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Regularity of patterns in the factorization of $n!$

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Abstract

Consider the multiplicities $e_{p_1}(n), e_{p_2}(n), \dots, e_{p_k}(n)$ in which the primes $p_1, p_2, \dots, p_k$ appear in the factorization of $n!$. We show that these multiplicities are jointly uniformly distributed modulo $(m_1, m_2, \dots, m_k)$ for any fixed integers $m_1, m_2, \dots, m_k$, thus improving a result of Luca and Stănică [F. Luca, P. Stănică, On the prime power factorization of $n!$, J. Number Theory 102 (2003) 298–305]. To prove the theorem, we obtain a result regarding the joint distribution of several completely q-additive functions, which seems to be of independent interest.

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1. Introduction and main theorems

Given a non-negative integer n, write

$$n! = 2^{e_2(n)} 3^{e_3(n)} 5^{e_5(n)} \cdots p_l^{e_{p_l}(n)} \tag{1}$$

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(where $l = \pi(n)$ is the number of primes not exceeding n). It is well known that for each prime p we have

$$e_p(n) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots. \tag{2}$$

Even though the formula gives in principle all relevant information regarding the factorization (1), it fails to answer in a simple way some basic questions. Thus, Erdős and Graham [6, p. 77] posed the following

Question 1.1. Does there exist, for any fixed l, some n with all the exponents $e_2(n), e_3(n), e_5(n), \ldots, e_{p_l}(n)$ even?

In [1] this question was answered in the affirmative. Moreover, the following result was obtained. (We actually change the formulation a little, as here it will be more convenient to let $p_1, p_2, \ldots, p_k$ denote any fixed primes, and not necessarily the first k primes.)

Theorem A. *For any primes $p_1, p_2, \ldots, p_k$ there exist infinitely many positive integers $1 = n_0 < n_1 < n_2 < \cdots$ such that, for each j, all the numbers $e_{p_1}(n_j), e_{p_2}(n_j), \ldots, e_{p_k}(n_j)$ are even. Moreover,*

$$n_{j+1} - n_j \leqslant C, \quad i = 1, 2, \ldots,$$

where C is an effectively computable constant.

This result was strengthened in [1] in two directions. First, the requirement that the exponents be even was replaced by the requirement that they be divisible by an arbitrary fixed integer m. Secondly, it turned out that the same is true when one looks at the factorizations of several factorials $n!, (2n)!, \ldots, (an)!$.

Several authors—Chen and Zhu [3], Sander [12], Chen [2] and Luca and Stănică [10]—continued working on this problem, each obtaining a stronger version of Theorem A. In the last of the above papers, the following result was proved.

Theorem B. [10] *Let $p_1, p_2, \ldots, p_l$ be any primes, $m_1, m_2, \ldots, m_l$ positive integers, and $a_1, a_2, \ldots, a_l$ integers with $p_j \nmid m_j$ and $0 \leqslant a_j \leqslant m_j - 1$ for $1 \leqslant j \leqslant l$. Then*

$$\left|\left\{0 \leqslant n < N\colon\ e_{p_j}(n) \equiv a_j \pmod{m_j},\ 1 \leqslant j \leqslant l\right\}\right| = \frac{N}{m_1 \cdots m_l} + O\left(N^{1-\delta}\right),$$

with $\delta = 1/(120 l^2 p^{3m} m^2)$, where $p = \max_{1 \leqslant i \leqslant l} p_i$ and $m = \max_{1 \leqslant i \leqslant l} m_i$.

Luca and Stănică note that the condition, whereby the primes p_j do not divide the corresponding moduli m_j, is probably superfluous. In this paper, we are indeed able to establish their result in full generality. Moreover, in the process we improve the value of the constant δ. Our proof goes according to lines similar to those Luca and Stănică suggested. Namely, we start from Kim's results [9] on the joint distribution of completely q-additive functions, and obtain a version of his result (weaker than the one proposed by Luca and Stănică) which enables us to prove the required result.

In Section 2 we present formally our main results—both the improvement of Theorem B and that of Kim's result. Section 3 is devoted to the proofs of both theorems.

2. The main results

To state our main result we need a few notations. As in Theorem B, let $p_1, \dots, p_l$ be distinct primes and $m_1, \dots, m_l$ any positive integers (which we may assume to be at least 2). Write $m_j = p_j^{\alpha_j} k_j$, with $(k_j, p_j) = 1$. Distinguish between four cases:

Case 1. $\alpha_j = 1$ and $k_j = 1$.

In this case put $u_j = 2$ and $\beta_j = 2$.

Case 2. $\alpha_j \geqslant 2$ and $k_j = 1$.

Put $u_j = 1$ and $\beta_j = \alpha_j$.

Case 3. $\alpha_j = 0$ and $k_j > 1$.

Let u_j be the least positive integer for which

$$\frac{p_j^{u_j} - 1}{p_j - 1} \equiv 0 \pmod{k_j}$$

and $\beta_j = u_j$.

Case 4. $\alpha_j \geqslant 1$ and $k_j > 1$.

Let u_j be as in the preceding case and $\beta_j = \operatorname{lcm}(\alpha_j, u_j)$.
In all cases, put

$$\bar{m} = \max_{1 \leqslant j \leqslant l} m_j, \qquad \bar{q} = \max_{1 \leqslant j \leqslant l} p_j^{\beta_j}.$$

Our main result is

Theorem 2.1. *For any integers $a_1, \dots, a_l$*

$$\left|\left\{0 \leqslant n \leqslant N-1\colon\ e_{p_j}(n) \equiv a_j \pmod{m_j},\ j = 1, 2, \dots, l\right\}\right| = \frac{N}{m_1 \cdots m_l} + O\left(N^{1-\delta}\right), \tag{3}$$

where $\delta = 4/(\bar{q}^2 \bar{m}^2 l \log \bar{q} + 8l + 8)$.

It is natural to inquire how tight the result is, namely whether the error term can be reduced. Now sometimes the error is indeed much lower, as in the following

Example 2.2. Consider the sequence $(e_2(n) \bmod 2)_{n=0}^{\infty}$. It is easy to see that, if $4 \mid n$, then $e_2(n+1) = e_2(n)$, while $e_2(n+2) = e_2(n+3) = e_2(n) + 1$. Hence, out of any 4 consecutive elements of our sequence, starting at an index divisible by 4, there are two 0's and two 1's. It follows that, in this case, the error term on the right-hand side of (3) is bounded by 1. The situation is similar for the sequence $(e_p(n) \bmod p)_{n=0}^{\infty}$ for any prime p.

However, while it is hardly conceivable that the error term $O(N^{1-\delta})$ in Theorem 2.1 cannot be reduced, it is nevertheless bounded below in general by some positive power of N, as is the case in

Example 2.3. Consider the sequence $(e_3(n) \bmod 2)_{n=0}^{\infty}$. Employing (2) we obtain

$$\begin{aligned} e_3(9n+r) &= \left[\frac{9n+r}{3}\right] + \left[\frac{9n+r}{9}\right] + \left[\frac{9n+r}{27}\right] + \cdots \\ &= 3n + \left[\frac{r}{3}\right] + n + e_3(n) \\ &\equiv e_3(n) + \left[\frac{r}{3}\right] \pmod 2, \quad r = 0, 1, \ldots, 8. \end{aligned}$$

In the range $[0, 8]$, six of the values assumed by $[r/3]$ are 0 modulo 2 and the other three are 1. It follows that, denoting by a_s and b_s the number of 0's and the number of 1's, respectively, in the finite sequence $(e_3(n) \bmod 2)_{n=0}^{9^s-1}$, we have:

$$a_{s+1} = 6a_s + 3b_s, \qquad b_{s+1} = 3a_s + 6b_s.$$

This yields

$$a_s = \frac{9^s + 3^s}{2}, \qquad b_s = \frac{9^s - 3^s}{2},$$

which implies that the error term on the right-hand side of (3) is $\Omega(\sqrt{N})$.

It is also interesting to note the following somewhat strengthened version of Theorem 2.1.

Theorem 2.4. *In the setup of Theorem* 2.1, *for every positive integer* $m \geqslant 2$ *and integer* a

$$\begin{aligned} &\left|\left\{0 \leqslant n \leqslant N-1:\ n \equiv a \pmod m,\ e_{p_j}(n) \equiv a_j \pmod{m_j},\ j = 1, 2, \ldots, l\right\}\right| \\ &\quad = \frac{N}{m m_1 \cdots m_l} + O\big(N^{1-\delta}\big), \end{aligned} \tag{4}$$

where $\delta = 4/(\bar{q}^2 \bar{m}^2 l \log \bar{q} + 8l + 8)$.

The proof of the theorem is similar to that of Theorem 2.1. Let us mention a particular example in which the error term in the last theorem has been calculated (although starting with a different motivation than ours).

Example 2.5. According to Theorem 2.4 we have

$$\{0 \leqslant n \leqslant N-1\colon\ n \equiv 0 \pmod 6,\ e_2(n) \equiv 0 \pmod 2\} = \frac{N}{6} + O\big(N^{1-1/(32\log 2+4)}\big). \tag{5}$$

Now, as is well known,

$$e_2(n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \cdots = n - \sigma_2(n).$$

Consequently, for $n \equiv 0 \pmod 2$ we have $e_2(n) \equiv \sigma_2(n) \pmod 2$. By [11], the error term in (5) is actually $O(N^{\log_4 3})$. Interestingly, the error is always positive, and is actually bounded between two positive constant multiples of $N^{\log_4 3}$. For similar results for other primes, as well as results showing that these primes are actually exceptional, we refer to [4,5,7,8].

As mentioned earlier, the proof depends on an improvement of a result of Kim [9], dealing with completely q-additive functions. Recall that a function $f : \mathbf{N} \cup \{0\} \to \mathbf{C}$ is *completely q-additive*, where $q \geqslant 2$ is an integer, if $f(0) = 0$ and $f(aq + b) = f(a) + f(b)$ for integers $a \geqslant 1$ and $0 \leqslant b \leqslant q - 1$. In Kim's setup there are pairwise prime integers $q_j \geqslant 2$ and corresponding completely q_j-additive integer-valued functions, $1 \leqslant j \leqslant l$, and we are interested in the asymptotic frequency of the set of integers for which these functions assume values in some prescribed residue classes modulo certain positive integers m_j. The proof of Theorem 2.1 relies on the following result, which allows for two completely q_j-additive functions for each j.

Theorem 2.6. *Let $q_1, \ldots, q_l \geqslant 3$ be pairwise primes integers. For each $j \leqslant l$, let f_j and f_{l+j} be completely q_j-additive integer-valued functions and m_j, m_{l+j} relatively prime positive integers such that*

$$\gcd\big(m_j, f_j(2) - 2f_j(1), \ldots, f_j(q_j - 1) - (q_j - 1)f_j(1)\big) = 1 \tag{6}$$

and

$$\gcd\big(m_{l+j}, f_{l+j}(2) - 2f_{l+j}(1), \ldots, f_{l+j}(q_j - 1) - (q_j - 1)f_{l+j}(1)\big) = 1. \tag{7}$$

Let $M = \max_{1 \leqslant j \leqslant l} m_j m_{l+j}$. Then for any integers $m \geqslant 2$ and $a, a_1, \ldots, a_{2l}$

$$\begin{aligned} &\big|\{0 \leqslant n \leqslant N-1\colon\ n \equiv a \pmod m,\ f_1(n) \equiv a_1 \pmod{m_1},\ \ldots,\ f_{2l}(n) \equiv a_{2l} \pmod{m_{2l}}\}\big| \\ &\quad = \frac{N}{m m_1 \cdots m_{2l}} + O\big(N^{1-\delta}\big), \end{aligned} \tag{8}$$

where $\delta = 4/(\bar{q}^2 M^2 l \log \bar{q} + 8l + 8)$.

Remark 2.7. It is probably possible to deal with the case where (6) and (7) do not hold, as done in [9]. However, this is unimportant for our application. We can also deal with several functions for each modulus.

3. Proofs

Lemma 3.1. *Let $q \geqslant 3$ be an integer, f_1 and f_2 integer-valued completely q-additive functions, m_1 and m_2 relatively prime positive integers such that*

$$\gcd\{m_j, f_j(2) - 2f_j(1), \dots, f_j(q-1) - (q-1)f_j(1)\} = 1, \quad j = 1, 2,$$

h_1 and h_2 integers, not both 0, satisfying $0 \leqslant h_j \leqslant m_j - 1$, and $K = q^s$ and $Q = q^t$ for some $1 \leqslant s \leqslant t$. Then:

$$\frac{1}{KQ}\sum_{k=0}^{K-1}\left|\sum_{n=0}^{Q-1} e\left(\frac{h_1}{m_1}\big(f_1(n+k) - f_1(n)\big) + \frac{h_2}{m_2}\big(f_2(n+k) - f_2(n)\big)\right)\right| \leqslant e^{-8s/q^2 m_1^2 m_2^2}. \tag{9}$$

Proof. Denote the left-hand side of (9) by S, and:

$$S_{K,Q}(u) = \frac{1}{KQ}\sum_{k=0}^{K-1}\left|\sum_{n=0}^{Q-1} e\left(\frac{h_1}{m_1}\big(f_1(n+k+u) - f_1(n)\big) + \frac{h_2}{m_2}\big(f_2(n+k+u) - f_2(n)\big)\right)\right|,$$
$$u = 0, 1.$$

Then:

$$\begin{aligned}
S_{Kq,Qq}(u) &= \frac{1}{KQq^2}\sum_{i=0}^{q-1}\sum_{k=0}^{K-1}\left|\sum_{j=0}^{q-1}\sum_{n=0}^{Q-1} e\left(\frac{h_1}{m_1}\big(f_1(kq+i+nq+j+u) - f_1(nq+j)\big)\right.\right. \\
&\quad \left.\left. + \frac{h_2}{m_2}\big(f_2(kq+i+nq+j+u) - f_2(nq+j)\big)\right)\right| \\
&= \frac{1}{KQq^2}\sum_{i=0}^{q-1}\sum_{k=0}^{K-1}\left|\sum_{j=0}^{q-i-u-1}\sum_{n=0}^{Q-1} e\left(\frac{h_1}{m_1}\big(f_1(k+n) - f_1(n) + f_1(i+j+u) - f_1(j)\big)\right.\right. \\
&\quad \left. + \frac{h_2}{m_2}\big(f_2(k+n) - f_2(n) + f_2(i+j+u) - f_2(j)\big)\right) \\
&\quad + \sum_{j=q-i-u}^{q-1}\sum_{n=0}^{Q-1} e\left(\frac{h_1}{m_1}\big(f_1(k+n+1) - f_1(n) + f_1(i+j+u-q) - f_1(j)\big)\right. \\
&\quad \left.\left. + \frac{h_2}{m_2}\big(f_2(k+n+1) - f_2(n) + f_2(i+j+u-q) - f_2(j)\big)\right)\right| \\
&\leqslant \alpha(0,u)S_{K,Q}(0) + \alpha(1,u)S_{K,Q}(1),
\end{aligned}$$

where

$$\alpha(v,u)=\frac{1}{q^2}\sum_{i=0}^{q-1}\left|\sum_{j=\max\{0,vq-i-u\}}^{\min\{q-1,(v+1)q-i-u-1\}} e\left(\frac{h_1}{m_1}\big(f_1(i+j+u)-f_1(j)\big)+\frac{h_2}{m_2}\big(f_2(i+j+u)-f_2(j)\big)\right)\right|.$$

Thus, putting $X_{K,Q}=(S_{K,Q}(0),S_{K,Q}(1))^T$ and

$$A=\begin{pmatrix}\alpha(0,0) & \alpha(1,0)\\ \alpha(0,1) & \alpha(1,1)\end{pmatrix},$$

we obtain

$$X_{Kq,Qq}\leqslant AX_{K,Q},$$
$$\beta(v,u)=\frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=\max\{0,vq-i-u\}}^{\min\{q-1,(v+1)q-i-u-1\}}1. \tag{10}$$

Obviously, we have $\alpha(v,u)\leqslant\beta(v,u)$ for every $v,u\in\{0,1\}$. We want to prove that, moreover,

$$\alpha(0,0)\leqslant\beta(0,0)-\varepsilon, \tag{11}$$

and

$$\alpha(0,1)\leqslant\beta(0,1)-\varepsilon, \tag{12}$$

with $\varepsilon=8/q^2m_1^2m_2^2$. Indeed, denote by Σ the portion of $q^2\alpha(0,0)$ corresponding to $i=1$. Then:

$$\Sigma=\left|\sum_{j=0}^{q-2}e\left(\frac{h_1}{m_1}\big(f_1(j+1)-f_1(j)\big)+\frac{h_2}{m_2}\big(f_2(j+1)-f_2(j)\big)\right)\right|.$$

Assume that $h_1\neq 0$, and let r be the smallest integer $\geqslant 2$ for which

$$h_1\big(f_1(r)-rf_1(1)\big)\not\equiv 0 \pmod{m_1}.$$

Such an r exists, as otherwise we would have

$$f_1(j)-jf_1(1)\equiv 0 \quad \big(\mathrm{mod}\ m_1/(m_1,h_1)\big),\quad j=1,\dots,q-1.$$

Then:

$$\begin{aligned}\Sigma&\leqslant q-3+\left|e\left(\frac{h_1}{m_1}f_1(1)+\frac{h_2}{m_2}f_2(1)\right)\right.\\&\quad\left.+e\left(\frac{h_1}{m_1}\big(f_1(r)-f_1(r-1)\big)+\frac{h_2}{m_2}\big(f_2(r)-f_2(r-1)\big)\right)\right|\\&=q-3+\left|1+e\left(\frac{h_1}{m_1}\big(f_1(r)-f_1(r-1)-f_1(1)\big)\right)+\frac{b}{m_2}\right|,\end{aligned}$$

where $b = h_2(f_2(r) - f_2(r-1) - f_2(1))$. Now

$$h_1 f_1(j) \equiv h_1 j f_1(1), \quad j = 2, \ldots, r-1,$$

so that, denoting $a = h_1(f_1(r) - f_1(r-1) - f_1(1))$, we have

$$a \equiv h_1\big(f_1(r) - r f_1(1)\big) \not\equiv 0 \pmod{m_1}.$$

Since $(m_1, m_2) = 1$ and at least one of a and b is non-zero, we have $\|a/m_1 + b/m_2\| \geqslant 1/m_1 m_2$ and

$$\Sigma \leqslant q - 3 + 2\cos\frac{\pi}{m_1 m_2} \leqslant q - 1 - \frac{8}{m_1^2 m_2^2}.$$

Evaluating the remaining portion of $\alpha(0,0)$ trivially, we obtain

$$\alpha(0,0) \leqslant \beta(0,0) - \frac{8}{q^2 m_1^2 m_2^2}.$$

Similarly we prove (12).

In view of (11)

$$\alpha(0,0) + \beta(1,0) \leqslant \beta(0,0) + \beta(1,0) - \varepsilon = 1 - \varepsilon,$$

and similarly, due to (12):

$$\alpha(0,1) + \beta(1,1) \leqslant \beta(0,1) + \beta(1,1) - \varepsilon = 1 - \varepsilon.$$

By (10), taking

$$A = \begin{pmatrix} \alpha(0,0) & \beta(1,0) \\ \alpha(0,1) & \beta(1,1) \end{pmatrix},$$

we have

$$X_{Kq,Qq} \leqslant A' X_{K,Q}.$$

Applying this inequality s times, and starting from $K_1 = 1$ and $Q_1 = q^{t-s}$ instead of K and Q, respectively, we obtain:

$$X_{K,Q} \leqslant A'^s X_{K_1,Q_1} \leqslant A'^s (1,1)^T.$$

By [9, Lemma 5], this yields

$$S_{K,Q}(0) \leqslant (1-\varepsilon)^s \leqslant e^{-8s/q^2 m_1^2 m_2^2},$$

which proves the lemma. □

Proof of Theorem 2.6. Denote the left-hand side of (8) by $N(a)$. Write:

$$N(a)=\frac{1}{mm_1\cdots m_{2l}}\sum_{r=0}^{m-1}\sum_{h_1=0}^{m_1-1}\cdots\sum_{h_{2l}=0}^{m_{2l}-1}\sum_{n=0}^{N-1}e\biggl(\frac{r(n-a)}{m}\biggr)e\Biggl(\sum_{j=1}^{2l}\frac{h_j}{m_j}\bigl(f_j(n)-a_j\bigr)\Biggr).$$

The portion of the last sum corresponding to $h_1=h_2=\cdots=h_{2l}=0$ is

$$\frac{1}{mm_1\cdots m_{2l}}\sum_{r=0}^{m-1}\sum_{n=0}^{N-1}e\biggl(\frac{r(n-a)}{m}\biggr)=\frac{N}{mm_1\cdots m_{2l}}+O(1).$$

Now consider the portion of the sum corresponding to arbitrary fixed $h_1,\ldots,h_{2l}$, not all 0. Set:

$$S=\Biggl|\sum_{n=0}^{N-1}e\biggl(\frac{rn}{m}\biggr)e\Biggl(\sum_{j=1}^{2l}\frac{h_j}{m_j}f_j(n)\Biggr)\Biggr|.$$

Suppose, say, $h_1\neq 0$. Put

$$s_1=\biggl[\frac{1-2\delta(l+1)}{l}\cdot\frac{\log N}{\log q_1}\biggr],\qquad K=q_1^{s_1},$$

and

$$t_j=\biggl[\frac{1-2\delta}{l}\cdot\frac{\log N}{\log q_j}\biggr],\qquad Q_j=q_j^{t_j},$$

for $j=1,\ldots,l$. By van-der-Corput's inequality:

$$|S|^2\leqslant\frac{2N^2}{K}+\frac{4N}{K}\sum_{k=0}^{K-1}\Biggl|\sum_{n=0}^{N-1}e\Biggl(\sum_{j=1}^{2l}\frac{h_j}{m_j}\bigl(f_j(k+n)-f_j(n)\bigr)\Biggr)\Biggr|.$$

Set:

$$S_{N,K}=\frac{1}{KN}\sum_{k=0}^{K-1}\Biggl|\sum_{n=0}^{N-1}e\Biggl(\sum_{j=1}^{2l}\frac{h_j}{m_j}\bigl(f_j(k+n)-f_j(n)\bigr)\Biggr)\Biggr|.$$

Divide the set $\{0,1,\ldots,N-1\}$, over which n varies, into residue classes modulo $Q_1\cdots Q_l$. For a typical $n\in[0,N-1]$, put:

$$r_j=n \mod Q_j,\quad j=1,\ldots,l.$$

Clearly, the number of n's in $[0,N-1]$, corresponding to any fixed $r_1,r_2,\ldots,r_l$ is $N/Q_1\cdots Q_l+O(1)$. Denote

$$R_0=\bigl\{(r_1,r_2,\ldots,r_l)\colon\ 0\leqslant r_j<Q_j-K,\ j=1,2,\ldots,l\bigr\},$$

and let R_1 be the complementary set. Obviously:

$$|R_1| \leqslant KQ_1 \cdots Q_l\left(\frac{1}{Q_1} + \cdots + \frac{1}{Q_l}\right).$$

If $(r_1, r_2, \ldots, r_l) \in R_0$, then, letting $r_{l+j} = r_j$ for $1 \leqslant j \leqslant l$, we have

$$f_j(k+n) - f_j(n) = f_j(k+r_j) - f_j(r_j), \quad j = 1, \ldots, 2l.$$

It follows that:

$$\begin{aligned}
S_{N,K} &= \frac{1}{KN}\sum_{k=0}^{K-1}\left|\sum_{(r_1,\ldots,r_l)\in R_0} e\left(\sum_{j=1}^{2l}\frac{h_j}{m_j}\big(f_j(k+r_j) - f_j(r_j)\big)\right)\left(\frac{N}{Q_1\cdots Q_l} + O(1)\right)\right| \\
&\quad + O(K/Q_1) \\
&= \frac{1}{KN}\cdot\frac{N}{Q_1\cdots Q_l}\sum_{k=0}^{K-1}\prod_{j=1}^{l}\left|\sum_{r_j=0}^{Q_j-1} e\left(\frac{h_j}{m_j}\big(f_j(k+r_j) - f_j(r_j)\big)\right.\right. \\
&\quad \left.\left. + \frac{h_{l+j}}{m_{l+j}}\big(f_{l+j}(k+r_j) - f_{l+j}(r_j)\big)\right)\right| + O\left(\frac{K}{Q_1} + \frac{Q_1\cdots Q_l}{N}\right) \\
&\leqslant \frac{1}{KQ_1}\sum_{k=0}^{K-1}\left|\sum_{r_1=0}^{Q_1-1} e\left(\frac{h_j}{m_j}\big(f_j(k+r_j) - f_j(r_j)\big) + \frac{h_{l+j}}{m_{l+j}}\big(f_{l+j}(k+r_j) - f_{l+j}(r_j)\big)\right)\right|.
\end{aligned}$$

Estimating the last sum by Lemma 3.1, we complete the proof. □

Proof of Theorem 2.1. For $1 \leqslant j \leqslant l$, define a completely q_j-additive function f_j by

$$f_j(n) = 0, \quad n = 0, 1, \ldots,$$

in case $k_j = 1$ and by

$$f_j(n) = e_{p_j}(n), \quad n = 0, 1, \ldots, q_j - 1, \tag{13}$$

otherwise. (Note that (13), together with the complete q_j-additivity requirement, determine f_j uniquely.) Put

$$v_j = \frac{p_j^{\alpha_j} - 1}{p_j - 1}, \quad 1 \leqslant j \leqslant l,$$

and define functions g_j by:

$$g_j(n) = e_{p_j}(n) + v_j n, \quad n = 0, 1, \ldots.$$

Also set $q_{l+j} = p_j^{\alpha_j}$ if $\alpha_j \geqslant 2$ and $q_{l+j} = p_j^2$ if $\alpha_j = 1$, and define completely q_{l+j}-additive functions f_{l+j} by

$$f_{l+j}(n) = g_j(n), \quad n = 0, 1, \ldots, q_{l+j} - 1, \tag{14}$$

in case $\alpha_j > 0$, and by

$$f_{l+j}(n) = 0, \quad n = 0, 1, \ldots,$$

otherwise. In the sequel we shall relate to the main definitions (13) and (14) of f_j and f_{l+j}, and omit the details of the cases where these functions vanish identically. We first claim that:

$$f_{l+j}(n) \equiv g_j(n) \pmod{q_{l+j}}, \quad 1 \leqslant j \leqslant l,\ n = 0, 1, \ldots. \tag{15}$$

In fact, (15) is trivial for integers up to q_{l+j}. Assume it holds for all integers less than n, and write $n = cq_{l+j} + d$ for integers $c \geqslant 1$ and $0 \leqslant d \leqslant q_{l+j} - 1$. Expand c and d in base p_j:

$$c = \sum_{i=0}^{t} c_i p_j^i, \qquad d = \sum_{i=0}^{\alpha_j - 1} d_i p_j^i.$$

Then

$$\begin{aligned} f_{l+j}(n) &= f_{l+j}(c) + f_{l+j}(d) \\ &\equiv \sum_{i=0}^{t} c_i \frac{p_j^i - 1}{p_j - 1} + v_j c + e_{p_j}(d) + v_j d \\ &\equiv \sum_{i=0}^{t} c_i \frac{p_j^i - 1}{p_j - 1} + \sum_{i=0}^{t} c_i p_j^i \frac{p_j^{\alpha_j} - 1}{p_j - 1} + v_j c q_{l+j} + e_{p_j}(d) + v_j d \\ &= \sum_{i=0}^{t} c_i \frac{p_j^{\alpha_j + i} - 1}{p_j - 1} + \sum_{i=0}^{\alpha_j - 1} d_i \frac{p_j^i - 1}{p_j - 1} + v_j (c q_{l+j} + d) \\ &\equiv e_{p_j}(c q_{l+j} + d) + v_j (c q_{l+j} + d) \pmod{q_{l+j}}, \end{aligned}$$

which proves (15).

Now put $q = \prod_{j=1}^{l} q_{l+j}$. For $a = 0, 1, \ldots, q - 1$, let:

$$R(a) = \bigl\{0 \leqslant n \leqslant N - 1\colon\ n \equiv a \pmod{q},\ e_{p_j}(n) \equiv a_j \pmod{m_j},\ j = 1, \ldots, l\bigr\}.$$

Denoting $\mathbf{a} = (a_1, \ldots, a_l)$ and denoting the left-hand side of (3) by $N(\mathbf{a})$, we clearly have $N(\mathbf{a}) = \sum_{a=0}^{q-1} |R(a)|$, so that it suffices to show that

$$\bigl|R(a)\bigr| = \frac{N}{q m_1 \cdots m_l} + O\bigl(N^{1-\delta}\bigr). \tag{16}$$

In fact, putting $b_j = a \bmod q_{l+j}$ for $1 \leqslant j \leqslant l$, we have:

$$R(a) = \{0 \leqslant n \leqslant N-1\colon\ n \equiv a \pmod{q},\ f_j(n) \equiv a_j \pmod{k_j},$$
$$f_{l+j}(n) \equiv a_j + v_j b_j \pmod{q_{l+j}}\ \forall j\}.$$

Since β_j is divisible by u_j and α_j, both functions f_j and f_{l+j} are completely $p_j^{\beta_j}$-additive. We want to verify that the conditions of Theorem 2.6 hold. First, since $u_j \geqslant 2$, each q_j is at least 3. The moduli k_j and $p_j^{\alpha_j}$, corresponding to m_j and m_{l+j}, corresponding to the moduli m_j and m_{l+j}, respectively, in that theorem, are relatively prime. Next, $f_j(p_j) = e_{p_j}(p_j) = 1$ and $f_j(n) = 0$ for $n < p_j$, so that (6) holds. Since

$$f_{l+j}(p_j) - p_j f_{l+j}(1) = e_{p_j}(p_j) + v + jp_j - p_j\big(e_{p_j}(1) + v_j\big) = 1,$$

we see that (7) holds as well. Finally, since the p_j's are distinct, the q_j's are pairwise prime. Thus we can use Theorem 2.6 to obtain (16), which proves the theorem. □

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References

[1] D. Berend, On the parity of exponents in the factorization of $n!$, J. Number Theory 64 (1997) 13–19.
[2] Y.G. Chen, On the parity of exponents in the standard factorization of $n!$, J. Number Theory 100 (2003) 326–331.
[3] Y.G. Chen, Y.C. Zhu, On the prime power factorization of $n!$, J. Number Theory 82 (2000) 1–11.
[4] J. Coquet, A summation formula related to the binary digits, Invent. Math. 73 (1983) 107–115.
[5] M. Drmota, M. Skałba, Rarified sums of the Thue–Morse sequence, Trans. Amer. Math. Soc. 352 (2000) 609–642.
[6] P. Erdős, R.L. Graham, Old and New Problems and Results in Combinatorial Number Theory, Enseign. Math., Imprimerie Kundig, Geneva, 1980.
[7] P.J. Grabner, A note on the parity of the sum-of-digits function, in: Séminaire Lotharingien de Combinatoire, Gerolfingen, 1993, in: Prépubl. Inst. Rech. Math. Av., vol. 1993/34, Univ. Louis Pasteur, Strasbourg, 1993, pp. 35–42.
[8] P.J. Grabner, T. Herendi, R.F. Tichy, Fractal digital sums and codes, Appl. Algebra Engrg. Comm. Comput. 1 (1997) 33–39.
[9] D.H. Kim, On the joint distribution of q-additive functions in residue classes, J. Number Theory 74 (1999) 307–336.
[10] F. Luca, P. Stănică, On the prime power factorization of $n!$, J. Number Theory 102 (2003) 298–305.
[11] D.J. Newman, On the number of binary digits in a multiple of three, Proc. Amer. Math. Soc. 21 (1969) 719–721.
[12] J.W. Sander, On the parity of exponents in the prime factorization of factorials, J. Number Theory 90 (2001) 316–328.