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Generalizations of reverse Bebiano–Lemos–Providência inequality

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Abstract

In our recent paper, we generalized Bebiano–Lemos–Providência inequality (BLP inequality) that for $A, B \geqslant 0$

$$\|A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}}\| \leqslant \|A^{\frac{1}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}}\|$$

for all $s \geqslant t \geqslant 0$. On the other hand, we also propose a reverse of BLP inequality, which is inspired by Araki–Cordes inequality; for $A, B > 0$

$$\|A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}}\| \geqslant \|A^{\frac{1}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}}\|$$

for all $t \geqslant s \geqslant 1$.

Based on our results, we discuss the reverse of BLP inequality in a general setting, in which we point out that the restriction $t \geqslant s \geqslant 1$ in the above is quite reasonable.

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1. Introduction

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space H. A positive operator A is denoted by $A \geqslant 0$. Löwner–Heinz inequality (cf. [14]) asserts

$$A \geqslant B \geqslant 0 \quad \text{implies} \quad A^p \geqslant B^p \qquad \text{for all } 0 \leqslant p \leqslant 1. \tag{1.1}$$

It is known that it is equivalent to the Araki–Cordes inequality that

$$\|A^{\frac{t}{2}} B^t A^{\frac{t}{2}}\| \leqslant \|(A^{\frac{1}{2}} B A^{\frac{1}{2}})^t\|$$

for $0 \leqslant t \leqslant 1$, [1,3]. Moreover, it is easily seen that so is the following reverse inequality:

$$\|A^{\frac{t}{2}} B^t A^{\frac{t}{2}}\| \geqslant \|(A^{\frac{1}{2}} B A^{\frac{1}{2}})^t\|$$

for $t \geqslant 1$.

By the way, Bebiano et al. [2] showed the following norm inequality, say BLP inequality; for $A, B \geqslant 0$

$$\|A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}}\| \leqslant \|A^{\frac{1}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}}\| \tag{1.2}$$

for all $s \geqslant t \geqslant 0$. Inspired by Araki–Cordes inequality, we showed a reverse of BLP inequality in our preceding paper [13] as follows. For $A, B > 0$

$$\|A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}}\| \geqslant \|A^{\frac{1}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}}\|$$

for all $t \geqslant s \geqslant 1$.

On the other hand, we generalized BLP inequality using Furuta inequality.

Let $A, B \geqslant 0$. Then

$$\|A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}}\|^{\frac{p+s}{p(1+s)}} \leqslant \|A^{\frac{1}{2}} (A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}}\| \tag{1.3}$$

for all $p \geqslant 1$ and $s \geqslant 0$.

In this note, we consider a reverse of generalized BLP inequality, in which Kamei's theorem [12] on complements of Furuta inequality corresponds to our results. As a corollary, we have our preceding theorem; in particular, the restriction $t \geqslant s \geqslant 1$ is well explained.

2. Preliminary-generalized BLP inequalities

In our recent paper [5], we generalized BLP inequality (1.2). For it we used Furuta inequality [7] (see also [4,8,11,15]).

For each $r \geqslant 0$

$$A \geqslant B \geqslant 0 \implies A^{1+r} \geqslant \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \tag{2.1}$$

holds for $p \geqslant 1$.

It is an essential part of Furuta inequality, whose whole picture is given in Fig. 1.

Theorem F (Furuta inequality, [6]). *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,*

(i) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$, *and*
(ii) $(A^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$ *hold for $p \geqslant 0$ and $q \geqslant 1$ with $(1 + r)q \geqslant p + r$.*

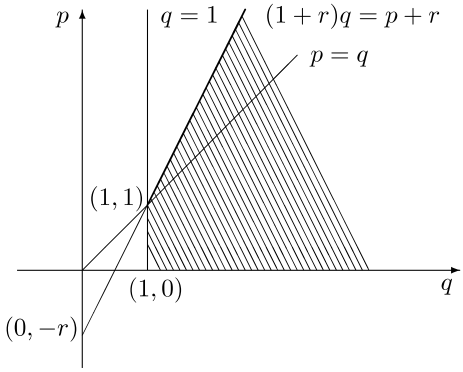

Fig. 1.

Now we review our results in the preceding paper [5]. First BLP inequality has the following representation by α-geometric mean $\sharp_\alpha$; for $A, B \geqslant 0$

$$A^s \sharp_{\frac{t}{s}} B^s \leqslant A^{1+s} \quad \text{for some } s \geqslant t \geqslant 0 \implies B^t \leqslant A^{1+t}, \tag{2.2}$$

where $A \sharp_\alpha B$ for $0 \leqslant \alpha \leqslant 1$ is defined by

$$A \sharp_\alpha B := A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}} \quad \text{for } A, B > 0. \tag{2.3}$$

Replacing here B by $B^{\frac{1+t}{t}}$, and putting $p := \frac{s}{t} (\geqslant 1)$ in (2.2), it is rewritten as follows: for $A, B \geqslant 0$

$$A^s \sharp_{\frac{1}{p}} B^{p+s} \leqslant A^{1+s} \quad \text{for some } p \geqslant 1 \text{ and } s \geqslant 0 \implies B^{1+\frac{s}{p}} \leqslant A^{1+\frac{s}{p}}. \tag{2.4}$$

Now Furuta inequality gives an improvement of (2.4). Let $A, B \geqslant 0$. Then

$$A^s \sharp_{\frac{1}{p}} B^{p+s} \leqslant A^{1+s} \quad \text{for some } p \geqslant 1 \text{ and } s \geqslant 0 \implies B^{1+s} \leqslant A^{1+s}. \tag{2.5}$$

As a consequence, we have the following norm inequality equivalent to (2.5) :

Generalized BLP inequality, [5]. *Let $A, B \geqslant 0$. Then*

$$\| A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}} \|^{\frac{p+s}{p(1+s)}} \leqslant \| A^{\frac{1}{2}} (A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}} \| \tag{2.6}$$

holds for all $p \geqslant 1$ and $s \geqslant 0$.

3. Reverse of generalized BLP inequality

Inspired by Araki–Cordes inequality and its reverse, we proposed in [13] the following reverse inequality with a slight restriction:

Theorem 3.1. *For $A, B > 0$*

$$\| A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}} \| \geqslant \| A^{\frac{1}{2}} \left(A^{\frac{s}{2}} B^s A^{\frac{s}{2}}\right)^{\frac{t}{s}} A^{\frac{1}{2}} \| \tag{3.1}$$

holds for all $t \geqslant s \geqslant 1$.

We remark that the original proof of Theorem 3.1 in [13] is constructive. On the other hand, BLP inequality as generalized in (1.3) is equivalent to Furuta inequality. Therefore, we expect that the reverse of generalized BLP inequality (1.3) will correspond to the following complement of Furuta inequality, due to Kamei [12]:

Theorem A. *If* $A \geqslant B > 0$, *then for* $0 < p \leqslant \frac{1}{2}$

$$A^t \natural_{\frac{2p-t}{p-t}} B^p \leqslant A^{2p} \quad \text{for } 0 \leqslant t \leqslant p \tag{3.2}$$

and for $\frac{1}{2} \leqslant p \leqslant 1$

$$A^t \natural_{\frac{1-t}{p-t}} B^p \leqslant A \quad \text{for } 0 \leqslant t \leqslant p. \tag{3.3}$$

Here $\natural_q$ *for* $q \notin [0, 1]$ *has been used as*

$$A \natural_q B := A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q A^{\frac{1}{2}} \quad \text{for } A, B > 0.$$

Now, we state our main theorem which is the reverse inequality of the generalized BLP inequality (1.3):

Theorem 3.2. *Let* $A, B \geqslant 0$ *and* $0 < p \leqslant 1$. *Then*

$$\|A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}}\|^{\frac{p+s}{p(1+s)}} \geqslant \|A^{\frac{1}{2}} (A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}}\| \tag{3.4}$$

for all $s \geqslant 0$ *with* $s \geqslant 1 - 2p$.

Proof. It suffices to show that

$$B^{1+s} \leqslant A^{-(1+s)} \Rightarrow A^{\frac{1}{2}} \left(A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}}\right)^{\frac{1}{p}} A^{\frac{1}{2}} \leqslant 1 \tag{3.5}$$

for $0 < p \leqslant 1$ and $s \geqslant 0$ with $s \geqslant 1 - 2p$. So we put

$$A_1 = A^{-(1+s)}, B_1 = B^{1+s}.$$

Then (3.5) is rephrased as

$$A_1 \geqslant B_1 > 0 \Rightarrow A_1^{\frac{s}{1+s}} \natural_{\frac{1}{p}} B_1^{\frac{p+s}{1+s}} \leqslant A_1$$

for $0 < p \leqslant 1$ and $s \geqslant 0$ with $s \geqslant 1 - 2p$. Moreover, if we replace

$$t_1 = \frac{s}{1+s}, \quad p_1 = \frac{p+s}{1+s},$$

then we have $\frac{1-t_1}{p_1-t_1} = \frac{1}{p}$, and $\frac{1}{2} \leqslant p_1 (\leqslant 1)$ if and only if $1 - 2p \leqslant s$, so that (3.5) has the following equivalent expression:

$$A_1 \geqslant B_1 > 0 \Rightarrow A_1^{t_1} \natural_{\frac{1-t_1}{p_1-t_1}} B_1^{p} \leqslant A_1 \quad \text{for } 0 \leqslant t_1 < p_1.$$

Since $\frac{1}{2} \leqslant p_1 \leqslant 1$, this is ensured by Theorem A due to Kamei. □

Next we show that Theorem 3.1 is obtained as a corollary of Theorem 3.2.

Proof of Theorem 3.1. We put $p = \frac{s}{t}$ for $t \geqslant s \geqslant 0$. Then we have $1 - 2p \leqslant s$ if and only if $\frac{t}{t+2} \leqslant s$. Since $s \geqslant 1$ is assumed, $\frac{t}{t+2} \leqslant s$ holds for arbitrary $t > 0$, so that Theorem 3.2 is applicable.

Now we take $B = B_1^{\frac{t}{1+t}}$ for a given arbitrary $B_1 \geqslant 0$, i.e., $B_1 = B^{\frac{1+t}{t}}$. Then Araki–Cordes inequality and Theorem 3.2 imply that

$$\|A^{\frac{1+t}{2}} B_1^t A^{\frac{1+t}{2}}\| \geqslant \|A^{\frac{1+s}{2}} B_1^{\frac{t(1+s)}{1+t}} A^{\frac{1+s}{2}}\|^{\frac{1+t}{1+s}} = \|A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}}\|^{\frac{p+s}{p(1+s)}}$$
$$\geqslant \|A^{\frac{1}{2}}(A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}}\| = \|A^{\frac{1}{2}}(A^{\frac{s}{2}} B_1^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}}\|,$$

which proves (3.1). □

Theorem 3.1 is slightly generalized as follows:

Corollary 3.3. *For $A, B > 0$ and $r \geqslant 0$*

$$\|A^{\frac{r+t}{2}} B^t A^{\frac{r+t}{2}}\| \geqslant \|A^{\frac{r}{2}}(A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{r}{2}}\| \tag{3.6}$$

holds for all $t \geqslant s \geqslant r$.

Proof. It is proved by applying Theorem 3.1 to $A_1 = A^r$, $B_1 = B^r$ and $t_1 = \frac{t}{r}$, $s_1 = \frac{s}{r}$. □

Finally, we consider a reverse inequality of generalized BLP inequality which corresponds to another Kamei's complement (3.2). If $A \geqslant B > 0$, then for $0 < p \leqslant \frac{1}{2}$

$$A^t \natural_{\frac{2p-t}{p-t}} B^p \leqslant A^{2p} \quad \text{for } 0 \leqslant t < p.$$

Theorem 3.4. *Let $A, B \geqslant 0$ and $0 < p \leqslant \frac{1}{2}$. Then*

$$\|A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}}\|^{\frac{(2p+s)(p+s)}{p(1+s)}} \geqslant \|A^{p+\frac{s}{2}}(A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{2p+s}{p}} A^{p+\frac{s}{2}}\| \tag{3.7}$$

for all $0 \leqslant s \leqslant 1 - 2p$.

Proof. The proof is quite similar to that of Theorem 3.2. We put

$$A_1 = A^{-(1+s)}, \quad B_1 = B^{1+s}; \quad t_1 = \frac{s}{1+s}, \quad p_1 = \frac{p+s}{1+s}.$$

Then (3.2) gives

$$A_1 \geqslant B_1 > 0 \Rightarrow A_1^{t_1} \natural_{\frac{2p_1-t_1}{p_1-t_1}} B_1^{p_1} \leqslant A_1^{2p_1}$$

for $0 \leqslant t_1 < p_1 \leqslant \frac{1}{2}$, so that

$$A^{-(1+s)} \geqslant B^{1+s} \Rightarrow A^{-s} \natural_{\frac{2p+s}{p}} B^{p+s} \leqslant A^{-2(p+s)}$$

for $0 \leqslant s \leqslant 1 - 2p$. Obviously, it implies the desired norm inequality (3.7). □

Remark. In Theorem 3.4, if we take $s = 0$, then we obtain Araki–Cordes inequality

$$\|A^{\frac{1}{2}} B A^{\frac{1}{2}}\|^{2p} \geqslant \|A^p B^{2p} A^p\|$$

for $0 \leqslant p \leqslant \frac{1}{2}$. Also it appears in Corollary 3.3 by taking $r = 0$. Actually (3.6) for $r = 0$ is expressed as

$$\|A^{\frac{t}{2}} B^t A^{\frac{t}{2}}\| \geqslant \|(A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}}\| = \|(A^{\frac{s}{2}} B^s A^{\frac{s}{2}})\|^{\frac{t}{s}}$$

for $t \geqslant s \geqslant 0$. So, replacing A^t (resp. B^t) by A (resp. B), we obtain this because $2p = \frac{t}{s} \geqslant 1$.

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