Abstract

The cutwidth problem is to find a linear layout of a network so that the maximal number of cuts of a line separating consecutive vertices is minimized (see e.g. [7]). A related and more natural problem is the cyclic cutwidth when a circular layout is considered. The main question is to compare both measures cw and ccw for specific networks, whether adding an edge to a path and forming a cycle reduces the cutwidth essentially. We prove exact values for the cyclic cutwidths of the two-dimensional ordinary and cylindrical meshes \( P_m \times P_n \) and \( P_m \times C_n \), respectively. Especially, if \( m \geq n + 3 \), then \( \text{ccw}(P_m \times P_n) = \text{cw}(P_m \times P_n) = n + 1 \) and if \( n \) is even then \( \text{ccw}(P_n \times P_n) = n - 1 \) while \( \text{cw}(P_n \times P_n) = n + 1 \) and if \( m \geq 2, n \geq 3 \), then \( \text{ccw}(P_m \times C_n) = \min\{m + 1, n + 2\} \).

Keywords: Cutwidth; Cyclic cutwidth; Ordinary mesh; Cylindrical mesh

1. Introduction

The underlying practical problem for this paper is the tradeoff between cost and speed of computer architectures: the linear array is the least expensive architecture and the ring architecture is probably only slightly more expensive. This is dependent on technology—in free space optics an additional connection might cost more than in fibre optics. Whether this small additional cost leads to significantly improved communication speed in the architecture depends on the communication pattern used by the parallel algorithm run on the linear array or ring (see also [3,14,16]). Communication patterns we are investigating in this paper are two-dimensional ordinary and cylindrical meshes. There are further motivations for studying these problems coming from rearrangeability [10], VLSI design [13], isoperimetric problems [4].

The cutwidth problem is to find a linear layout of an interconnection network so that the number of cuts of a line separating consecutive vertices is minimized (see e.g. [7]). A related and more natural problem is the cyclic cutwidth when a circular layout is considered. Both problems are NP-hard [8,12]. One of the main questions is to compare both measures, denoted by \( \text{cw}(G) \) and \( \text{ccw}(G) \), respectively, for a specific network \( G \), whether adding an edge to a path and forming a cycle reduces the cutwidth essentially. In [6,12], it is shown that the cyclic cutwidth equals the cutwidth in case of trees. For the toroidal mesh, the cyclic cutwidth is roughly half of the cutwidth [15]. A partial result for the \( \text{ccw} \) of the hypercube is in [2] showing that the cyclic cutwidth is less than \( \frac{5}{4} \) of the cutwidth. Note that another related problem is to compare the average \( \text{ccw} \) with the average cutwidth [5] and the cyclic bandwidth with the bandwidth [11].
In this paper we show the following results:

**Theorem 1.** For \( m \geq n \geq 3 \)

\[
\text{ccw}(P_m \times P_n) = \begin{cases} 
  n - 1 & \text{if } m = n \text{ is even}, \\
  n & \text{if } (m = n \text{ is odd}) \text{ or } (m = n + 1) \\
  n + 1 & \text{or } (m = n + 2 \text{ is even}) 
\end{cases}
\]

**Theorem 2.** For \( m \geq 2, n \geq 3 \)

\[
\text{ccw}(P_m \times C_n) = \min\{m + 1, n + 2\}.
\]

Actually, we prove the lower bounds only. The upper bounds are in [15] as well as the remaining cases \( \text{ccw}(P_m \times P_2) = 2 \), for \( m = 3, 4 \) and \( \text{ccw}(P_m \times P_2) = 3 \), for \( m \geq 5 \). Compare the above results with the results for the cutwidth of two-dimensional ordinary mesh: \( \text{cw}(P_m \times P_2) = n + 1 \), for \( m \geq n \geq 3 \) and \( \text{cw}(P_m \times P_2) = 3 \), for \( m \geq 3 \), and for the cutwidth of 2-dimensional cylindrical mesh: \( \text{cw}(P_m \times C_n) = \min\{2m + 1, n + 2\} \), for \( m \geq 2, n \geq 3 \) except \( \text{cw}(P_2 \times C_3) = 4 \) [15].

Our results completely solve the cyclic cutwidth problem for two-dimensional meshes. Generally saying the ratio of \( \text{cw}/\text{ccw} \) for ordinary mesh and toroidal mesh is roughly 1 and 2, respectively, while for the cylindrical mesh it lies inbetween depending on the relative sizes of \( m \) and \( n \).

Our method could be used to other mesh-like or product graphs.

In terms of the underlying practical problem, which hardware architecture to choose, our results are: If the communication pattern is a hypercube, torus, or “high” cylinder the ring architecture leads to a significant improvement over the linear array; if the communication pattern is a mesh or a “flat” cylinder the improvement is not significant or zero.

## 2. Definitions and notations

The cyclic cutwidth is a special case of the so-called congestion, an important concept from the theory of interconnection networks. Therefore, it is more convenient to define it by means of the congestion. Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be graphs such that \( |V_1| = |V_2| \). An embedding of \( G_1 \) in \( G_2 \) is a couple of mappings \((\phi, \psi)\) satisfying

\[
\phi : V_1 \to V_2 \quad \text{is a bijection, } \psi : E_1 \to \{\text{set of all paths in } G_2\},
\]

such that if \( uv \in E_1 \) then \( \psi(uv) \) is a path between \( \phi(u) \) and \( \phi(v) \). Define the congestion of an edge \( e \in E_2 \) under the embedding \((\phi, \psi)\) of \( G_1 \) in \( G_2 \) as

\[
\text{cg}(G_1, G_2, \phi, \psi, e) = |\{f \in E_1 : e \in \psi(f)\}|,
\]

the congestion of \( G_1 \) in \( G_2 \) under \((\phi, \psi)\) as

\[
\text{cg}(G_1, G_2, \phi, \psi) = \max_{e \in E_2} \{\text{cg}(G_1, G_2, \phi, \psi, e)\}
\]

and the congestion of \( G_1 \) in \( G_2 \) as

\[
\text{cg}(G_1, G_2) = \min_{(\phi, \psi)} \{\text{cg}(G_1, G_2, \phi, \psi)\}.
\]

Let \( P_n \) denote the \( n \)-vertex path. Let \( V_{P_n} = \{1, 2, \ldots, n\} \), with edges between \( i \) and \( i + 1 \), for \( i = 1, 2, \ldots, n - 1 \). Let \( C_n \) denote the \( n \)-vertex cycle. Vertices and edges of \( C_n \) are defined similarly.

Let \( G = (V, E) \) be an \( n \)-vertex graph. Define the cutwidth of \( G \) as \( \text{cw}(G) = \text{cg}(G, P_n) \), and the cyclic cutwidth of \( G \) as \( \text{ccw}(G) = \text{cg}(G, C_n) \).

For \( A \subseteq V \), let \( \partial(A) \) denote the edge boundary of \( A \), i.e., the set of all edges having one end in \( A \) and the second one in \( V - A \).

Let \( P_m \times P_n \) and \( P_m \times C_n \) denote the 2-dimensional ordinary and cylindrical meshes defined as the Cartesian products of two paths and a path and a cycle, respectively.
3. Proof of Theorem 1

The proof follows from a series of propositions and corollaries. The main idea consists of finding two edges on the target cycle whose sum of congestion under any embedding is sufficiently large. Then at least one of the edges will have large enough congestion. We start with a lemma.

Lemma 3.1. Let \(k, l \geq 2\) be integers s.t. \(k + l\) is even. Consider a cycle of length at least \(k + l\). Colour \(k\) vertices of the cycle by black and \(l\) vertices by white. Then the cycle can be cut by two edge cuts such that one part contains exactly \([k/2]\) black vertices and exactly \([l/2]\) white vertices.

Proof. We may assume that the length of the cycle is \(k + l\). The extension for any cycle is straightforward. Denote \(S_i = \{i, i + 1, \ldots, i + (k + l)/2 - 1\}\), where the arithmetic operations are taken mod \(k + l\). Wlog assume that \(S_0\) contains more than \([k/2]\) black vertices. Then \(S_{k + l/2}\) contains less than \([k/2]\) black vertices. Observe that numbers of black vertices in \(S_i\) and \(S_{i+1}\) differ by at most 1. Hence there must exist an \(i_0\), \(0 < i_0 < (k + l)/2\), for which the number of black vertices in \(S_{i_0}\) equals \([k/2]\). \(\Box\)

Remark. Stronger lemmata of this kind were also proved in [1,9].

Proposition 3.1. For any \(n \geq 3\)

\[
ccw(P_n \times P_n) \geq \begin{cases} 
   n - 1 & \text{if } n \text{ is even}, \\
   n & \text{if } n \text{ is odd}.
\end{cases}
\]

Proof. Let us have any embedding \((\phi, \psi)\) of \(P_n \times P_n\) into the cycle \(C_{n^2}\). Consider now only the boundary vertices of the mesh, i.e. the vertices of degree not greater than 3. Colour the embedded vertices \((i,1), i = 1, 2, \ldots, n\) and the vertices \((n,j), j = 2, 3, \ldots, n - 1\) by black and the remaining boundary vertices by white. According to Lemma 3.1 we can cut two edges of the cycle so that each of the two resulting paths contains exactly \(n - 1\) black and \(n - 1\) white vertices. This cut induces a cut of \(P_n \times P_n\) into two parts I and II such that each part contains exactly \(n - 1\) black and \(n - 1\) white vertices. We estimate the size of the cut. Clearly the size of the cut is at least the number of edge disjoint paths between I and II in the mesh.

Consider the subgraph \(T_n\) of \(P_n \times P_n\) induced by vertices \((i,j)\), where \(1 \leq j \leq i \leq n\), except for the vertex \((n,n)\). Note that \(T_n\) contains all black vertices and no white vertex.

Claim on \(T_n\). There are \(n - 1\) edge disjoint paths between the black vertices with one vertex in \(I\) and the second in \(II\). Moreover, these paths can be routed in \(T_n\) only.

Proof of the claim. We prove the claim by induction on \(n\). The cases \(n = 3, 4\) are trivial. Assume \(n \geq 5\) and let the claim hold for every subgraph \(T_m, 3 \leq m < n\).

If there exist two vertices \((i,1)\) and \((n,i)\), \(1 < i < n\) belonging to different sets I and II then join the vertices by the shortest path through \((i,i)\). Deleting the edges of the \(i\)th row and the edges of the \(i\)th column, if \(i < n\), we reduce the problem to the subgraph \(T_{n-1}\).

In the opposite case we must find \(i\) and \(j, 1 < i < j < n\), such that \((i,1)\) and \((n,i)\) belong to one set, say I, and \((j,1)\) and \((n,j)\) belong to II. Then join \((i,1)\) and \((j,1)\) by the path \((i,1) - (i,i) - (j,i) - (j,1)\). Similarly join \((n,j)\) and \((n,i)\). Deleting the edges of the \(i\)th and the \(j\)th row and column we reduce the problem to \(T_{n-2}\).

See the illustration of creating the disjoint paths in the Fig. 1 for \(n = 8\) and a random distribution of the black vertices over I and II. The paths are shown by heavy lines.

By the above procedure we get \(n - 3\) edge disjoint paths and \(T_1\) in which at least 2 other edge disjoint paths can be constructed. There may be found still other edge disjoint paths. It depends on into how many subpaths the path of black vertices \((1,1)(2,1)(3,1) - (n,1)(n,2) - (n,n - 1)\) is cut. If it is cut into two subpaths, we can construct altogether \(n - 1\) edge disjoint paths, if into three or more subpaths, we can construct at least \(n\) edge disjoint paths. \(\Box\)

Doing the same construction of edge disjoint paths with white end vertices in the subgraph induced by \((i,j), 1 \leq i \leq j \leq n\), except for the vertex \((1,1)\), we get other \(n - 1\) edge disjoint paths between I and II. So we have in total at least \(2n - 2\) edge disjoint paths between I and II in \(P_n \times P_n\). This implies that the cut of \(P_n \times P_n\) into the parts I and II has at least \(2n - 2\) edges which in turn immediately implies that the congestion is at least \((2n - 2)/2 = n - 1\).
Let the set of black vertices of and \([1,n] \subseteq V\) contains the edge between \(v_1,v_2\). Note that some edges may be counted twice. Finally, this cut corresponds to a cut of \(G\) in at least one vertex \(k,l\). Wlog we may assume that \(l \geq (n+1)/2\). Consider two shortest paths in \(C \cup C'\) which contain vertices \(v,[1,1],[k,l]\) and vertices \(v,[n-1,1],[k,l]\), respectively. As \(l \geq (n+1)/2\), the sum of lengths of the paths is at least \(2n-1\). The edges in the paths correspond to an edge cut that separates vertices \((i,1),i=1,2,\ldots,n-1\) and the rest of boundary vertices in \(P_n \times P_n\). Note that some edges may be counted twice. Finally, this cut corresponds to a cut of \(C'\) s.t. one part of the cycle contains vertices \((i,1),i=1,2,\ldots,n-1\) and the rest of boundary vertices is in the second part. Hence \(\text{ccw}(P_n \times P_n) \geq \lceil (2n-1)/2 \rceil = n\). Illustration of the argument for \(n=7,k=4,l=5\) is in the Fig. 2.

The dual graph \(G\) is shown by heavier lines and vertices as the original mesh. Cycles \(C\) and \(C'\) are shown by the heaviest lines. \(\square\)

**Proposition 3.2.** For odd \(n \geq 3\)

\[
\text{ccw}(P_{n+2} \times P_n) \geq n + 1.
\]

**Proof.** The proof is the same as above but we start with the black vertices defined to be \((i,1),i=2,3,\ldots,n+2\) and \((n+2,j),j=2,3,\ldots,n\), and white vertices defined to be \((1,i),i=1,2,\ldots,n\) and \((j,n),j=2,3,\ldots,n+1\). The only case, when we cannot use the methods of Proposition 3.1 is when one part, say part II, contains these and only these vertices \((n+2,j),j=1,2,3,\ldots,n\), and \((1,i),i=1,2,\ldots,n\). In this case we move cut so that we enlarge part II by one vertex. The number of edge disjoint paths increases at least by one and we are done. \(\square\)

The above propositions immediately imply:

**Corollary 3.1.** For \(n \geq 3\)

\[
\text{ccw}(P_{n+1} \times P_n) \geq n.
\]
Proposition 3.3. For even $n \geq 4$
\[ \text{ccw}(P_{n+3} \times P_n) \geq n + 1. \]

Proof. Follow the proof of Proposition 3.1. Colour the vertices: $(i, 1), i = 3, 4, \ldots, n + 3$ and $(n + 3, j), j = 2, 3, \ldots, n$ by black and vertices $(1, i), i = 1, 2, \ldots, n$ and $(j, n), j = 2, 3, \ldots, n + 1$ by white. In similar way as above we construct at least $2n$ edge disjoint paths between I and II. If there are only $2n$ such paths, i.e. exactly $n$ edge disjoint paths in each $T_{n+1}$, then two cases are possible:

- either the vertices $(i, 1), i = 3, 4, \ldots, n + 2$ and $(1, j), j = 1, 2, \ldots, n$ are in one part e.g. I, and the rest of coloured vertices is in II and therefore one can construct a new path between the vertices $(3, 1) \in I$ and $(n + 1, n) \in II$ which is edge disjoint with the previously constructed paths;
- or the vertices $(i, 1), i = 3, 4, \ldots, n + 2$ and $(i, n), i = 2, 3, \ldots, n + 1$ are in one part and the vertices $(1, j), j = 1, 2, \ldots, n$ and $(n + 3, j), j = 1, 2, \ldots, n$ are in the other part and one can construct a new edge disjoint path between the vertices $(3, 1)$ and $(n + 3, n)$. □

4. Proof of Theorem 2

The basic idea is the same as in the previous proof: we will look for two edges on the cycle whose sum of congestions is sufficiently large. The details are different. In [15] we proved the following useful:

Lemma 4.1. For $m \geq 2, n \geq 3$, let $A$ be an arbitrary subset of vertices of $P_m \times C_n$, s.t. $|A| = \lfloor mn/2 + 1 \rfloor$. Then $|\hat{c}(A)| \geq \min\{2m + 1, n + 2\}$.

To prove the lower bound in Theorem 2 it is sufficient to show that for $n \geq 3$, $\text{ccw}(P_{n+1} \times C_n) \geq n + 2$, as if $m \geq n + 1$ then
\[ \text{ccw}(P_m \times C_n) \geq \text{ccw}(P_{n+1} \times C_n), \]
and if $m \leq n + 1$ then
\[ \text{ccw}(P_m \times C_n) \geq \text{ccw}(P_m \times C_{n-1}). \]

Delete the $n$ “middle” edges from $P_{n+1} \times C_n$ to get two distinct graphs $G_1 = (V_1, E_1) = P_{\lfloor (n+1)/2 \rfloor} \times C_n$ and $G_2 = (V_2, E_2) = P_{\lfloor (n+2)/2 \rfloor} \times C_n$. Assume $G_1 \cup G_2$ is embedded in $C_{n+2}$. We will identify the vertices of $G_1 \cup G_2$ with the vertices of $C_{n+2}$. We show that $\text{ccw}(G_1 \cup G_2) \geq n + 2$. 

Fig. 2. Illustration of the dual graph argument.
Distinguish 2 cases:

Case 1: Let \( n = 4t + 1 \), for \( t \geq 1 \). Then \( |V_1| = \lceil (n + 1)/2 \rceil n \) is an odd number. Colour the vertices of \( V_1 \) by black. According to Lemma 3.1 there exist a set \( S \) of \( n(n + 1)/2 \) consecutive vertices of \( C_{n^2} \) containing exactly \\

\[
\left\lceil \frac{n+1}{2} \right\rceil n + 1
\]
black vertices. Denote these vertices by \( A_1 \). Define \( A_2 = V_2 - (S - A_1) \). Note that \( A_2 \subseteq V_2 \) and \( A_2 \) lies in the complement of \( S \). Then

\[
|A_2| = \left\lfloor \frac{n+1}{2} \right\rfloor n + 1.
\]

Now consider the congestions on the two edges joining \( S \) with its complement. Because neighbours of \( A_1 \) are in the complement of \( S \) and neighbours of \( A_2 \) are in \( S \) we have

\[
\text{ccw}(G_1 \cup G_2) \geq \left\lceil \frac{|\partial(A_1)| + |\partial(A_2)|}{2} \right\rceil \geq n + 2,
\]
where we applied Lemma 4.1 for estimating the sizes of edge boundaries of \( A_1 \) and \( A_2 \).

Case 2: Let \( n \neq 4t + 1 \). Then \( |V_1| = \lceil (n + 1)/2 \rceil n \) is an even number.

(a) Assume that there exists a set of \( (n + 1)n/2 \) consecutive vertices of \( C_{n^2} \) containing at least \( \lceil (n + 1)/2 \rceil n/2 + 1 \) black vertices. Then similarly as in the proof of Lemma 3.1 we show that there must exist a set \( S \) of \( (n + 1)n/2 \) consecutive vertices of \( C_{n^2} \) containing exactly \( \lceil (n + 1)/2 \rceil n/2 + 1 \) black vertices. Denote this set by \( A_1 \). Define \( A_2 = V_2 - (S - A_1) \). Then

\[
|A_2| = \left\lfloor \frac{n+1}{2} \right\rfloor n + 1.
\]

Similarly as above we have

\[
\text{ccw}(G_1 \cup G_2) \geq \left\lceil \frac{|\partial(A_1)| + |\partial(A_2)|}{2} \right\rceil \geq \frac{n + 2 + n + 1}{2} = n + 2.
\]

(b) If (a) does not hold then any set of \( (n + 1)n/2 \) consecutive vertices of \( C_{n^2} \) contains precisely \( \lceil (n + 1)/2 \rceil n/2 \) black vertices. Then one can easily find a set \( S \) of \( (n + 1)n/2 + 2 \) consecutive vertices of \( C_{n^2} \) containing precisely \( \lceil (n + 1)/2 \rceil n/2 + 1 \) black vertices. Define \( A_2 = S - A_1 \). Then

\[
|A_2| = \left\lfloor \frac{n+1}{2} \right\rfloor n + 1
\]
and the rest is similar as above.

5. Conclusions

We created a new method to show exact results for the cyclic cutwidth of the two-dimensional ordinary and cylindrical meshes. The method is applicable to mesh-like and product graphs. At the time being there are only a few classes of interconnection networks including meshes, complete graphs and trees, for which the cyclic cutwidth is known precisely. Cyclic cutwidth of the hypercube and the complete bipartite graph is an open problem. Another interesting question is to characterize the class of interconnection networks for which the cutwidth and the cyclic cutwidth is the same.

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References