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# On Relative Regular Sequences

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## 1. INTRODUCTION

In the article (Ref. [5]) the authors studied the ideal $\mathfrak{A}$, of a unitary commutative ring $R$, generated by $n$ elements $a_1 ,..., a_n$ satisfying the following property (*): if $b_1 ,..., b_n$ are elements of the ideal $\mathfrak{A} = (a_1 ,..., a_n)R$ such that $b_1a_1 + \cdots + b_na_n = 0$, then the syzygy $(b_1 ,..., b_n)$ is trivial, i.e., it is a combination of the elementary syzygies $(0,..., 0, a_j , 0,..., 0, -a_i , 0,..., 0)$.

It is well-known that if $a_1 ,..., a_n$ is an $R$-regular sequence every syzygy is trivial, so (*) is verified. It is also known that the property "every syzygy is trivial" can be expressed by the relation $H_1(K(a_1 ,..., a_n; R)) = 0$, where $K(a_1 ,..., a_n; R)$ is the Koszul complex determined by $a_1 ,..., a_n$ .

In [5] some examples are given of elements $a_1 ,..., a_n$ verifying (*) which are not a regular sequence and, therefore, $H_1(K(a_1 ,..., a_n)) \neq 0$.

The condition (*) can also be stated by means of the Koszul complex. In fact, let $\mathfrak{A} = (a_1 ,..., a_n)R$ and let

$$\varphi : K(a_1 ,..., a_n; \mathfrak{A}) \to K(a_1 ,..., a_n; R)$$

be the canonical mapping induced by the inclusion $\mathfrak{A} \subset R$. Then we observe in Section 2 that (*) is equivalent to the condition $H_1(\varphi) = 0$, where $H_1(\varphi)$ is the map induced on homology.

By the remarks above we are led to consider the following problem: Given an $R$-module $M$ and a submodule $N$, under what conditions will a sequence $a_1 ,..., a_n$ of elements in $R$, not necessarily a regular sequence, be such that $H_1(\varphi) = 0$, $\varphi$ being the canonical mapping $K(a_1 ,..., a_n; N) \to K(a_1 ,..., a_n; M)$ induced by the inclusion $N \subset M$?

Thus in Section 3 we define a notion of relative $M$-regular sequence with respect to $N$ and verify that if $a_1 ,..., a_n$ is a relative $M$-regular sequence then we have $H_1(\varphi) = 0$ as wanted.

Finally we verify in Section 4 that there exist interesting examples of ideals

$\mathfrak{A}$ generated by a relative contained in $\mathrm{Rad}_J$ $(R)$ $R$-regular sequence with respect to $\mathfrak{A}$. Namely, if $h_{11}, \ldots, h_{q-1,q}$ are elements of a noetherian ring $R$ which form a regular sequence, then such an ideal is the ideal $\mathfrak{A}$ generated by the $(q-1)$ minors of the matrix $H = \| h_{ij} \|$ $(1 \leqslant i \leqslant q-1,\ 1 \leqslant j \leqslant q)$.

## 2. Relative $M$-Regular Sequences with Respect to a Submodule $N$

Let $R$ be a commutative unitary ring and $M$ an $R$-module. A sequence of elements $a_1, \ldots, a_n$ of $R$ is said to be $M$-regular if for each $i = 1, \ldots, n$, $a_i$ is not a zero-divisor in the module $M/(a_1, \ldots, a_{i-1})M$. (In particular, this means $a_1$ is not a zero-divisor in $M$).

Definition 1. Let $R$ be a commutative unitary ring, $M$ an $R$-module and $N$ a submodule of $M$. A sequence $a_1, \ldots, a_n$ of elements in $R$ is said to be a relative $M$-regular sequence with respect to $N$, if for all $i$, $a_{i+1}x \in \sum_{j=1}^{i} a_j N$ and $x \in N$ imply $x \in \sum_{j=1}^{i} a_j M$, $0 \leqslant i < n$. (For $i = 0$, this means $a_1$ is not a zero-divisor in $N$).

Definition 2. Moreover, if every arbitrary permutation $a_{i_1}, \ldots, a_{i_n}$ of elements $a_1, \ldots, a_n$ also gives a relative $M$-regular sequence with respect to the submodule $N$, we say that $a_1, \ldots, a_n$ form an unconditioned relative $M$-regular sequence with respect to $N$.

*Remark* 1. Let $a_1, \ldots, a_n \in R$. The property of being a relative $M$-regular sequence with respect to a submodule $N$ is preserved under flat extension of $R$. Indeed, let $f: R \to R'$ be a ring homomorphism making $R'$ into a flat $R$-module. Put $M' = M \otimes_R R'$, $N' = N \otimes_R R'$. It is obvious that $N' \subset M'$. The fact that $a_1, \ldots, a_n$ is a relative $M$-regular sequence with respect to $N$ can be expressed by the following commutative exact diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \ker(\omega_i) & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \ker(\omega_i) & \longrightarrow & N_i & \xrightarrow{\alpha_i} & M_i, \\ & & & & \downarrow{\scriptstyle \omega_i} & & \\ & & & & N_i & & \end{array}$$

where

$$N_1 = N, \qquad M_1 = M, \qquad N_i = N/(a_1N + \cdots + a_{i-1}N),$$
$$M'_i = M/(a_1M + \cdots + a_{i-1}M), \qquad i = 2,\dots, n;$$

$\alpha_i$ is induced by $\alpha$ and $\omega_i(x) = a_i x (x \in N_i)$, $i = 1,\dots, n$. Tensoring with $R'$ over $R$ this diagram is transformed into the corresponding diagram for $M', N', a_i' = f(a_i), i = 1,\dots, n$, and the exactness is preserved.

Let $K(a_1 ,\dots, a_n; E) = K(a_1 ,\dots, a_n) \otimes_R E$, for all $R$-module $E$, where $K(a_1 ,\dots, a_n)$ denotes the Koszul complex associated to the sequence $a_1 ,\dots, a_n$.

THEOREM. *Let $N$ be a submodule of $M$ such that $\sum_{i=1}^{n} a_iM \subset N$, with $a_i \in R$. Let $\alpha : N \to M$ be the inclusion mapping, and let*

$$\varphi_\alpha : K(a_1 ,\dots, a_n; N) \to K(a_1 ,\dots, a_n; M)$$

*be the mapping induced by $\alpha$. If the sequence of elements $a_1 ,\dots, a_n$ is a relative $M$-regular sequence with respect to $N$, then $H_p(\varphi_\alpha) = 0$, $\forall p > 0$.*

*Proof.* We proceed by induction on $n$.

*Case* (a)

If $n = 1$, then $xa_1 = 0$ and $x \in N$ imply $x = 0$; hence $a_1$ is an $N$-regular element. However, $H_1(K(a_1; N)) = \mathrm{Ann}_N\, a_1$. It follows that $H_1(K(a_1; N)) = 0$, and, "a fortiori," $H_1(\varphi_\alpha) = 0$.

*Case* (b)

If $n > 1$, then we suppose the statement true for $n - 1$. We have

$$K(a_1 ,\dots, a_n; E) = K(a_n) \otimes_R K(a_1 ,\dots, a_{n-1}; E),$$

for all $R$-modules $E$ [7]; we also have the following exact sequence:

$$0 \to H_0(K(a_n) \otimes_R H_p(a_1 ,\dots, a_{n-1}; E)) \to H_p(a_1 ,\dots, a_n; E)$$
$$\to H_1(K(a_n) \otimes H_{p-1}(a_1 ,\dots, a_{n-1}; E)) \to 0.$$

Choosing $E = N, M$, we get the commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow & H_0(K(a_n) \otimes_R H_p(a_1 ,\dots, a_{n-1}; N)) & \longrightarrow & H_p(a_1 ,\dots, a_n; N) & \\ & \downarrow{\scriptstyle \alpha_p} & & \downarrow{\scriptstyle \beta_p} & \\ 0 \longrightarrow & H_0(K(a_n) \otimes_R H_p(a_1 ,\dots, a_{n-1}; M)) & \longrightarrow & H_p(a_1 ,\dots, a_n; M) & \end{array}$$

$$\begin{array}{ccc} \longrightarrow & H_1(K(a_n) \otimes H_{p-1}(a_1 ,\dots, a_{n-1}; N)) & \longrightarrow 0 \\ & \downarrow{\scriptstyle \gamma_p} & \\ \longrightarrow & H_1(K(a_n) \otimes H_{p-1}(a_1 ,\dots, a_{n-1}; M)) & \longrightarrow 0, \end{array}$$

where the vertical arrows are defined by the inclusion $\alpha : N \to M$.

If $p > 1$, by the induction hypothesis, $\alpha_p = \gamma_p = 0$.

If $p = 1$, also by the induction hypothesis, $\alpha_1 = 0$. We prove now that even in this case $\gamma_1 = 0$.

In fact, since $H_0(a_1, ..., a_{n-1}; E) = E/(a_1E + \cdots + a_{n-1}E)$, for all $R$-modules $E$, we have that $H_1(K(a_n) \otimes H_0(a_1, ..., a_{n-1}; E)) = \mathrm{Ann}_{\bar{E}}\, a_n$, with $\bar{E} = E/(a_1E + \cdots + a_{n-1}E)$.

On the other hand, $\gamma_1 : \mathrm{Ann}_{\bar{N}}\, a_n \to \mathrm{Ann}_{\bar{M}}\, a_n$ is the homomorphism defined by $\bar{\alpha} : \bar{N} \to \bar{M}$, where $\bar{\alpha}$ is the homomorphism induced by the inclusion $\alpha : N \to M$. Every element of $\mathrm{Ann}_{\bar{N}}\, a_n$ is the class $\bar{z}$ of an element $z \in N$, such that $a_n z \in \sum_{j=1}^{n-1} a_j N$. From the hypothesis, this implies that $z \in \sum_{j=1}^{n-1} a_j M$, hence $\gamma_1(\bar{z}) = 0$. Therefore for any $p > 0$, we have $\alpha_p = \gamma_p = 0$.

We shall prove next that $\beta_p = 0$.

By the above diagram,

$$\mathrm{Im}\, \beta_p \subseteq H_0(K(a_n) \otimes_R H_p(a_1, ..., a_{n-1}; N)).$$

Hence every cycle $c$ of $K_p(a_1, ..., a_n; N)$ is homologous to a cycle $e \in K_p(a_1, ..., a_{n-1}; M)$; that means, $c - e \in dK_{p+1}(a_1, ..., a_n; M)$. But

$$dK_{p+1}(a_1, ..., a_n; M) \subseteq K_p(a_1, ..., a_{n-1}; N),$$

since, by hypothesis, $\sum_{i=1}^{n} a_i M \subset N$. Consequently,

$$e \in K_p(a_1, ..., a_{n-1}; M) \cap K_p(a_1, ..., a_{n-1}; N) = K_p(a_1, ..., a_{n-1}; N).$$

Taking the homology classes, it follows that $\mathrm{Im}\, \beta_p \subseteq \mathrm{Im}\, \alpha_p$. But $\mathrm{Im}\, \alpha_p = 0$, so $\beta_p = 0$, and the theorem is proved.

*Remark* 2. The condition $H_p(\varphi_\alpha) = 0$, indicated in the above theorem, is equivalent to

$$Z(K_p(a_1, ..., a_n; N)) = dK_{p+1}(a_1, ..., a_n; M), \qquad (**)$$

where $Z(K_p(a_1, ..., a_n; N))$ is the module of cycles of $K_p(a_1, ..., a_n; M)$, identified as a submodule of $K_p(a_1, ..., a_n; M)$.

The property (**) is equivalent to

$$Z(K_p(a_1, ..., a_n; M)) \cap K_p(a_1, ..., a_n; N) = dK_{p+1}(a_1, ..., a_n; M).$$

In particular, if $p = 1$, we have $\mathrm{syz}(a_1, ..., a_n) \cap N^n = T$, where $T = dK_2(a_1, ..., a_n; M)$ is the submodule of trivial syzygies [4, Chapter IV].

## 3. One Example

Let $k$ be a field and $y_{11},..., y_{q-1,q}$ indeterminates over $k$. In the polynomial ring $A = K[y_{11},..., y_{q-1,q}]$ we consider the ideal $p = \sum_{i=1}^{q} F_i A$, where $(-1)|^{i+1}F_i$ is the $(q-1)$-minor obtained by deleting the $i$-th column in the matrix $Y = \| y_{ij} \|$, $(i = 1,..., q-1; j = 1,..., q)$.

We know [2] that every pair $F_i$, $F_j$ with $i \neq j$, is a maximal $A$-regular sequence contained in $p$. It follows that if $q > 2, F_1,..., F_q$ is not an $A$-regular sequence. Our purpose is to verify that $F_1,..., F_q$ is a relative $A$-regular sequence with respect to $p$.

Let us suppose that $yF_q \in (F_1,..., F_{q-1})$, with $y \in p$, i.e., $y = \sum_{i=1}^{q} r_i F_i (r_i \in A)$. Then $y \in (F_1,..., F_{q-1})$ is equivalent to $r_q F_q \in (F_1,..., F_{q-1})$.

$I = \sum_{j=1}^{q} y_{jq} A$ is a prime ideal of $A$, and one can easily see that (i) $(F_1,..., F_{q-1}) \subset I$, and (ii) $F_q \notin I$. On the other hand $\sum_{j=1}^{q} y_{ij} F_j = 0$, $(1 \leqslant i \leqslant q-1)$, which shows that $y_{iq} F_q \in I$, for $i = 1,..., q-1$, hence (iii) $IF_q \subset (F_1,..., F_{q-1})$. $yF_q \in (F_1,..., F_{q-1})$ implies $r_q F_q^2 \in (F_1,..., F_{q-1})$, hence, by (i) $r_q F_q^2 \in I$; therefore by (ii) $r_q \in I$; also, by (iii) $r_q F_q \in (F_1,..., F_{q-1})$.

We have thus verified that $yF_q \in (F_1,..., F_{q-1})$ and $y \in p$ implies $y \in (F_1,..., F_{q-1})$, i.e., $p \cap [(F_1,..., F_{q-1}) : F_q] = (F_1,..., F_{q-1})$.

Suppose now that $yF_{q-1} \in (F_1,..., F_{q-2})$, where $y = \sum_{i=1}^{q} r_i F_i$. Since a permutation of the columns of the matrix $Y$ leads to a permutation of the elements of the system $\{F_1,..., F_q\}$, we get

$$[(F_1,..., F_{q-2}, F_q) : F_{q-1}] \cap p = (F_1,..., F_{q-2}, F_q),$$

and, therefore, we can suppose $r_{q-1} = 0$.

We have to prove that $r_q F_q \in (F_1,..., F_{q-2})$.

Let $J$ be the ideal of $A$ generated by all the minors of order 2 of the matrix formed by the last two columns of the matrix $Y$. We know [2] that $J$ is a prime ideal and it is easy to see that

(j) $(F_1,..., F_{q-2}) \subset J$; (jj) $F_{q-1}, F_q \notin J$; (jjj) $JF_q \subset (F_1,..., F_{q-2})$.

In a similar way to that employed above for the ideal $I$, we can now conclude that $y \in (F_1,..., F_{q-2})$. By repeating this procedure one can verify that $yF_i \in (F_1,..., F_{i-1})$ and $y \in p$ imply $y \in (F_1,..., F_{i-1})$, for $i = 1,..., q-1$.

We have therefore proved that *the elements $F_1,..., F_q$ form a relative $A$-regular sequence with respect to $p$ and that this sequence is unconditioned* since we can permute arbitrarily the elements $F_1,..., F_q$ by simply permuting the columns of $Y$.

*Remark* 3. We have the relation

$$T = \ker(\psi) \cap pA^q = \text{syz}\,(F_1,..., F_q) \cap pA^q,$$

where $T \subset A^q$ is the submodule of the trivial syzygies of the system $\{F_1, ..., F_q\}$, and $\psi : A^q \to A$, is the ring homomorphism defined by $\psi(e_i) = F_i$, where $e_1, ..., e_q$ is the natural basis of the free $A$-module $A^q$.

*Remark* 4. The property of being an unconditioned relative regular sequence also holds for the elements $G_1, ...., G_q$ which are the $(q - 1)$-minors of a matrix $H = \| h_{ij} \|$, $(1 \leqslant i \leqslant q - 1, 1 \leqslant j \leqslant q)$, where $h_{ij}$ are elements of a noetherian ring $R$ which contains a field $k$, provided $h_{11}, ..., h_{q-1,q}$ form a regular sequence, contained in $\text{Rad}_J(R)$.

Indeed, is this an easy consequence of Remark 1, of the example above, and of the following

LEMMA. *Let $R$ be a unitary commutative noetherian ring containing a field $k$ and $y_1, ..., y_m \in Rad_J(R)$. Let $x_1, ..., x_m$ be indeterminates over $k$ and $\theta : A = k[x_1, ..., x_m] \to R$ the homomorphism of $k$-algebras for which $\theta(x_i) = y_i$ $(i = 1, ..., m)$. Suppose that $y_1, ..., y_m$ form a regular sequence. Then $\theta$ is flat, i.e., $\theta$ makes $R$ into a flat $A$-module, and consequently* $\ker(\theta) = 0$.

Note: this Lemma slightly extends Proposition 1 of [4].

*Proof.* Let $I = y_1R + \cdots + y_mR$, $J = x_1A + \cdots + x_mA$. One knows [3] that $R$ is $A$-flat iff (i) $R/JR$ is flat over $A/J$, and (ii) $\text{Tor}_1^A(R, A/J) = 0$.

We observe that (i) is trivially verified as $A/J = k$ is a field.

To prove (ii) consider the Koszul complex associated to $y_1, ..., y_m$. But $y_1, ..., y_m$ forms a regular sequence; hence $H_p(K(y_1, ..., y_m)) = 0$, for $p > 0$. Thus (ii) is true and $R$ is a flat $A$-module.

$\text{Ker}(\theta) = 0$ follows from the fact that for any flat homomorphism of rings $f : B \to C$, any non zero-divisor $b$ of $B$ is mapped onto a non zero-divisor $c$ of $C$. Indeed the homothety $z \mapsto bz$ of $B$ tensored over $B$ with $C$ leads to the homothety $u \mapsto cu$ of $C$. So the Lemma is proved.

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