delta\to 0^+}\frac{d^\beta}{d\delta^\beta}\left[\frac{d^n g(\delta)}{d\delta^n}\right] = \lim_{\delta\to 0^+}\sum_{k=0}^{n}\frac{d^\beta(\varphi^{(k)}(x)[f^{(n-k)}(x+\delta)-f^{(n-k)}(x)])}{d\delta^\beta}$$
$$= \sum_{k=0}^{n}\varphi^{(k)}(x)\mathbb{D}_+^\beta f^{(n-k)}(x). \tag{27}$$

But $f^{(n-k)}(x)\in C^k[a,b]$, for $k=1,2,\ldots,n$. Thus, in view of Lemma 3.1, $\mathbb{D}_+^\beta f^{(n-k)}(x)=0$, for $k=1,2,\ldots,n$.

Hence

$$\lim_{\delta\to 0^+}\frac{d^\beta}{d\delta^\beta}\left[\frac{d^n g(\delta)}{d\delta^n}\right] = \varphi(x)\mathbb{D}_+^{\alpha-n} f^{(n)}(x). \tag{28}$$

On the other hand,

$$\left.\frac{d^j g(\delta)}{d\delta^j}\right|_{\delta=0} = \sum_{r=0}^{j}\binom{j}{r} f^{(r)}(x)\varphi^{(j-r)}(x) - \frac{d^j(f(x)\varphi(x))}{dx^j} = 0. \tag{29}$$

Using Eqs. (24), (28) and (29) we get

$$\mathbb{D}_+^\alpha\big(\varphi(x)f(x)\big) = \lim_{\delta\to 0^+}\frac{d^\alpha g(\delta)}{d\delta^\alpha} = \varphi(x)\mathbb{D}_+^{\alpha-n}f^{(n)}(x), \quad n<\alpha<n+1.$$

A similar proof can be given for $\mathbb{D}_-^\alpha f$. Further note that $\mathbb{D}_+^\alpha((\varphi f))(a) = \varphi(a)\mathbb{D}_+^{\alpha-n}f^{(n)}(a)$ and $\mathbb{D}_-^\alpha((\varphi f))(b)=\varphi(b)\mathbb{D}_-^{\alpha-n}f^{(n)}(b)$ whenever $\mathbb{D}_+^{\alpha-n}f^{(n)}(a)$ and $\mathbb{D}_-^{\alpha-n}f^{(n)}(b)$ exist. Hence the theorem. □

Note: In case $\varphi(x)=\text{const}$, the above theorem is valid even if $f^{(r)}$ exists, without being continuous.

4. Chain rule for LFD

The Leibniz rule enables us to find LFD of a composite function, i.e., the chain rule, which we derive below.

Theorem 4.1. *Let $h:[a,b]\to\mathbb{R}$ be a function of class C^{n+3} on $[a,b]$, f be a function of class C^n on $h[a,b]$, and $\mathbb{D}_\pm^{\alpha-n}[f^{(n)}(h(x))]$ exists. Then*

$$\mathbb{D}_\pm^\alpha\big[f\big(h(x)\big)\big] = \left(\frac{dh}{dx}\right)^n\mathbb{D}_\pm^{\alpha-n}\big[f^{(n)}\big(h(x)\big)\big], \tag{30}$$

where $n<\alpha<n+1$, $n\in\mathbb{N}\cup\{0\}$.

Proof. In view of the Leibniz rule for LFD,

$$\mathbb{D}_{\pm}^{\alpha}\big[f\big(h(x)\big)\big]=\mathbb{D}_{\pm}^{\alpha-n}\big[f\big(h(x)\big)\big]^{(n)}. \tag{31}$$

The nth-order derivative of $f(h(x))$ is evaluated with the help of the Faá di Bruno formula [2], which states

$$\big[f\big(h(x)\big)\big]^{(n)}=n!\sum_{m=1}^{n}f^{(m)}\big(h(x)\big)\sum\prod_{r=1}^{n}\frac{1}{a_r!}\left(\frac{h^{(r)}(x)}{r!}\right)^{a_r}, \tag{32}$$

where the sum $\sum$ extends over all combinations of non-negative integer values of $a_1, a_2, \ldots, a_n$ such that $\sum_{r=1}^{n} r a_r = n$ and $\sum_{r=1}^{n} a_r = m$. As $h \in C^{n+3}[a,b]$ by assumption and $\prod_{r=1}^{n}(1/a_r!)(h^{(r)}(x)/r!)^{a_r} \in C^3[a,b]$, using Eqs. (32) and (16) we obtain

$$\begin{aligned}
&\mathbb{D}_{\pm}^{\alpha-n}\Big(\big[f\big(h(x)\big)\big]^{(n)}\Big)\\
&\quad=\mathbb{D}_{\pm}^{\alpha-n}\left[n!\sum_{m=1}^{n}f^{(m)}\big(h(x)\big)\sum\prod_{r=1}^{n}\frac{1}{a_r!}\left(\frac{h^{(r)}(x)}{r!}\right)^{a_r}\right]\\
&\quad=n!\left[\sum_{m=1}^{n}\mathbb{D}_{\pm}^{\alpha-n}\big[f^{(m)}\big(h(x)\big)\big]\sum\prod_{r=1}^{n}\frac{1}{a_r!}\left(\frac{h^{(r)}(x)}{r!}\right)^{a_r}\right].
\end{aligned}$$

Note that $0<\alpha-n<1,\ f\in C^n[a,b]$; hence in view of Lemma 3.1 we get

$$\mathbb{D}_{\pm}^{\alpha-n}\big(f^{(m)}\big(h(x)\big)\big)=0,\quad m=1,2,\ldots,n-1.$$

Therefore

$$\mathbb{D}_{\pm}^{\alpha-n}\Big(\big[f\big(h(x)\big)\big]^{(n)}\Big)=n!\mathbb{D}_{\pm}^{\alpha-n}\big[f^{(n)}\big(h(x)\big)\big]\left(\sum\prod_{r=1}^{n}\frac{1}{a_r!}\left(\frac{h^{(r)}(x)}{r!}\right)^{a_r}\right), \tag{33}$$

where $\sum$ extends over all combinations satisfying $\sum_{r=1}^{n} r a_r = n$ and $\sum_{r=1}^{n} a_r = n$, which has unique possibility, viz. $a_1 = n,\ a_2 = a_3 = \cdots = a_n = 0$. Hence in view of (31) and (33) we get

$$\begin{aligned}
&\mathbb{D}_{\pm}^{\alpha}\big[f\big(h(x)\big)\big]=\left(\frac{dh}{dx}\right)^{n}\mathbb{D}_{\pm}^{\alpha-n}\big[f^{(n)}\big(h(x)\big)\big],\\
&\quad n<\alpha<n+1,\ n\in\mathbb{N}\cup\{0\}. \qquad \square
\end{aligned}$$

This theorem yields the following interesting corollary.

Corollary 4.1. *If $f:\mathbb{R}^n\to\mathbb{R}$ is a function of class C^1, $x_i(t):\mathbb{R}\to\mathbb{R}$ are of class C^r, $r\geqslant 4$, and $\mathbb{D}_{\pm}^{\alpha-1}[(\partial f/\partial x_i)(X(t))]$ exist for $i=1,2,\ldots,n$, where $X(t)=(x_1(t),\ldots,x_n(t))$, then*

$$\mathbb{D}_{\pm}^{\alpha} f\big(X(t)\big) = \sum_{i=1}^{n} \frac{dx_i}{dt} \mathbb{D}_{\pm}^{\alpha-1}\left(\frac{\partial f}{\partial x_i}\big(X(t)\big)\right), \quad \textit{for } 1 < \alpha < 2. \tag{34}$$

Proof. Using Theorem 4.1,

$$\mathbb{D}_{\pm}^{\alpha} f\big(x_1(t), x_2(t), \ldots, x_n(t)\big) = \mathbb{D}_{\pm}^{\alpha-1}\left(\frac{d[f(x_1(t), x_2(t), \ldots, x_n(t))]}{dt}\right)$$
$$= \mathbb{D}_{\pm}^{\alpha-1}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big(X(t)\big)\frac{dx_i}{dt}\right) = \sum_{i=1}^{n} \frac{dx_i}{dt} \mathbb{D}_{\pm}^{\alpha-1}\left(\frac{\partial f}{\partial x_i}\big(X(t)\big)\right). \qquad \square$$

5. Higher-order directional LFD

The definition of LFD has been extended to directional LFD for functions of many variables, in case where the order of differentiation α is between 0 and 1 [8].

We generalize the definition of the directional LFD for order α, for $n < \alpha < n+1$, as follows.

Definition 5.1. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function such that all its partial derivatives of order n exist. If

$$\big(\mathbb{D}_V^{\alpha}\big)_{\pm} f(X) = \lim_{t \to 0^+} \frac{d^{\alpha}\big(f(X \pm tV) - \sum_{k=0}^{n} \frac{1}{\Gamma(k+1)}\big(\sum_{i=1}^{m} tv_i \frac{\partial}{\partial x_i}\big)^k f(X)\big)}{dt^{\alpha}}$$

exists, where $n < \alpha < n+1$, then $(\mathbb{D}_V^{\alpha})_{\pm} f(X)$ is called as the right (left) directional LFD of order α at X in the direction of V for $X, V \in \mathbb{R}^m$.

These definitions render the following theorem.

Theorem 5.1. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function whose all the partial derivatives of order n exist. Then*

$$\big(\mathbb{D}_V^{\alpha}\big)_{\pm} f(X) = \big(\mathbb{D}_V^{\alpha-n}\big)_{\pm}\left(\left(\sum_{i=1}^{m} v_i \frac{\partial}{\partial x_i}\right)^n f(X)\right), \quad n < \alpha < n+1, \tag{35}$$

provided the expression on the R.H.S. exists.

Proof.

$$\big(\mathbb{D}_V^{\alpha}\big)_{+} f(X) = \frac{1}{\Gamma(n+1-\alpha)} \lim_{\delta \to 0^+} \frac{d^{n+1} I_1(\delta)}{d\delta^{n+1}},$$

where

$$I_1(\delta) = \int_0^{\delta} \frac{f(X + sV) - \sum_{k=0}^{n} \frac{1}{\Gamma(k+1)}\big(\sum_{i=1}^{m} sv_i \frac{\partial}{\partial x_i}\big)^k f(X)}{(\delta - s)^{\alpha-n}}\, ds.$$

By performing integration by parts we get

$$I_1(\delta) = \int_0^\delta \frac{\sum_{i=1}^m v_i f_{x_i}(X+sV) - \sum_{k=1}^n \frac{ks^{k-1}}{\Gamma(k+1)}\left(\sum_{i=1}^m v_i \frac{\partial}{\partial x_i}\right)^k f(X)}{(n-\alpha+1)(\delta-s)^{\alpha-n-1}}\, ds.$$

Further we define

$$\begin{aligned} I_2(\delta) &= \frac{dI_1(\delta)}{d\delta} \\ &= \int_0^\delta \frac{\sum_{i=1}^m v_i f_{x_i}(X+sV) - \sum_{k=1}^n \frac{ks^{k-1}}{\Gamma(k+1)}\left(\sum_{i=1}^m v_i \frac{\partial}{\partial x_i}\right)^k f(X)}{(\delta-s)^{\alpha-n}}\, ds. \end{aligned}$$

Again integration by parts leads to

$$\begin{aligned} I_2(\delta) = \frac{1}{n+1-\alpha} \int_0^\delta \Bigg(& \frac{\sum_{i=1}^m v_i \left(\sum_{j=1}^m v_j f_{x_i x_j}(X+sV)\right)}{(\delta-s)^{\alpha-n-1}} \\ & - \frac{\sum_{k=2}^n \frac{k(k-1)s^{k-2}}{\Gamma(k+1)}\left(\sum_{i=1}^m v_i \frac{\partial}{\partial x_i}\right)^k f(X)}{(\delta-s)^{\alpha-n-1}} \Bigg)\, ds. \end{aligned}$$

Hence

$$\begin{aligned} \frac{dI_2(\delta)}{d\delta} = \int_0^\delta \Bigg(& \frac{\left(\sum_{i=1}^m v_i \frac{\partial}{\partial x_i}\right)^2 f(X+sV)}{(\delta-s)^{\alpha-n}} \\ & - \frac{\sum_{k=2}^n \frac{k(k-1)s^{k-2}}{\Gamma(k+1)}\left(\sum_{i=1}^m v_i \frac{\partial}{\partial x_i}\right)^k f(X)}{(\delta-s)^{\alpha-n}} \Bigg)\, ds. \end{aligned}$$

Repeating this process n times we get

$$\begin{aligned} & \left(\mathbb{D}_V^\alpha\right)_+ f(X) \\ & \quad = \lim_{\delta\to 0^+} \frac{d}{d\delta} \int_0^\delta \frac{\left(\sum_{i=1}^m v_i \frac{\partial}{\partial x_i}\right)^n f(X+sV) - \left(\sum_{i=1}^m v_i \frac{\partial}{\partial x_i}\right)^n f(X)}{\Gamma(n+1-\alpha)(\delta-s)^{\alpha-n}}\, ds \\ & \quad = \left(\mathbb{D}_V^{\alpha-n}\right)_+ \left(\left(\sum_{i=1}^m v_i \frac{\partial}{\partial x_i}\right)^n f(X)\right). \end{aligned}$$

A similar proof can be given for $(\mathbb{D}_V^\alpha)_- f$. □

6. Local fractional Taylor series

Local fractional Taylor series for a real function involving LFD has been constructed [4,5]. Here we derive higher-order local fractional Taylor series for multivariable case. To begin with we prove the following lemma.

Lemma 6.1. *Let $f:\mathbb{R}^m \to \mathbb{R}$ be a function of class C^{r+1}. Then*

$$\left(\mathbb{D}_V^{\alpha}\right)_{\pm} f(X) = 0, \quad n < \alpha < n+1 \leqslant r+1. \tag{36}$$

Proof.

$$\left(\mathbb{D}_V^{\alpha}\right)_{+} f(X) = \lim_{t\to 0^+} \frac{d^{\alpha} g(t)}{dt^{\alpha}},$$

where

$$g(t) = f(X+tV) - \sum_{k=0}^{n} \frac{1}{\Gamma(k+1)} \left[\sum_{i=1}^{m} t v_i \frac{\partial}{\partial x_i}\right]^k f(X).$$

So

$$\left(\mathbb{D}_V^{\alpha}\right)_{+} f(X) = \lim_{t\to 0^+} \left[\sum_{r=0}^{n} \frac{g^{(r)}(0) t^{r-\alpha}}{\Gamma(r-\alpha+1)} + \int_0^t \frac{(t-u)^{n-\alpha} g^{(n+1)}(u)}{\Gamma(n-\alpha+1)}\, du\right]. \tag{37}$$

Note

$$g^{(r)}(t) = \left(\sum_{i=1}^{m} v_i \frac{\partial}{\partial x_i}\right)^r f(X+tV) - \sum_{k=r}^{n} \frac{k(k-1)\ldots(k-r+1)t^{k-r}}{\Gamma(k+1)} \left(\sum_{i=1}^{m} v_i \frac{\partial}{\partial x_i}\right)^k f(X).$$

Hence $g^{(r)}(0) = 0$, which after substituting in (37) gives

$$\left(\mathbb{D}_V^{\alpha}\right)_{+} f(X) = \lim_{t\to 0^+} \frac{1}{\Gamma(n-\alpha+1)} \int_0^t (t-u)^{n-\alpha} g^{(n+1)}(u)\, du = 0,$$

as $g^{(n+1)}(u)$ is continuous and $\lim_{t\to 0^+} \int_0^t (t-u)^{n-\alpha}\, du = 0$. A similar proof can be given for $(\mathbb{D}_V^{\alpha})_- f(X)$. □

Theorem 6.1. *Let $f:\mathbb{R}^m \to \mathbb{R}$ be a function of class C^n. If $(\mathbb{D}_V^{\alpha})_{\pm} f(X)$ exists, where $n < \alpha < n+1$, $X, V \in \mathbb{R}^m$, then*

$$f(X \pm tV) = f(X) + \sum_{k=1}^{n} \frac{1}{\Gamma(k+1)} \left(\sum_{i=1}^{m} t v_i \frac{\partial}{\partial x_i} \right)^k f(X) + \frac{(\mathbb{D}_V^\alpha)_\pm f(X)}{\Gamma(\alpha+1)} t^\alpha + \frac{1}{\Gamma(\alpha+1)} \int_0^t \frac{dF(X \pm sV, s, \alpha)}{ds} (t-s)^\alpha \, ds, \quad t > 0, \tag{38}$$

where

$$F(X \pm sV, s, \alpha) = \frac{d^\alpha \left(f(X \pm sV) - \sum_{k=0}^{n} \frac{1}{\Gamma(k+1)} \left(\sum_{i=1}^{m} s v_i \frac{\partial}{\partial x_i} \right)^k f(X) \right)}{ds^\alpha}. \tag{39}$$

Proof. As $(\mathbb{D}_V^\alpha)_\pm f(X) = F(X, 0, \alpha)$, in view of (7) and (39) we get

$$\begin{aligned}
& f(X \pm tV) - \sum_{k=0}^{n} \frac{1}{\Gamma(k+1)} \left(\sum_{i=1}^{m} t v_i \frac{\partial}{\partial x_i} \right)^k f(X) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{F(X \pm sV, s, \alpha)}{(t-s)^{-\alpha+1}} \, ds + \sum_{j=1}^{n} \left[\left(\mathbb{D}_V^{\alpha-j} \right)_\pm f(X) \right] \frac{t^{\alpha-j}}{\Gamma(\alpha-j+1)} \\
&\quad + \frac{t^{\alpha-n-1}}{\Gamma(\alpha-n)} \\
&\qquad \times \left[\int_0^t \frac{f(X \pm sV) - \sum_{k=0}^{n} \frac{1}{\Gamma(k+1)} \left(\sum_{i=1}^{m} s v_i \frac{\partial}{\partial x_i} \right)^k f(X)}{\Gamma(n+1-\alpha)(t-s)^{\alpha-n}} \, ds \right]_{t=0} \\
&= \frac{-F(X \pm sV, s, \alpha)(t-s)^\alpha}{\Gamma(\alpha+1)} \bigg|_0^t \\
&\quad + \int_0^t \left(\frac{dF(X \pm sV, s, \alpha)}{ds} \right) \frac{(t-s)^\alpha \, ds}{\Gamma(\alpha+1)} \\
&= \frac{(\mathbb{D}_V^\alpha)_\pm f(X)}{\Gamma(\alpha+1)} t^\alpha + \frac{1}{\Gamma(\alpha+1)} \int_0^t \left(\frac{dF(X \pm sV, s, \alpha)}{ds} \right) (t-s)^\alpha \, ds.
\end{aligned}$$

From Lemma 6.1, $(\mathbb{D}_V^{\alpha-j})_\pm f(X) = 0$ for $j = 1, \ldots, n$ and

$$\left[\int_0^t \frac{f(X \pm sV) - \sum_{k=0}^{n} \frac{1}{\Gamma(k+1)} \left(\sum_{i=1}^{m} s v_i \frac{\partial}{\partial x_i}\right)^k f(X)}{\Gamma(n+1-\alpha)(t-s)^{\alpha-n}} \, ds\right]_{t=0} = 0,$$

as

$$f(X \pm sV) - \sum_{k=0}^{n} \frac{1}{\Gamma(k+1)} \left(\sum_{i=1}^{m} s v_i \frac{\partial}{\partial x_i}\right)^k f(X)$$

is continuous. Further observe that $\lim_{t\to 0^+} \int_0^t (t-s)^{n-\alpha} \, ds = 0$. Therefore

$$\begin{aligned} f(X \pm tV) = f(X) &+ \sum_{k=1}^{n} \frac{1}{\Gamma(k+1)} \left(\sum_{i=1}^{m} t v_i \frac{\partial}{\partial x_i}\right)^k f(X) \\ &+ \frac{(\mathbb{D}_V^{\alpha})_\pm f(X)}{\Gamma(\alpha+1)} t^\alpha \\ &+ \frac{1}{\Gamma(\alpha+1)} \int_0^t \left(\frac{dF(X \pm sV, s, \alpha)}{ds}\right)(t-s)^\alpha \, ds, \end{aligned}$$

$t > 0$. □

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References

[1] K. Oldham, J. Spanier, The Fractional Calculus, Academic Press, London, 1970.
[2] I. Podlubny, Fractional Differential Equations, Academic Press, London, 1999.
[3] H.M. Srivastava, S. Owa, Univalent Functions, Fractional Calculus and Their Applications, Halsted Press/Wiley, New York, 1989.
[4] K.M. Kolwankar, A.D. Gangal, Fractional differentiability of nowhere differentiable functions and dimension, Chaos 6 (1996) 505–513.
[5] K.M. Kolwankar, A.D. Gangal, Hölder exponents of irregular signals and local fractional derivatives, Pramana J. Phys. 48 (1997) 49–68.
[6] K.M. Kolwankar, A.D. Gangal, Local fractional Fokker–Planck equation, Phys. Rev. Lett. 48 (1997) 49–52.
[7] H. Risken, The Fokker–Planck Equation, Springer-Verlag, Berlin, 1984.
[8] K.M. Kolwankar, A.D. Gangal, Local fractional derivatives and fractal functions of several variables, in: Proceedings of the Conference on Fractals in Engineering, Archanon, 1997.