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ScienceDirect
Journal of Functional Analysis 261 (2011) 1028–1082
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Local entropy theory for a countable discrete amenable group action

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Received 4 January 2011; accepted 25 April 2011

Available online 11 May 2011

Communicated by D. Voiculescu

Abstract

The local properties of entropy for a countable discrete amenable group action are studied. For such an action, a local variational principle for a given finite open cover is established, from which the variational relation between the topological and measure-theoretic entropy tuples is deduced. While doing this it is shown that two kinds of measure-theoretic entropy for finite Borel covers coincide. Moreover, two special classes of such an action: systems with uniformly positive entropy and completely positive entropy are investigated.

Keywords: Entropy; Amenable group; Variational principle; Entropy tuple

1. Introduction

Rohlin and Sinai [38] introduced the notion of completely positive entropy (c.p.e.) for $\mathbb{Z}$-actions on a Lebesgue space. It is also known as K-actions of $\mathbb{Z}$. K-actions played an important role in the classic ergodic theory. In 1992, Blanchard introduced the notions of uniformly positive entropy (u.p.e.) and c.p.e. as topological analogues of the K-actions in topological dynamics of $\mathbb{Z}$-actions [1]. By localizing the concepts of u.p.e. and c.p.e., he defined the notion of entropy

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2011.04.014

pairs, and used it to show that a u.p.e. system is disjoint from all minimal zero entropy systems [2] and to obtain the maximal zero entropy factor for any topological dynamical system of $\mathbb{Z}$-actions (namely the topological Pinsker factor) [5]. From then, on the local entropy theory of $\mathbb{Z}$-actions there have been made great achievements [1–5,11,16,21,23–25,27,39,45], see also the relevant chapters in [17] and the survey papers [19,20]. A key point in the local entropy theory of $\mathbb{Z}$-actions is the local variational principle for finite open covers.

Note that for each dynamical system (X, T) of $\mathbb{Z}$-actions (or call it TDS), there always exist T-invariant Borel probability measures on X so that the classic ergodic theory involves the study of the entropy theory of (X, T). Whereas, there are some groups G such that there exists no any invariant Borel probability measures on some compact metric space with G-actions, for example the rank two free group F_2. It is well known that, for a dynamical system of group actions, the amenability of the group ensures the existence of invariant Borel probability measures, which includes all finite groups, solvable groups and compact groups.

Comparing to dynamical systems of $\mathbb{Z}$-actions, the level of development of dynamical systems of an amenable group action lagged behind. However, this situation is rapidly changing in recent years. A turning point occurred with Ornstein and Weiss's pioneering paper [34] in 1987 which laid a foundation of an amenable group action. In 2000, Rudolph and Weiss [40] showed that K-actions for a countable discrete amenable group is mixing of all orders (an open important question for years) by using methods from orbit equivalence. Inspired by this, Danilenko [7] pushed further the idea used by Rudolph and Weiss providing new short proofs of results in [18,34,40,43]. Meanwhile, based on the result of [40] and with the help of the results from [6], Dooley and Golodets in [9] proved that every free ergodic actions of a countable discrete amenable group with c.p.e. has a countable Lebesgue spectrum. Another long standing open problem is the generalization of pointwise convergence results, even such basic theorems as the L^1-pointwise ergodic theorem and the Shannon–McMillan–Breiman (SMB) Theorem for general amenable groups, for related results see for example [13,29,35]. In [30] Lindenstrauss gave a satisfactory answer to the question by proving the pointwise ergodic theorem for general locally compact amenable groups along Følner sequences obeying some restrictions (such sequences must exist for all amenable groups) and obtaining a generalization of the SMB Theorem to all countable discrete amenable groups (see also the survey [44] written by Weiss). Moreover, using the tools built in [30] Lindenstrauss also proved other pointwise results, for example [35] and so on.

Along with the development of the local entropy theory for $\mathbb{Z}$-actions, a natural question arises: to what extends the theory can be generalized to an amenable group action? In [27] Kerr and Li studied the local entropy theory of an amenable group action for topological dynamics via independence. In this paper we try to study systematically the local properties of entropy for actions of a countable discrete amenable group both in topological and measure theoretical settings.

First, we shall establish a local variational principle for a given finite open cover of a countable discrete amenable group action. Note that the classical variational principle of a countable discrete amenable group action (see [33,41]) can be deduced from our result by proceeding some simple arguments. In the way to build the local variational principle, we also introduce two kinds of measure-theoretic entropy for finite Borel covers following the ideas of [39], prove the upper semi-continuity (u.s.c.) of them (when considering a finite open cover) on the set of invariant measures, and show that they coincide. We note that completely different from the case of $\mathbb{Z}$-actions, in our proving of the u.s.c. we need a deep convergence lemma related to a countable discrete amenable group; and in our proving of the equivalence of these two kinds of entropy, we need the result that they are equivalent for $\mathbb{Z}$-actions, and Danilenko's orbital approach method (since we can't obtain a universal Rohlin Lemma and a result similar to Glasner–Weiss Theorem [19] in this setting). Meanwhile, inspired by [44, Lemma 5.11] we shall give a local version of the well-known Katok's result [26, Theorem I.I] for a countable discrete amenable group action.

Then we introduce entropy tuples in both topological and measure-theoretic settings. The set of measure-theoretic entropy tuples for an invariant measure is characterized, the variational relation between these two kinds of entropy tuples is obtained as an application of the local variational principle for a given finite open cover. Based on the ideas of topological entropy pairs, we discuss two classes of dynamical systems: having u.p.e. and having c.p.e. Precisely speaking, for a countable discrete amenable group action, it is proved: u.p.e. and c.p.e. are both preserved under a finite production; u.p.e. implies c.p.e.; c.p.e. implies the existence of an invariant measure with full support; u.p.e. implies mild mixing; and minimal topological K implies strong mixing if the group considered is commutative.

We note that when we finished our writing of the paper, we received a preprint by Kerr and Li [28], where the authors investigated the local entropy theory of an amenable group action for measure-preserving systems via independence. They obtained the variational relation between these two kinds of entropy tuples defined by them, and stated the local variational principle for a given finite open cover as an open question, see [28, Question 2.10]. Moreover, the results obtained in this paper have been applied to consider the co-induction of dynamical systems in [10].

The paper is organized as following. In Section 2, we introduce the terminology from [34,43] that we shall use, and obtain some convergence lemmas which play key roles in the following sections. In Section 3, for a countable discrete amenable group action we introduce the entropy theory of it, including two kinds of measure-theoretic entropy for a finite Borel cover, and establish some basic properties of them, such as u.s.c., affinity and so on. Then in Section 4 we prove the equivalence of those two kinds of entropy introduced for a finite Borel cover, and give a local version of the well-known Katok's result [26, Theorem I.I] for a countable discrete amenable group action. In Section 5, we aim to establish the local variational principle for a finite open cover. In Section 6, we introduce entropy tuples in both topological and measure-theoretic settings and establish the variational relation between them. Based on the ideas of topological entropy pairs, in Section 7 we discuss two special classes of dynamical systems: having u.p.e. and having c.p.e., respectively.

2. Backgrounds of a countable discrete amenable group

Let G be a countable discrete infinite group and $F(G)$ the set of all finite non-empty subsets of G. G is called *amenable*, if for each $K \in F(G)$ and $\delta > 0$ there exists $F \in F(G)$ such that

$$\frac{|F \Delta K F|}{|F|} < \delta,$$

where $|\cdot|$ is the counting measure, $KF = \{kf\colon\ k \in K,\ \ f \in F\}$ and $F \Delta KF = (F \setminus KF) \cup (KF \setminus F)$. Let $K \in F(G)$ and $\delta > 0$. Set $K^{-1} = \{k^{-1}\colon\ k \in K\}$. $A \in F(G)$ is (K,δ)-*invariant* if

$$\frac{|B(A, K)|}{|A|} < \delta,$$

where $B(A,K) \doteq \{g \in G\colon\ Kg \cap A \neq \emptyset \text{ and } Kg \cap (G \setminus A) \neq \emptyset\} = K^{-1}A \cap K^{-1}(G \setminus A)$. A sequence $\{F_n\}_{n\in\mathbb{N}} \subseteq F(G)$ is called a *Følner sequence*, if for each $K \in F(G)$ and $\delta > 0$,

F_n is (K,δ)-invariant when n is large enough. It is not hard to obtain the following asymptotic invariance property that G is amenable if and only if G has a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$. For example, for $\mathbb{Z}$ we may take Følner sequence $F_n=\{0,1,\dots,n-1\}$, or for that matter $\{a_n,a_n+1,\dots,a_n+n-1\}$ for any sequence $\{a_n\}_{n\in\mathbb{N}}\subseteq\mathbb{Z}$.

Throughout the paper, any amenable group considered is assumed to be a countable discrete amenable infinite group, and G will always be such a group with the unit e_G.

2.1. *Quasi-tiling for an amenable group*

The following terminology and results are due to Ornstein and Weiss [34] (see also [40,43]). Let $\{A_1,\dots,A_k\}\subseteq F(G)$ and $\epsilon\in(0,1)$. Subsets $A_1,\dots,A_k$ are *ϵ-disjoint* if there are $\{B_1,\dots,B_k\}\subseteq F(G)$ such that

(1) $B_i\subseteq A_i$ and $\frac{|B_i|}{|A_i|}>1-\epsilon$ for $i=1,\dots,k$,
(2) $B_i\cap B_j=\emptyset$ if $1\leqslant i\neq j\leqslant k$.

For $\alpha\in(0,1]$, we say that $\{A_1,\dots,A_k\}$ *α-covers* $A\in F(G)$ if

$$\frac{|A\cap(\bigcup_{i=1}^k A_i)|}{|A|}\geqslant\alpha.$$

For $\delta\in[0,1)$, $\{A_1,\dots,A_k\}$ is called a *δ-even cover* of $A\in F(G)$ if

(1) $A_i\subseteq A$ for $i=1,\dots,k$,
(2) there is $M\in\mathbb{N}$ such that $\sum_{i=1}^k 1_{A_i}(g)\leqslant M$ for each $g\in G$ and $\sum_{i=1}^k|A_i|\geqslant(1-\delta)M|A|$.

We say that $A_1,\dots,A_k$ *ϵ-quasi-tile* $A\in F(G)$ if there exists $\{C_1,\dots,C_k\}\subseteq F(G)$ such that

(1) for $i=1,\dots,k$, $A_iC_i\subseteq A$ and $\{A_ic\colon c\in C_i\}$ forms an ϵ-disjoint family,
(2) $A_iC_i\cap A_jC_j=\emptyset$ if $1\leqslant i\neq j\leqslant k$,
(3) $\{A_iC_i\colon i=1,\dots,k\}$ forms a $(1-\epsilon)$-cover of A.

The subsets $C_1,\dots,C_k$ are called the *tiling centers*.

The following lemmas are proved in [34, §1.2].

Lemma 2.1. *Let $\delta\in[0,1)$, $e_G\in S\in F(G)$ and $A\in F(G)$ satisfy that A is (SS^{-1},δ)-invariant. Then the right translates of S that lie in A, $\{Sg\colon g\in G,\ Sg\subseteq A\}$, form a δ-even cover of A.*

Lemma 2.2. *Let $\delta\in[0,1)$ and let $\mathcal{A}\subseteq F(G)$ be a δ-even cover of $A\in F(G)$. Then for each $\epsilon\in(0,1)$ there is an ϵ-disjoint sub-collection of $\mathcal{A}$ which $\epsilon(1-\delta)$-covers A.*

Then we can claim the following proposition (see [34] or [43, Theorem 2.6]).

Proposition 2.3. *Let $\{F_n\}_{n\in\mathbb{N}}$ with $e_G\in F_1\subseteq F_2\subseteq\cdots$ and $\{F'_n\}_{n\in\mathbb{N}}$ be two Følner sequences of G. Then for any $\epsilon\in(0,\frac14)$ and $N\in\mathbb{N}$, there exist integers $n_1,\dots,n_k$ with $N\leqslant n_1<\cdots<n_k$ such that $F_{n_1},\dots,F_{n_k}$ ϵ-quasi-tile F'_m when m is large enough.*

Proof. We follow the arguments in the proof of [43, Theorem 2.6]. Fix $\epsilon \in (0, \frac{1}{4})$ and $N \in \mathbb{N}$.

Let $k \in \mathbb{N}$ and $\delta > 0$ such that $(1 - \frac{\epsilon}{2})^k < \epsilon$ and $6^k \delta < \frac{\epsilon}{2}$. We can choose integers $n_1, \ldots, n_k$ with $N \leqslant n_1 < \cdots < n_k$ such that $F_{n_{i+1}}$ is $(F_{n_i} F_{n_i}^{-1}, \delta)$-invariant and $\frac{|F_{n_i}|}{|F_{n_{i+1}}|} < \delta$, $i = 1, \ldots, k-1$.

Now for each enough large m, F'_m is $(F_{n_k} F_{n_k}^{-1}, \delta)$-invariant and $\frac{|F_{n_k}|}{|F'_m|} < \delta$, thus by Lemma 2.1 the right translates of F_{n_k} that lie in F'_m form a δ-even cover of F'_m, and so by Lemma 2.2 there exists $C_k \in F(G)$ such that $F_{n_k} C_k \subseteq F'_m$ and the family $\{F_{n_k} c\colon c \in C_k\}$ is ϵ-disjoint and $\epsilon(1-\delta)$-covers F'_m. Let $c_k \in C_k$. Without loss of generality assume that $|F_{n_k} C_k \setminus F_{n_k} c_k| < \epsilon(1-\delta)|F'_m|$ (if necessary we may take a subset of C_k to replace with C_k). Then $(1-\epsilon)|F_{n_k}||C_k| < |F'_m|$ and

$$
\begin{aligned}
1 - \epsilon(1-\delta) \geqslant \frac{|F'_m \setminus F_{n_k} C_k|}{|F'_m|} &= 1 - \frac{|F_{n_k} C_k \setminus F_{n_k} c_k| + |F_{n_k} c_k|}{|F'_m|} \\
&\geqslant 1 - \epsilon(1-\delta) - \delta.
\end{aligned} \tag{2.1}
$$

Set $A_{k-1} = F'_m \setminus F_{n_k} C_k$, $K_{k-1} = F_{n_{k-1}} F_{n_{k-1}}^{-1}$. We have

$$
\begin{aligned}
B(A_{k-1}, K_{k-1}) &= K_{k-1}^{-1}\big(F'_m \setminus F_{n_k} C_k\big) \cap K_{k-1}^{-1}\big(\big(G \setminus F'_m\big) \cup F_{n_k} C_k\big) \\
&\subseteq B\big(F'_m, K_{k-1}\big) \cup \bigcup_{c \in C_k} B(F_{n_k} c, K_{k-1}) \\
&\subseteq B\big(F'_m, F_{n_k} F_{n_k}^{-1}\big) \cup \bigcup_{c \in C_k} B(F_{n_k}, K_{k-1}) c \quad \big(\text{as } K_{k-1} \subseteq F_{n_k} F_{n_k}^{-1}\big),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\frac{|B(A_{k-1}, K_{k-1})|}{|A_{k-1}|} &\leqslant \frac{|B(F'_m, F_{n_k} F_{n_k}^{-1})|}{|A_{k-1}|} + |C_k| \frac{|B(F_{n_k}, K_{k-1})|}{|A_{k-1}|} \\
&< \frac{\delta}{|F'_m \setminus F_{n_k} C_k|} \big(\big|F'_m\big| + |C_k||F_{n_k}|\big) \\
&< \delta\left(1 + \frac{1}{1-\epsilon}\right) \frac{|F'_m|}{|F'_m \setminus F_{n_k} C_k|} \quad \big(\text{as } (1-\epsilon)|F_{n_k}||C_k| < \big|F'_m\big|\big) \\
&\leqslant \delta\left(1 + \frac{1}{1-\epsilon}\right) \frac{1}{1 - \epsilon(1-\delta) - \delta} \quad \big(\text{by (2.1)}\big) \\
&< 6\delta \quad \left(\text{as } \epsilon \in \left(0, \frac{1}{4}\right)\right).
\end{aligned}
$$

That is, A_{k-1} is $(F_{n_{k-1}} F_{n_{k-1}}^{-1}, 6\delta)$-invariant. Moreover, using (2.1) one has

$$
\frac{|F_{n_{k-1}}|}{|A_{k-1}|} = \frac{|F_{n_{k-1}}|}{|F_{n_k}|} \cdot \frac{|F_{n_k}|}{|F'_m|} \cdot \frac{|F'_m|}{|F'_m \setminus F_{n_k} C_k|} < \frac{\delta^2}{1 - \epsilon(1-\delta) - \delta} < \delta.
$$

By the same reasoning there exists $C_{k-1} \in F(G)$ such that $F_{n_{k-1}} C_{k-1} \subseteq A_{k-1}$, the family $\{F_{n_{k-1}} c\colon c \in C_{k-1}\}$ is ϵ-disjoint and $\epsilon(1-6\delta)$-covers A_{k-1} and

$$1-\epsilon(1-6\delta) \geqslant \frac{|A_{k-1} \setminus F_{n_{k-1}} C_{k-1}|}{|A_{k-1}|} \geqslant 1-\epsilon(1-6\delta)-6\delta. \tag{2.2}$$

Moreover, by (2.1) and (2.2) we have

$$\frac{|A_{k-1} \setminus F_{n_{k-1}} C_{k-1}|}{|F'_m|} = \frac{|A_{k-1} \setminus F_{n_{k-1}} C_{k-1}|}{|A_{k-1}|} \cdot \frac{|F'_m \setminus F_{n_k} C_k|}{|F'_m|}$$
$$\leqslant \big(1-\epsilon(1-6\delta)\big)\big(1-\epsilon(1-\delta)\big) < \left(1-\frac{\epsilon}{2}\right)^2.$$

Inductively, we get $\{C_k, \ldots, C_1\} \subseteq F(G)$ such that if $1 \leqslant i \neq j \leqslant k$ then $F_{n_i} C_i \cap F_{n_j} C_j = \emptyset$, and if $i = 1, \ldots, k$ then $F_{n_i} C_i \subseteq F'_m$ and the family $\{F_{n_i} c\colon c \in C_i\}$ is ϵ-disjoint. Moreover,

$$\frac{|F'_m \setminus \bigcup_{i=1}^k F_{n_i} C_i|}{|F'_m|} < \left(1-\frac{\epsilon}{2}\right)^k < \epsilon.$$

Thus, $\{F_{n_i} C_i\colon i = 1, \ldots, k\}$ forms a $(1-\epsilon)$-cover of F'_m. This ends the proof. □

2.2. Convergence key lemmas

Let $f : F(G) \to \mathbb{R}$ be a function. We say that f is

(1) *monotone*, if $f(E) \leqslant f(F)$ for any $E, F \in F(G)$ satisfying $E \subseteq F$;
(2) *non-negative*, if $f(F) \geqslant 0$ for any $F \in F(G)$;
(3) *G-invariant*, if $f(Fg) = f(F)$ for any $F \in F(G)$ and $g \in G$;
(4) *sub-additive*, if $f(E \cup F) \leqslant f(E) + f(F)$ for any $E, F \in F(G)$.

The following lemma is proved in [31, Theorem 6.1].

Lemma 2.4. *Let $f : F(G) \to \mathbb{R}$ be a monotone non-negative G-invariant sub-additive (m.n.i.s.a.) function. Then for any Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ of G, the sequence $\{\frac{f(F_n)}{|F_n|}\}_{n\in\mathbb{N}}$ converges and the value of the limit is independent of the selection of the Følner sequence $\{F_n\}_{n\in\mathbb{N}}$.*

Proof. We give a proof for the completion. Since f is G-invariant, there exists $M \in \mathbb{R}_+$ such that $f(\{g\}) = M$ for all $g \in G$.

Now first we claim that if $\{F_n\}_{n\in\mathbb{N}}$ with $e_G \in F_1 \subseteq F_2 \subseteq \cdots$ and $\{F'_n\}_{n\in\mathbb{N}}$ are two Følner sequences of G then

$$\limsup_{n\to+\infty} \frac{f(F'_n)}{|F'_n|} \leqslant \limsup_{n\to+\infty} \frac{f(F_n)}{|F_n|}. \tag{2.3}$$

Let $\epsilon \in (0, \frac{1}{4})$ and $N \in \mathbb{N}$. By Proposition 2.3 there exist integers $n_1, \ldots, n_k$ with $N \leqslant n_1 < \cdots < n_k$ such that when n is large enough then $F_{n_1}, \ldots, F_{n_k}$ ϵ-quasi-tile F'_n with tiling centers $C_1^n, \ldots, C_k^n$. Thus, when n is large enough then

$$F_n' \supseteq \bigcup_{i=1}^k F_{n_i} C_i^n \quad \text{and} \quad \left|\bigcup_{i=1}^k F_{n_i} C_i^n\right| \geqslant \max\left\{(1-\epsilon)\left|F_n'\right|, (1-\epsilon)\sum_{i=1}^k \left|C_i^n\right| \cdot |F_{n_i}|\right\}, \tag{2.4}$$

which implies

$$\begin{aligned}
\frac{f(F_n')}{|F_n'|} &\leqslant \frac{f(F_n' \setminus \bigcup_{i=1}^k F_{n_i} C_i^n) + f(\bigcup_{i=1}^k F_{n_i} C_i^n)}{|F_n'|} \\
&\leqslant M \frac{|F_n' \setminus \bigcup_{i=1}^k F_{n_i} C_i^n|}{|F_n'|} + \frac{f(\bigcup_{i=1}^k F_{n_i} C_i^n)}{|\bigcup_{i=1}^k F_{n_i} C_i^n|} \\
&\leqslant M\epsilon + \frac{f(\bigcup_{i=1}^k F_{n_i} C_i^n)}{|\bigcup_{i=1}^k F_{n_i} C_i^n|} \\
&\leqslant M\epsilon + \sum_{i=1}^k \frac{|C_i^n| f(F_{n_i})}{(1-\epsilon)\sum_{i=1}^k |C_i^n| \cdot |F_{n_i}|} \quad \bigl(\text{using (2.4)}\bigr) \\
&\leqslant M\epsilon + \frac{1}{1-\epsilon} \max_{1\leqslant i\leqslant k} \frac{f(F_{n_i})}{|F_{n_i}|} \leqslant M\epsilon + \frac{1}{1-\epsilon} \sup_{m\geqslant N} \frac{f(F_m)}{|F_m|}.
\end{aligned} \tag{2.5}$$

Now letting $\epsilon \to 0+$ and $N \to +\infty$, we conclude the inequality (2.3).

Now let $\{H_n\}_{n\in\mathbb{N}}$ with $e_G \in H_1 \subseteq H_2 \subseteq \cdots$ be a Følner sequence of G. Clearly, there is a sub-sequence $\{H_{n_m}\}_{m\in\mathbb{N}}$ of $\{H_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m\to+\infty} \frac{f(H_{n_m})}{|H_{n_m}|} = \liminf_{n\to+\infty} \frac{f(H_n)}{|H_n|}. \tag{2.6}$$

Applying the above claim to Følner sequences $\{H_{n_m}\}_{m\in\mathbb{N}}$ and $\{H_n\}_{n\in\mathbb{N}}$ (see (2.3)), we obtain

$$\limsup_{n\to+\infty} \frac{f(H_n)}{|H_n|} \leqslant \limsup_{m\to+\infty} \frac{f(H_{n_m})}{|H_{n_m}|} = \liminf_{n\to+\infty} \frac{f(H_n)}{|H_n|} \quad \bigl(\text{by (2.6)}\bigr).$$

Thus, the sequence $\{\frac{f(H_n)}{|H_n|}\}_{n\in\mathbb{N}}$ converges (say $N(f)$ to be the value of the limit). Then for any Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ with $e_G \in F_1 \subseteq F_2 \subseteq \cdots$ of G, the sequence $\{\frac{f(F_n)}{|F_n|}\}_{n\in\mathbb{N}}$ converges to $N(f)$ (by (2.3)).

Finally, in order to complete the proof, we only need to check that, for any given Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ of G, if $\{F_n'\}_{n\in\mathbb{N}}$ is any sub-sequence of $\{F_n\}_{n\in\mathbb{N}}$ such that the sequence $\{\frac{f(F_n')}{|F_n'|}\}_{n\in\mathbb{N}}$ converges, then it converges to $N(f)$, which implies $\{\frac{f(F_n)}{|F_n|}\}_{n\in\mathbb{N}}$ converges to $N(f)$. Let $\{F_n'\}_{n\in\mathbb{N}}$ be such a sub-sequence. With no loss of generality we assume $\lim_{n\to+\infty} \frac{|F_n^*|}{|F_{n+1}'|} = 0$ (if necessary we take a sub-sequence of $\{F_n'\}_{n\in\mathbb{N}}$), where $F_n^* = \{e_G\} \cup \bigcup_{i=1}^n F_i'$ for each n. It is easy to check that $e_G \in F_1^* \subseteq F_2^* \subseteq \cdots$ forms a Følner sequence of G and so the sequence $\{\frac{f(F_n^*)}{|F_n^*|}\}_{n\in\mathbb{N}}$ converges to $N(f)$ from the above discussion. Note that, for each $n \in \mathbb{N}$,

$$\begin{aligned}\left|\frac{f(F_{n+1}^*)}{|F_{n+1}^*|}-\frac{f(F'_{n+1})}{|F'_{n+1}|}\right| &\leqslant \frac{f(F_n^*)}{|F_{n+1}^*|}+\left|\frac{f(F'_{n+1})}{|F_{n+1}^*|}-\frac{f(F'_{n+1})}{|F'_{n+1}|}\right| \\ &\leqslant M\left(\frac{|F_n^*|}{|F_{n+1}^*|}+\left|F'_{n+1}\right|\cdot\left|\frac{1}{|F_{n+1}^*|}-\frac{1}{|F'_{n+1}|}\right|\right) \\ &\leqslant M\left(\frac{|F_n^*|}{|F'_{n+1}|}+1-\frac{1}{1+\frac{|F_n^*|}{|F'_{n+1}|}}\right).\end{aligned}$$

By letting $n \to +\infty$ one has $\lim_{n\to+\infty}\frac{f(F_{n+1}^*)}{|F_{n+1}^*|}=\lim_{n\to+\infty}\frac{f(F'_{n+1})}{|F'_{n+1}|}=N(f)$, that is, the sequence $\{\frac{f(F'_n)}{|F'_n|}\}_{n\in\mathbb{N}}$ converges also to $N(f)$. $\square$

Remark 2.5. Recall that we say a set T tiles G if there is a subset C such that $\{Tc\colon c\in C\}$ is a partition of G. It's proved that if G admits a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ of tiling sets then for each f as in Lemma 2.4 the sequence $\{\frac{f(F_n)}{|F_n|}\}_{n\in\mathbb{N}}$ converges to $\inf_{n\in\mathbb{N}}\frac{f(F_n)}{|F_n|}$ and the value of the limit is independent of the choice of such a Følner sequence, which is stated as [44, Theorem 5.9].

The following useful lemma is an alternative version of (2.5) in the proof of Lemma 2.4.

Lemma 2.6. *Let $e_G\in F_1\subseteq F_2\subseteq\cdots$ be a Følner sequence of G. Then for any $\epsilon\in(0,\frac{1}{4})$ and $N\in\mathbb{N}$ there exist integers $n_1,\ldots,n_k$ with $N\leqslant n_1<\cdots<n_k$ such that if $f: F(G)\to\mathbb{R}$ is a m.n.i.s.a. function with $M=f(\{g\})$ for all $g\in G$ then*

$$\lim_{n\to+\infty}\frac{f(F_n)}{|F_n|}\leqslant M\epsilon+\frac{1}{1-\epsilon}\max_{1\leqslant i\leqslant k}\frac{f(F_{n_i})}{|F_{n_i}|}\leqslant M\epsilon\left(1+\frac{1}{1-\epsilon}\right)+\max_{1\leqslant i\leqslant k}\frac{f(F_{n_i})}{|F_{n_i}|}.$$

3. Entropy of an amenable group action

Let $\{F_n\}_{n\in\mathbb{N}}$ be a Følner sequence of G and fix it in the section. In this section, we aim to introduce the entropy theory of a G-system. By a *G-system* (X,G) we mean that X is a compact metric space and $\Gamma: G\times X\to X$, $(g,x)\mapsto gx$ is a continuous mapping satisfying

(1) $\Gamma(e_G,x)=x$ for each $x\in X$,
(2) $\Gamma(g_1,\Gamma(g_2,x))=\Gamma(g_1g_2,x)$ for each $g_1,g_2\in G$ and $x\in X$.

Moreover, if a non-empty compact subset $W\subseteq X$ is G-invariant (i.e. $gW=W$ for any $g\in G$) then (W,G) is called a *sub-G-system* of (X,G).

From now on, we let (X,G) always be a G-system if there is no any special statement. Denote by $\mathcal{B}_X$ the collection of all Borel subsets of X. A cover of X is a finite family of Borel subsets of X, whose union is X. A partition of X is a cover of X whose elements are pairwise disjoint. Denote by $\mathcal{C}_X$ (resp. $\mathcal{C}_X^o$) the set of all covers (resp. finite open covers) of X. Denote by $\mathcal{P}_X$ the set of all partitions of X. Given two covers $\mathcal{U},\mathcal{V}\in\mathcal{C}_X$, $\mathcal{U}$ is said to be *finer* than $\mathcal{V}$ (denoted by $\mathcal{U}\succcurlyeq\mathcal{V}$ or $\mathcal{V}\preccurlyeq\mathcal{U}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$; set $\mathcal{U}\vee\mathcal{V}=\{U\cap V\colon U\in\mathcal{U},\ V\in\mathcal{V}\}$.

3.1. Topological entropy

Let $\mathcal{U} \in \mathcal{C}_X$. Set $N(\mathcal{U})$ to be the minimum among the cardinalities of all sub-families of $\mathcal{U}$ covering X and denote by $\#(\mathcal{U})$ the cardinality of $\mathcal{U}$. Define $H(\mathcal{U}) = \log N(\mathcal{U})$. Clearly, if $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, then $H(\mathcal{U} \vee \mathcal{V}) \leqslant H(\mathcal{U}) + H(\mathcal{V})$ and $H(\mathcal{V}) \geqslant H(\mathcal{U})$ when $\mathcal{V} \succcurlyeq \mathcal{U}$.

Let $F \in F(G)$ and $\mathcal{U} \in \mathcal{C}_X$, set $\mathcal{U}_F = \bigvee_{g \in F} g^{-1}\mathcal{U}$ (letting $\mathcal{U}_\emptyset = \{X\}$). It is not hard to check that $F \in F(G) \mapsto H(\mathcal{U}_F)$ is a m.n.i.a.s. function, and so by Lemma 2.4, the quantity

$$h_{\mathrm{top}}(G, \mathcal{U}) \doteq \lim_{n \to +\infty} \frac{1}{|F_n|} H(\mathcal{U}_{F_n})$$

exists and $h_{\mathrm{top}}(G, \mathcal{U})$ is independent of the choice of $\{F_n\}_{n \in \mathbb{N}}$. $h_{\mathrm{top}}(G, \mathcal{U})$ is called the *topological entropy of* $\mathcal{U}$. It is clear that $h_{\mathrm{top}}(G, \mathcal{U}) \leqslant H(\mathcal{U})$. Note that if $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{C}_X$, then $h_{\mathrm{top}}(G, \mathcal{U}_1 \vee \mathcal{U}_2) \leqslant h_{\mathrm{top}}(G, \mathcal{U}_1) + h_{\mathrm{top}}(G, \mathcal{U}_2)$ and $h_{\mathrm{top}}(G, \mathcal{U}_2) \geqslant h_{\mathrm{top}}(G, \mathcal{U}_1)$ when $\mathcal{U}_2 \succcurlyeq \mathcal{U}_1$. The *topological entropy of* (X, G) is defined by

$$h_{\mathrm{top}}(G, X) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_{\mathrm{top}}(G, \mathcal{U}).$$

3.2. Measure-theoretic entropy

Denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X. For $\mu \in \mathcal{M}(X)$, denote by $\mathrm{supp}(\mu)$ the *support* of μ, i.e. the smallest closed subset $W \subseteq X$ such that $\mu(W) = 1$. $\mu \in \mathcal{M}(X)$ is called *G-invariant* if $g\mu = \mu$ for each $g \in G$; G-invariant $\nu \in \mathcal{M}(X)$ is called *ergodic* if $\nu(\bigcup_{g \in G} gA) = 0$ or 1 for any $A \in \mathcal{B}_X$. Denote by $\mathcal{M}(X, G)$ (resp. $\mathcal{M}^e(X, G)$) the set of all G-invariant (resp. ergodic G-invariant) elements in $\mathcal{M}(X)$. Note that the amenability of G ensures that $\emptyset \neq \mathcal{M}^e(X, G)$ and both $\mathcal{M}(X)$ and $\mathcal{M}(X, G)$ are convex compact metric spaces when they are endowed with the weak*-topology.

Given $\alpha \in \mathcal{P}_X$, $\mu \in \mathcal{M}(X)$ and a sub-σ-algebra $\mathcal{A} \subseteq \mathcal{B}_X$, define

$$H_\mu(\alpha|\mathcal{A}) = \sum_{A \in \alpha} \int_X -\mathbb{E}(1_A|\mathcal{A}) \log \mathbb{E}(1_A|\mathcal{A})\, d\mu,$$

where $\mathbb{E}(1_A|\mathcal{A})$ is the expectation of 1_A with respect to (w.r.t.) $\mathcal{A}$. One standard fact is that $H_\mu(\alpha|\mathcal{A})$ increases w.r.t. α and decreases w.r.t. $\mathcal{A}$. Set $\mathcal{N} = \{\emptyset, X\}$. Define

$$H_\mu(\alpha) = H_\mu(\alpha|\mathcal{N}) = \sum_{A \in \alpha} -\mu(A) \log \mu(A).$$

Let $\beta \in \mathcal{P}_X$. Note that β generates naturally a sub-σ-algebra $\mathcal{F}(\beta)$ of $\mathcal{B}_X$, define

$$H_\mu(\alpha|\beta) = H_\mu\big(\alpha|\mathcal{F}(\beta)\big) = H_\mu(\alpha \vee \beta) - H_\mu(\beta).$$

Now let $\mu \in \mathcal{M}(X, G)$, it is not hard to see that $F \in F(G) \mapsto H_\mu(\alpha_F)$ is a m.n.i.a.s. function. Thus by Lemma 2.4 we can define the *measure-theoretic μ-entropy of* α as

$$h_\mu(G, \alpha) = \lim_{n \to +\infty} \frac{1}{|F_n|} H_\mu(\alpha_{F_n}) \left(= \inf_{F \in F(G)} \frac{1}{|F|} H_\mu(\alpha_F) \right), \tag{3.1}$$

where the last identity is to be proved in Lemma 3.1(4). In particular, $h_\mu(G,\alpha)$ is independent of the choice of Følner sequence $\{F_n\}_{n\in\mathbb{N}}$. The *measure-theoretic μ-entropy of* (X,G) is defined by

$$h_\mu(G,X)=\sup_{\alpha\in\mathcal{P}_X} h_\mu(G,\alpha). \tag{3.2}$$

3.2.1. The proof of the second identity in (3.1)

Lemma 3.1. *Let $\alpha\in\mathcal{P}_X$, $\mu\in\mathcal{M}(X)$, $m\in\mathbb{N}$ and $E,F,B,E_1,\dots,E_k\in F(G)$. Then*

1. $H_\mu(\alpha_{E\cup F})+H_\mu(\alpha_{E\cap F})\leqslant H_\mu(\alpha_E)+H_\mu(\alpha_F)$.
2. *If $1_E(g)=\frac{1}{m}\sum_{i=1}^k 1_{E_i}(g)$ holds for each $g\in G$, then $H_\mu(\alpha_E)\leqslant\frac{1}{m}\sum_{i=1}^k H_\mu(\alpha_{E_i})$.*
3.
$$H_\mu(\alpha_F)\leqslant\sum_{g\in F}\frac{1}{|B|}H_\mu(\alpha_{Bg})+\left|F\setminus\left\{g\in G\colon B^{-1}g\subseteq F\right\}\right|\cdot\log\#(\alpha).$$
4. *If in addition $\mu\in\mathcal{M}(X,G)$, then $h_\mu(G,\alpha)=\inf_{B\in F(G)}\frac{H_\mu(\alpha_B)}{|B|}$.*

Proof. 1. The conclusion follows directly from the following simple observation:

$$\begin{aligned}H_\mu(\alpha_{E\cup F})+H_\mu(\alpha_{E\cap F})&=H_\mu(\alpha_E)+H_\mu(\alpha_F|\alpha_E)+H_\mu(\alpha_{E\cap F})\\&\leqslant H_\mu(\alpha_E)+H_\mu(\alpha_F|\alpha_{E\cap F})+H_\mu(\alpha_{E\cap F})\\&=H_\mu(\alpha_E)+H_\mu(\alpha_F).\end{aligned}$$

2. Clearly, $\bigcup_{i=1}^k E_i=E$. Say $\{A_1,\dots,A_n\}=\bigvee_{i=1}^k\{E_i,E\setminus E_i\}$ (neglecting all empty elements). Set $K_0=\emptyset$, $K_i=\bigcup_{j=1}^i A_j$, $i=1,\dots,n$. Then $\emptyset=K_0\subsetneq K_1\subsetneq\cdots\subsetneq K_n=E$. Moreover, if for some $i=1,\dots,n$ and $j=1,\dots,k$ with $E_j\cap(K_i\setminus K_{i-1})\neq\emptyset$ then $K_i\setminus K_{i-1}\subseteq E_j$ and so $K_i=K_{i-1}\cup(K_i\cap E_j)$, thus $H_\mu(\alpha_{K_i})+H_\mu(\alpha_{K_{i-1}\cap E_j})\leqslant H_\mu(\alpha_{K_{i-1}})+H_\mu(\alpha_{K_i\cap E_j})$ (using 1), i.e.

$$H_\mu(\alpha_{K_i})-H_\mu(\alpha_{K_{i-1}})\leqslant H_\mu(\alpha_{K_i\cap E_j})-H_\mu(\alpha_{K_{i-1}\cap E_j}). \tag{3.3}$$

Now for each $i=1,\dots,n$ we select $k_i\in K_i\setminus K_{i-1}$, one has

$$\begin{aligned}H_\mu(\alpha_E)&=\sum_{i=1}^n\left(\frac{1}{m}\sum_{j=1}^k 1_{E_j}(k_i)\right)\big(H_\mu(\alpha_{K_i})-H_\mu(\alpha_{K_{i-1}})\big)\quad\text{(by assumptions)}\\&=\frac{1}{m}\sum_{j=1}^k\sum_{1\leqslant i\leqslant n\colon k_i\in E_j}\big(H_\mu(\alpha_{K_i})-H_\mu(\alpha_{K_{i-1}})\big)\\&\leqslant\frac{1}{m}\sum_{j=1}^k\sum_{1\leqslant i\leqslant n\colon k_i\in E_j}\big(H_\mu(\alpha_{K_i\cap E_j})-H_\mu(\alpha_{K_{i-1}\cap E_j})\big)\quad\big(\text{using (3.3)}\big)\end{aligned}$$

$$\leqslant \frac{1}{m}\sum_{j=1}^{k}\sum_{i=1}^{n}\big(H_\mu(\alpha_{K_i\cap E_j}) - H_\mu(\alpha_{K_{i-1}\cap E_j})\big)$$
$$= \frac{1}{m}\sum_{j=1}^{k} H_\mu(\alpha_{E_j}).$$

3. Note that $1_{\{h\in BF:\ B^{-1}h\subseteq F\}}(f) = \frac{1}{|B|}\sum_{g\in F} 1_{\{h\in Bg:\ B^{-1}h\subseteq F\}}(f)$ for each $f\in G$. By 2, one has

$$H_\mu(\alpha_{\{h\in BF:\ B^{-1}h\subseteq F\}}) \leqslant \frac{1}{|B|}\sum_{g\in F} H_\mu(\alpha_{\{h\in Bg:\ B^{-1}h\subseteq F\}}) \leqslant \frac{1}{|B|}\sum_{g\in F} H_\mu(\alpha_{Bg}), \tag{3.4}$$

which implies

$$H_\mu(\alpha_F) \leqslant H_\mu(\alpha_{\{h\in BF:\ B^{-1}h\subseteq F\}}) + H_\mu(\alpha_{F\setminus\{h\in BF:\ B^{-1}h\subseteq F\}})$$
$$\leqslant \frac{1}{|B|}\sum_{g\in F} H_\mu(\alpha_{Bg}) + \big|F\setminus\big\{h\in BF:\ B^{-1}h\subseteq F\big\}\big|\cdot\log\#\alpha \quad \big(\text{using (3.4)}\big)$$
$$= \frac{1}{|B|}\sum_{g\in F} H_\mu(\alpha_{Bg}) + \big|F\setminus\big\{h\in G:\ B^{-1}h\subseteq F\big\}\big|\cdot\log\#\alpha.$$

4. If in addition μ is G-invariant, then by 3, for each $n\in\mathbb{N}$ we have

$$\frac{1}{|F_n|}H_\mu(\alpha_{F_n}) \leqslant \frac{1}{|F_n|}\sum_{g\in F_n}\frac{1}{|B|}H_\mu(\alpha_{Bg}) + \frac{1}{|F_n|}\big|F_n\setminus\big\{g\in G:\ B^{-1}g\subseteq F_n\big\}\big|\cdot\log\#\alpha$$
$$= \frac{1}{|F_n|}\sum_{g\in F_n}\frac{1}{|B|}H_\mu\big(g^{-1}(\alpha_B)\big) + \frac{1}{|F_n|}\big|F_n\setminus\big\{g\in G:\ B^{-1}g\subseteq F_n\big\}\big|\cdot\log\#\alpha$$
$$= \frac{1}{|B|}H_\mu(\alpha_B) + \frac{1}{|F_n|}\big|F_n\setminus\big\{g\in G:\ B^{-1}g\subseteq F_n\big\}\big|\cdot\log\#\alpha. \tag{3.5}$$

Set $B' = B^{-1}\cup\{e_G\}$. Note that for each $\delta>0$, F_n is (B',δ)-invariant if n is large enough and

$$F_n\setminus\big\{g\in G:\ B^{-1}g\subseteq F_n\big\} = F_n\cap B(G\setminus F_n) \subseteq \big(B'\big)^{-1}F_n\cap\big(B'\big)^{-1}(G\setminus F_n) = B\big(F_n,B'\big),$$

letting $n\to+\infty$ we get

$$\lim_{n\to+\infty}\frac{1}{|F_n|}\big|F_n\setminus\big\{g\in G:\ B^{-1}g\subseteq F_n\big\}\big| = \lim_{n\to+\infty}\frac{|B(F_n,B')|}{|F_n|} = 0, \tag{3.6}$$

and so $h_\mu(G,\alpha)\leqslant\frac{1}{|B|}H_\mu(\alpha_B)$ (using (3.5) and (3.6)). Since B is arbitrary, 4 is proved. □

Remark 3.2. In [32], Lemma 3.1(1) is called the strong sub-additivity of entropy. In his treatment of entropy for amenable group actions [32, Chapter 4], Ollagnier used the property rather heavily.

3.2.2. Measure-theoretic entropy for covers

Following Romagnoli's ideas [39], we define a new notion that extends definition (3.1) to covers. Let $\mu \in \mathcal{M}(X)$ and $\mathcal{A} \subseteq \mathcal{B}_X$ be a sub-σ-algebra. For $\mathcal{U} \in \mathcal{C}_X$, we define

$$H_\mu(\mathcal{U}|\mathcal{A}) = \inf_{\alpha \in \mathcal{P}_X:\, \alpha \succcurlyeq \mathcal{U}} H_\mu(\alpha|\mathcal{A}) \quad \text{and} \quad H_\mu(\mathcal{U}) = H_\mu(\mathcal{U}|\mathcal{N}).$$

Many properties of the function $H_\mu(\alpha)$ are extended to $H_\mu(\mathcal{U})$ from partitions to covers.

Lemma 3.3. *Let $\mu \in \mathcal{M}(X)$, $\mathcal{A} \subseteq \mathcal{B}_X$ be a sub-σ-algebra, $g \in G$ and $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{C}_X$. Then*

1. $0 \leqslant H_\mu(g^{-1}\mathcal{U}_1|g^{-1}\mathcal{A}) = H_{g\mu}(\mathcal{U}_1|\mathcal{A}) \leqslant H(\mathcal{U}_1)$.
2. *If $\mathcal{U}_1 \succcurlyeq \mathcal{U}_2$, then $H_\mu(\mathcal{U}_1|\mathcal{A}) \geqslant H_\mu(\mathcal{U}_2|\mathcal{A})$.*
3. $H_\mu(\mathcal{U}_1 \vee \mathcal{U}_2|\mathcal{A}) \leqslant H_\mu(\mathcal{U}_1|\mathcal{A}) + H_\mu(\mathcal{U}_2|\mathcal{A})$.

Using Lemma 3.3, one gets easily that if $\mu \in \mathcal{M}(X, G)$ then $F \in F(G) \mapsto H_\mu(\mathcal{U}_F)$ is a m.n.i.s.a. function. So we may define the *measure-theoretic μ^--entropy of $\mathcal{U}$* as

$$h_\mu^-(G, \mathcal{U}) = \lim_{n \to +\infty} \frac{1}{|F_n|} H_\mu(\mathcal{U}_{F_n})$$

and $h_\mu^-(G, \mathcal{U})$ is independent of the choice of Følner sequence $\{F_n\}_{n \in \mathbb{N}}$ (see Lemma 2.4). At the same time, we define the *measure-theoretic μ-entropy of $\mathcal{U}$* as

$$h_\mu(G, \mathcal{U}) = \inf_{\alpha \in \mathcal{P}_X:\, \alpha \succcurlyeq \mathcal{U}} h_\mu(G, \alpha).$$

We obtain directly the following easy facts.

Lemma 3.4. *Let $\mu \in \mathcal{M}(X, G)$ and $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$. Then*

1. *$h_\mu^-(G, \mathcal{U}) \leqslant h_\mu(G, \mathcal{U})$ and $h_\mu^-(G, \mathcal{U}) \leqslant h_{\mathrm{top}}(G, \mathcal{U})$.*
2. *$h_\mu(G, \mathcal{U} \vee \mathcal{V}) \leqslant h_\mu(G, \mathcal{U}) + h_\mu(G, \mathcal{V})$ and $h_\mu^-(G, \mathcal{U} \vee \mathcal{V}) \leqslant h_\mu^-(G, \mathcal{U}) + h_\mu^-(G, \mathcal{V})$.*
3. *If $\mathcal{U} \succcurlyeq \mathcal{V}$, then $h_\mu(G, \mathcal{U}) \geqslant h_\mu(G, \mathcal{V})$ and $h_\mu^-(G, \mathcal{U}) \geqslant h_\mu^-(G, \mathcal{V})$.*

3.2.3. An alternative formula for (3.2)

Let $\mu \in \mathcal{M}(X, G)$. Since $\mathcal{P}_X \subseteq \mathcal{C}_X$, we have

$$h_\mu(G, X) = \sup_{\mathcal{U} \in \mathcal{C}_X} h_\mu^-(G, \mathcal{U}) = \sup_{\mathcal{U} \in \mathcal{C}_X} h_\mu(G, \mathcal{U}). \tag{3.7}$$

In fact, the above extension of local measure-theoretic entropy from partitions to covers allows us to give another alternative formula for (3.2).

Theorem 3.5. *Let $\mu \in \mathcal{M}(X, G)$. Then*

$$h_\mu(G, X) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_\mu^-(G, \mathcal{U}) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_\mu(G, \mathcal{U}). \tag{3.8}$$

Proof. By (3.7), $h_\mu(G,X) \geqslant \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_\mu(G,\mathcal{U})$. For the other direction, let $\alpha = \{A_1,\dots,A_k\} \in \mathcal{P}_X$ and $\epsilon > 0$.

Claim. *There exists $\mathcal{U} \in \mathcal{C}_X^o$ such that $H_\mu(g^{-1}\alpha|\beta) \leqslant \epsilon$ if $g \in G$ and $\beta \in \mathcal{P}_X$ satisfy $\beta \succcurlyeq g^{-1}\mathcal{U}$.*

Proof. By [42, Lemma 4.15], there exists $\delta_1 = \delta_1(k,\epsilon) > 0$ such that if $\beta_i = \{B_1^i,\dots,B_k^i\} \in \mathcal{P}_X$, $i = 1,2$ satisfy $\sum_{i=1}^k \mu(B_i^1 \Delta B_i^2) < \delta_1$ then $H_\mu(\beta_1|\beta_2) \leqslant \epsilon$. Since μ is regular, we can take closed subsets $B_i \subseteq A_i$ with $\mu(A_i \setminus B_i) < \frac{\delta_1}{2k^2}$, $i = 1,\dots,k$. Let $B_0 = X \setminus \bigcup_{i=1}^k B_i$, $U_i = B_0 \cup B_i$, $i = 1,\dots,k$. Then $\mu(B_0) < \frac{\delta_1}{2k}$ and $\mathcal{U} = \{U_1,\dots,U_k\} \in \mathcal{C}_X^o$.

Let $g \in G$. If $\beta \in \mathcal{P}_X$ is finer than $g^{-1}\mathcal{U}$, we can find $\beta' = \{C_1,\dots,C_k\} \in \mathcal{P}_X$ satisfying $C_i \subseteq g^{-1}U_i$, $i = 1,\dots,k$ and $\beta \succcurlyeq \beta'$, and so $H_\mu(g^{-1}\alpha|\beta) \leqslant H_\mu(g^{-1}\alpha|\beta')$. For each $i = 1,\dots,k$, as $g^{-1}U_i \supseteq C_i \supseteq X \setminus \bigcup_{l\neq i} g^{-1}U_l = g^{-1}B_i$ and $g^{-1}A_i \supseteq g^{-1}B_i$, one has

$$\mu\bigl(C_i \Delta g^{-1}A_i\bigr) \leqslant \mu\bigl(g^{-1}A_i \setminus g^{-1}B_i\bigr) + \mu\bigl(g^{-1}B_0\bigr) = \mu(A_i \setminus B_i) + \mu(B_0) < \frac{\delta_1}{2k} + \frac{\delta_1}{2k^2} \leqslant \frac{\delta_1}{k}.$$

Thus $\sum_{i=1}^k \mu(C_i \Delta g^{-1}A_i) < \delta_1$. It follows that $H_\mu(g^{-1}\alpha|\beta') \leqslant \epsilon$ and hence $H_\mu(g^{-1}\alpha|\beta) \leqslant \epsilon$. □

Let $F \in F(G)$. If $\beta \in \mathcal{P}_X$ is finer than $\mathcal{U}_F$, then $\beta \succcurlyeq g^{-1}\mathcal{U}$ for each $g \in F$, and so using the above Claim one has

$$H_\mu(\alpha_F) \leqslant H_\mu(\beta) + H_\mu(\alpha_F|\beta) \leqslant H_\mu(\beta) + \sum_{g\in F} H_\mu\bigl(g^{-1}\alpha|\beta\bigr) \leqslant H_\mu(\beta) + |F|\epsilon.$$

Moreover, $H_\mu(\alpha_F) \leqslant H_\mu(\mathcal{U}_F) + |F|\epsilon$. Now letting F range over $\{F_n\}_{n\in\mathbb{N}}$ one has

$$\begin{aligned} h_\mu(G,\alpha) &= \lim_{n\to+\infty} \frac{1}{|F_n|} H_\mu(\alpha_{F_n}) \leqslant \limsup_{n\to+\infty} \frac{1}{|F_n|} H_\mu(\mathcal{U}_{F_n}) + \epsilon \\ &= h_\mu^-(G,\mathcal{U}) + \epsilon \leqslant \sup_{\mathcal{V}\in\mathcal{C}_X^o} h_\mu^-(G,\mathcal{V}) + \epsilon. \end{aligned}$$

Since α and ϵ are arbitrary, $h_\mu(G,X) \leqslant \sup_{\mathcal{V}\in\mathcal{C}_X^o} h_\mu^-(G,\mathcal{V})$ and so

$$h_\mu(G,X) \leqslant \sup_{\mathcal{V}\in\mathcal{C}_X^o} h_\mu^-(G,\mathcal{V}) \leqslant \sup_{\mathcal{V}\in\mathcal{C}_X^o} h_\mu(G,\mathcal{V}) \quad \bigl(\text{by Lemma 3.4(1)}\bigr). \qquad \square$$

3.2.4. U.s.c. of measure-theoretic entropy of open covers

A real-valued function f defined on a compact metric space Z is called *upper semi-continuous* (u.s.c.) if one of the following equivalent conditions holds:

(A1) $\limsup_{z'\to z} f(z') \leqslant f(z)$ for each $z \in Z$;
(A2) for each $r \in \mathbb{R}$, the set $\{z \in Z\colon f(z) \geqslant r\}$ is closed.

Using (A2), the infimum of any family of u.s.c. functions is again a u.s.c. one; both the sum and the supremum of finitely many u.s.c. functions are u.s.c. ones.

In this sub-section, we aim to prove that those two kinds entropy of open covers over $\mathcal{M}(X, G)$ are both u.s.c. First, we need

Lemma 3.6. *Let $\mathcal{U} = \{U_1, \ldots, U_M\} \in \mathcal{C}_X^o$ and $F \in F(G)$. Then the function $\psi : \mathcal{M}(X) \to \mathbb{R}_+$ with $\psi(\mu) = \inf_{\alpha \in \mathcal{P}_X:\ \alpha \succcurlyeq \mathcal{U}} H_\mu(\alpha_F)$ is u.s.c.*

Proof. Fix $\mu \in \mathcal{M}(X)$ and $\epsilon > 0$. It is sufficient to prove that

$$\limsup_{\mu' \to \mu:\ \mu' \in \mathcal{M}(X)} \psi\big(\mu'\big) \leqslant \psi(\mu) + \epsilon. \tag{3.9}$$

We choose $\alpha \in \mathcal{P}_X$ such that $\alpha \succcurlyeq \mathcal{U}$ and $H_\mu(\alpha_F) \leqslant \psi(\mu) + \frac{\epsilon}{2}$. With no loss of generality we assume $\alpha = \{A_1, \ldots, A_M\}$ with $A_i \subseteq U_i$, $1 \leqslant i \leqslant M$. Then there exists $\delta = \delta(M, F, \epsilon) > 0$ such that if $\beta^i = \{B_1^i, \ldots, B_M^i\} \in \mathcal{P}_X$, $i = 1, 2$ satisfy $\sum_{i=1}^M \sum_{g \in F} g\mu(B_i^1 \Delta B_i^2) < \delta$ then $H_\mu(\beta_F^1 | \beta_F^2) \leqslant \sum_{g \in F} H_{g\mu}(\beta^1 | \beta^2) < \frac{\epsilon}{2}$ [42, Lemma 4.15]. Set $\mathcal{U}_{\mu,F}^* = \{\beta \in \mathcal{P}_X:\ \beta \succcurlyeq \mathcal{U},\ \mu(\bigcup_{B \in \beta_F} \partial B) = 0\}$.

Claim. *There exists $\beta = \{B_1, \ldots, B_M\} \in \mathcal{U}_{\mu,F}^*$ such that $H_\mu(\beta_F | \alpha_F) < \frac{\epsilon}{2}$.*

Proof. Let $\delta_1 \in (0, \frac{\delta}{2M})$. By the regularity of μ, there exists compact $C_j \subseteq A_j$ such that

$$\sum_{g \in F} g\mu(A_j \setminus C_j) < \frac{\delta_1}{M}, \quad j = 1, \ldots, M. \tag{3.10}$$

For $j \in \{1, \ldots, M\}$, set $O_j = U_j \cap (X \setminus \bigcup_{i \neq j} C_i)$, then O_j is an open subset of X satisfying

$$A_j \subseteq O_j \subseteq U_j \quad \text{and}$$
$$\sum_{g \in F} g\mu(O_j \setminus A_j) \leqslant \sum_{i \neq j} \sum_{g \in F} g\mu(A_i \setminus C_i) < \delta_1, \quad \text{as } O_j \setminus A_j \subseteq \bigcup_{i \neq j} A_i \setminus C_i. \tag{3.11}$$

Note that if $x \in X$ then there exist at most countably many $\gamma > 0$ such that $\{y \in X:\ d(x, y) = \gamma\}$ has positive $g\mu$-measure for some $g \in F$. Moreover, as $O_1, \ldots, O_M$ are open subsets of X and $\bigcup_{i=1}^M O_i = X$, it is not hard to obtain Borel subsets $C_1^*, \ldots, C_M^*$ such that $C_i^* \subseteq O_i$, $1 \leqslant i \leqslant M$, $\bigcup_{i=1}^M C_i^* = X$ and $\sum_{i=1}^M \sum_{g \in F} g\mu(\partial C_i^*) = 0$.

Set $B_1 = C_1^*$, $B_j = C_j^* \setminus (\bigcup_{i=1}^{j-1} C_i^*)$, $2 \leqslant j \leqslant M$. Then $\beta \doteq \{B_1, \ldots, B_M\} \in \mathcal{P}_X$ and $\beta \succcurlyeq \mathcal{U}$. As $g^{-1}(\partial D) = \partial(g^{-1} D)$ for each $g \in F$ and $D \subseteq X$, by the construction of $C_1^*, \ldots, C_M^*$ it's easy to check that $\mu(\bigcup_{B \in \beta_F} \partial B) = 0$ and so $\beta \in \mathcal{U}_{\mu,F}^*$. Note that if $1 \leqslant j \neq i \leqslant M$ then $B_j \cap C_i \subseteq O_j \cap C_i = \emptyset$, which implies $C_i \subseteq B_i \subseteq O_i$ for all $1 \leqslant i \leqslant M$. By (3.10) and (3.11),

$$\sum_{i=1}^M \sum_{g \in F} g\mu(A_i \Delta B_i) \leqslant \sum_{i=1}^M \sum_{g \in F} \big(g\mu(A_i \setminus C_i) + g\mu(O_i \setminus A_i)\big) \leqslant \sum_{i=1}^M 2\delta_1 < \delta.$$

Thus $H_\mu(\beta_F | \alpha_F) < \frac{\epsilon}{2}$ (by the selection of δ). This finishes the proof of the claim. □

Now, note that $\beta \in \mathcal{P}_X$ satisfies $\beta \succcurlyeq \mathcal{U}$ and $\mu(\bigcup_{B\in\beta_F} \partial B) = 0$, one has

$$\limsup_{\mu'\to\mu:\ \mu'\in\mathcal{M}(X)} \psi(\mu') \leqslant \limsup_{\mu'\to\mu,\ \mu'\in\mathcal{M}(X)} H_{\mu'}(\beta_F) = H_\mu(\beta_F) \leqslant H_\mu(\alpha_F) + H_\mu(\beta_F|\alpha_F) \leqslant \psi(\mu) + \epsilon \quad \text{(by Claim)}.$$

This establishes (3.9) and so completes the proof of the lemma. □

Lemma 3.7. *Let $\mu \in \mathcal{M}(X,G)$, $M \in \mathbb{N}$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that if $\mathcal{U} = \{U_1, \ldots, U_M\} \in \mathcal{C}_X$, $\mathcal{V} = \{V_1, \ldots, V_M\} \in \mathcal{C}_X$ satisfy $\mu(\mathcal{U}\Delta\mathcal{V}) \doteq \sum_{m=1}^M \mu(U_m \Delta V_m) < \delta$ then $|h_\mu(G,\mathcal{U}) - h_\mu(G,\mathcal{V})| \leqslant \epsilon$.*

Proof. We follow the arguments in the proof of [21, Lemma 5]. Fix $M \in \mathbb{N}$ and $\epsilon > 0$. Then there exists $\delta' = \delta'(M, \varepsilon) > 0$ such that for M-sets partitions α, β of X, if $\mu(\alpha\Delta\beta) < \delta'$ then $H_\mu(\beta|\alpha) < \epsilon$ (see for example [42, Lemma 4.15]). Let $\mathcal{U} = \{U_1, \ldots, U_M\}$ and $\mathcal{V} = \{V_1, \ldots, V_M\}$ be any two M-sets covers of X with $\mu(\mathcal{U}\Delta\mathcal{V}) < \frac{\delta'}{M} = \delta$.

Claim. *For every finite partition $\alpha \succcurlyeq \mathcal{U}$ there exists a finite partition $\beta \succcurlyeq \mathcal{V}$ with $H_\mu(\beta|\alpha) < \epsilon$.*

Proof. Since $\alpha \succcurlyeq \mathcal{U}$, there exists a partition $\alpha' = \{A_1, \ldots, A_M\}$ with $A_i \subseteq U_i$, $i = 1, \ldots, M$ and $\alpha \succcurlyeq \alpha'$, where A_i may be empty. Let

$$B_1 = V_1 \setminus \bigcup_{k>1} (A_k \cap V_k),$$

$$B_i = V_i \setminus \left(\bigcup_{k>i} (A_k \cap V_k) \cup \bigcup_{j<i} B_j \right), \quad i \in \{2, \ldots, M\}.$$

Then $\beta = \{B_1, \ldots, B_M\} \in \mathcal{P}_X$ which satisfies $B_m \subseteq V_m$ and $A_m \cap V_m \subseteq B_m$ for $m \in \{1, \ldots, M\}$. It is clear that $A_m \setminus B_m \subseteq U_m \setminus V_m$ and

$$B_m \setminus A_m = \left(X \setminus \bigcup_{k\neq m} B_k \right) \setminus A_m = \bigcup_{j\neq m} A_j \setminus \bigcup_{k\neq m} B_k \subseteq \bigcup_{k\neq m} (A_k \setminus B_k) \subseteq \bigcup_{k\neq m} (U_k \setminus V_k).$$

Hence for every $m \in \{1, \ldots, M\}$, $A_m \Delta B_m \subseteq \bigcup_{k=1}^M (U_k \Delta V_k)$ and $\mu(\alpha'\Delta\beta) \leqslant M \cdot \mu(\mathcal{U}\Delta\mathcal{V}) < \delta'$. This implies that $H_\mu(\beta|\alpha') < \epsilon$. Moreover, $H_\mu(\beta|\alpha) \leqslant H_\mu(\beta|\alpha') < \epsilon$. □

Fix $n \in \mathbb{N}$. For any $\alpha \in \mathcal{P}_X$ with $\alpha \succcurlyeq \mathcal{U}_{F_n}$, we have $g\alpha \succcurlyeq \mathcal{U}$ for $g \in F_n$. By the above Claim, there exists $\beta_g \in \mathcal{P}_X$ such that $\beta_g \succcurlyeq \mathcal{V}$ and $H_\mu(\beta_g|g\alpha) < \epsilon$, i.e., $H_\mu(g^{-1}\beta_g|\alpha) < \epsilon$. Let $\beta = \bigvee_{g\in F_n} g^{-1}\beta_g$. Then $\beta \in \mathcal{P}_X$ with $\beta \succcurlyeq \mathcal{V}_{F_n}$. Now

$$H_\mu(\mathcal{V}_{F_n}) \leqslant H_\mu(\beta) \leqslant H_\mu(\beta \vee \alpha) = H_\mu(\alpha) + H_\mu(\beta|\alpha)$$
$$\leqslant H_\mu(\alpha) + \sum_{g \in F_n} H_\mu\big(g^{-1}\beta_g|\alpha\big) < H_\mu(\alpha) + n\epsilon.$$

Since this is true for any $\alpha \in \mathcal{P}_X$ with $\alpha \succcurlyeq \mathcal{U}_{F_n}$, we get $\frac{1}{|F_n|} H_\mu(\mathcal{V}_{F_n}) \leqslant \frac{1}{|F_n|} H_\mu(\mathcal{U}_{F_n}) + \epsilon$.

Exchanging the roles of $\mathcal{U}$ and $\mathcal{V}$ we get

$$\frac{1}{|F_n|} H_\mu(\mathcal{U}_{F_n}) \leqslant \frac{1}{|F_n|} H_\mu(\mathcal{V}_{F_n}) + \epsilon.$$

This shows $\frac{1}{|F_n|}|H_\mu(\mathcal{U}_{F_n}) - H_\mu(\mathcal{V}_{F_n})| \leqslant \epsilon$. Letting $n \to +\infty$, one has $|h_\mu(G,\mathcal{U}) - h_\mu(G, \mathcal{V})| \leqslant \epsilon$. □

Now we can prove the u.s.c. property of those two kinds of measure-theoretic entropy of open covers over $\mathcal{M}(X, G)$.

Proposition 3.8. *Let $\mathcal{U} \in \mathcal{C}_X^o$. Then $h_{\{\cdot\}}(G,\mathcal{U}) : \mathcal{M}(X, G) \to \mathbb{R}_+$ is u.s.c. on $\mathcal{M}(X, G)$.*

Proof. Note that

$$h_\mu(G,\mathcal{U}) = \inf_{\alpha \in \mathcal{P}_X:\, \alpha \succcurlyeq \mathcal{U}} h_\mu(G,\alpha) = \inf_{\alpha \in \mathcal{P}_X:\, \alpha \succcurlyeq \mathcal{U}} \inf_{B \in F(G)} \frac{H_\mu(\alpha_B)}{|B|} \quad \big(\text{by Lemma 3.1(4)}\big)$$
$$= \inf_{B \in F(G)} \inf_{\alpha \in \mathcal{P}_X:\, \alpha \succcurlyeq \mathcal{U}} \frac{H_\mu(\alpha_B)}{|B|}.$$

Since $\mu \mapsto \inf_{\alpha \in \mathcal{P}_X:\, \alpha \succcurlyeq \mathcal{U}} H_\mu(\alpha_B)$ is u.s.c. (see Lemma 3.6) and the infimum of any family of u.s.c. functions is again u.s.c., one has $h_{\{\cdot\}}(G,\mathcal{U}) : \mathcal{M}(X, G) \to \mathbb{R}_+$ is u.s.c. on $\mathcal{M}(X, G)$. □

Proposition 3.9. *Let $\mathcal{U} \in \mathcal{C}_X^o$. Then $h^-_{\{\cdot\}}(G,\mathcal{U}) : \mathcal{M}(X, G) \to \mathbb{R}_+$ is u.s.c. on $\mathcal{M}(X, G)$.*

Proof. With no loss of generality we assume $e_G \in F_1 \subseteq F_2 \subseteq \cdots$ by Lemma 2.4. Let $\mu \in \mathcal{M}(X, G)$ and $\epsilon \in (0, \frac{1}{4})$. Then there exists $N \in \mathbb{N}$ with

$$\sup_{n \geqslant N} \frac{H_\mu(\mathcal{U}_{F_n})}{|F_n|} \leqslant h^-_\mu(G,\mathcal{U}) + \frac{\epsilon}{2}. \tag{3.12}$$

By Lemma 2.6, there exist integers $n_1, \dots, n_k$ with $N \leqslant n_1 < \cdots < n_k$ such that

$$h^-_\nu(G,\mathcal{U}) = \lim_{n \to +\infty} \frac{H_\nu(\mathcal{U}_{F_n})}{|F_n|}$$
$$\leqslant \max_{1 \leqslant i \leqslant k} \frac{H_\nu(\mathcal{U}_{F_{n_i}})}{|F_{n_i}|} + \frac{\epsilon H_\nu(\mathcal{U})}{2\log(N(\mathcal{U}) + 1)}$$
$$\leqslant \max_{1 \leqslant i \leqslant k} \frac{H_\nu(\mathcal{U}_{F_{n_i}})}{|F_{n_i}|} + \frac{\epsilon}{2} \quad \text{for each } \nu \in \mathcal{M}(X, G). \tag{3.13}$$

Then we have

$$\begin{aligned}
\limsup_{\mu'\to\mu,\, \mu'\in\mathcal{M}(X,G)} h^-_{\mu'}(G,\mathcal{U}) &\leqslant \frac{\epsilon}{2} + \limsup_{\mu'\to\mu,\, \mu'\in\mathcal{M}(X,G)} \max_{1\leqslant i\leqslant k} \frac{H_{\mu'}(\mathcal{U}_{F_{n_i}})}{|F_{n_i}|} \quad \bigl(\text{using (3.13)}\bigr) \\
&= \frac{\epsilon}{2} + \max_{1\leqslant i\leqslant k} \limsup_{\mu'\to\mu,\, \mu'\in\mathcal{M}(X,G)} \frac{H_{\mu'}(\mathcal{U}_{F_{n_i}})}{|F_{n_i}|} \\
&\leqslant \frac{\epsilon}{2} + \max_{1\leqslant i\leqslant k} \frac{H_{\mu}(\mathcal{U}_{F_{n_i}})}{|F_{n_i}|} \quad \text{(Lemma 3.6)} \\
&\leqslant \frac{\epsilon}{2} + \sup_{n\geqslant N} \frac{H_{\mu}(\mathcal{U}_{F_{n}})}{|F_{n}|} \leqslant h^-_{\mu}(G,\mathcal{U}) + \epsilon \quad \bigl(\text{using (3.12)}\bigr). \qquad (3.14)
\end{aligned}$$

Thus, we claim the conclusion from the arbitrariness of $\mu \in \mathcal{M}(X,G)$ and $\epsilon \in (0, \frac{1}{4})$ in (3.14). □

3.2.5. Affinity of measure-theoretic entropy of covers

Let $\mu = a\nu + (1-a)\eta$, where $\nu, \eta \in \mathcal{M}(X,G)$ and $0 < a < 1$. Using the concavity of $\phi(t) = -t\log t$ on $[0,1]$ with $\phi(0) = 0$ (fix it in the remainder of the paper), one has if $\beta \in \mathcal{P}_X$ and $F \in F(G)$ then $0 \leqslant H_\mu(\beta_F) - aH_\nu(\beta_F) - (1-a)H_\eta(\beta_F) \leqslant \phi(a) + \phi(1-a)$ (see for example the proof of [42, Theorem 8.1]) and so

$$h_\mu(G,\beta) = ah_\nu(G,\beta) + (1-a)h_\eta(G,\beta), \qquad (3.15)$$

i.e. the function $h_{\{\cdot\}}(G,\beta) : \mathcal{M}(X,G) \to \mathbb{R}_+$ is affine. In the following, we shall show the affinity of $h_{\{\cdot\}}(G,\mathcal{U})$ and $h^-_{\{\cdot\}}(G,\mathcal{U})$ on $\mathcal{M}(X,G)$ for each $\mathcal{U} \in \mathcal{C}_X$.

Let $\mu \in \mathcal{M}(X,G)$ and $\mathcal{B}_X^\mu$ be the completion of $\mathcal{B}_X$ under μ. Then $(X, \mathcal{B}_X^\mu, \mu, G)$ is a Lebesgue system. If $\{\alpha_i\}_{i\in I}$ is a countable family in $\mathcal{P}_X$, the partition $\alpha = \bigvee_{i\in I}\alpha_i \doteq \{\bigcap_{i\in I} A_i\colon A_i \in \alpha_i,\ i \in I\}$ is called a *measurable partition*. Note that the sets $A \in \mathcal{B}_X^\mu$, which are unions of atoms of α, form a sub-σ-algebra of $\mathcal{B}_X^\mu$, which is denoted by $\widehat{\alpha}$ or α if there is no ambiguity. In fact, every sub-σ-algebra of $\mathcal{B}_X^\mu$ coincides with a σ-algebra constructed in this way in the sense of mod μ [37]. We consider the sub-σ-algebra $I_\mu = \{A \in \mathcal{B}_X^\mu\colon \mu(gA\Delta A) = 0$ for each $g \in G\}$. Clearly, I_μ is G-invariant since G is countable. Let α be the measurable partition of X with $\widehat{\alpha} = I_\mu$ (mod μ). With no loss of generality we may require that α is G-invariant, i.e. $g\alpha = \alpha$ for any $g \in G$. Let $\mu = \int_X \mu_x\, d\mu(x)$ be the disintegration of μ over I_μ, where $\mu_x \in \mathcal{M}^e(X,G)$ and $\mu_x(\alpha(x)) = 1$ for μ-a.e. $x \in X$, here $\alpha(x)$ denotes the atom of α containing x. This disintegration is known as the *ergodic decomposition* of μ (see for example [17, Theorem 3.22]).

The disintegration is characterized by properties (3.16) and (3.17) below:

$$\begin{gathered}
\text{for every } f \in L^1(X,\mathcal{B}_X,\mu),\ f \in L^1(X,\mathcal{B}_X,\mu_x) \text{ for } \mu\text{-a.e. } x \in X, \\
\text{and the map } x \mapsto \int_X f(y)\, d\mu_x(y) \text{ is in } L^1(X, I_\mu, \mu); \qquad (3.16)
\end{gathered}$$

$$\text{for every } f \in L^1(X,\mathcal{B}_X,\mu),\ \mathbb{E}_\mu(f|I_\mu)(x) = \int_X f\, d\mu_x \text{ for } \mu\text{-a.e. } x \in X. \qquad (3.17)$$

Then for $f \in L^1(X, \mathcal{B}_X, \mu)$,

$$\int_X \left(\int_X f \, d\mu_x \right) d\mu(x) = \int_X f \, d\mu. \tag{3.18}$$

Note that the disintegration exists uniquely in the sense that if $\mu = \int_X \mu_x \, d\mu(x)$ and $\mu = \int_X \mu'_x \, d\mu(x)$ are both the disintegrations of μ over I_μ, then $\mu_x = \mu'_x$ for μ-a.e. $x \in X$. Moreover, there exists a G-invariant subset $X_0 \subseteq X$ such that $\mu(X_0) = 1$ and if for $x \in X_0$ we define $\Gamma_x = \{y \in X_0\colon \mu_x = \mu_y\}$ then $\Gamma_x = \alpha(x) \cap X_0$ and Γ_x is G-invariant.

Lemma 3.10. *Let $\mu \in \mathcal{M}(X, G)$ with $\mu = \int_X \mu_x \, d\mu(x)$ the ergodic decomposition of μ and $\mathcal{V} \in \mathcal{C}_X$. Then $H_\mu(\mathcal{V}|I_\mu) = \int_X H_{\mu_x}(\mathcal{V}) \, d\mu(x)$.*

Proof. Let $\mathcal{V} = \{V_1, \ldots, V_n\}$. For any $s = (s(1), \ldots, s(n)) \in \{0, 1\}^n$, set $V_s = \bigcap_{i=1}^n V_i(s(i))$, where $V_i(0) = V_i$ and $V_i(1) = X \setminus V_i$. Let $\alpha = \{V_s\colon s \in \{0, 1\}^n\}$. Then α is the Borel partition generated by $\mathcal{V}$ and put $P(\mathcal{V}) = \{\beta \in \mathcal{P}_X\colon \alpha \succcurlyeq \beta \succcurlyeq \mathcal{V}\}$, which is a finite family of partitions. It is well known that, for each $\theta \in \mathcal{M}(X)$ one has

$$H_\theta(\mathcal{V}) = \min_{\beta \in P(\mathcal{V})} H_\theta(\beta), \tag{3.19}$$

see for example the proof of [39, Proposition 6]. Now denote $P(\mathcal{V}) = \{\beta_1, \ldots, \beta_l\}$ and put

$$A_i = \Big\{ x \in X\colon H_{\mu_x}(\beta_i) = \min_{\beta \in P(\mathcal{V})} H_{\mu_x}(\beta) \Big\}, \quad i \in \{1, \ldots, l\}.$$

Let $B_1 = A_1$, $B_2 = A_2 \setminus B_1$, $\ldots$, $B_l = A_l \setminus \bigcup_{i=1}^{l-1} B_i$ and $B_0 = X \setminus \bigcup_{i=1}^{l} A_i$. By (3.19), $\mu(B_0) = 0$.

Set $\beta^* = \{B_0 \cap \beta_1\} \cup \{B_i \cap \beta_i\colon i = 1, \ldots, l\} \in \mathcal{P}_X \pmod{\mu}$. Then $\beta^* \succcurlyeq \mathcal{V}$. Clearly, for $i \in \{1, \ldots, l\}$ and μ-a.e. $x \in B_i$, $H_{\mu_x}(\beta^*) = H_{\mu_x}(\beta_i) = \min_{\beta \in P(\mathcal{V})} H_{\mu_x}(\beta) = H_{\mu_x}(\mathcal{V})$ where the last equality follows from (3.19). Combining this fact with $\mu(B_0) = 0$ one gets $H_{\mu_x}(\beta^*) = H_{\mu_x}(\mathcal{V})$ for μ-a.e. $x \in X$. This implies

$$\begin{aligned} H_\mu(\mathcal{V}|I_\mu) &\leqslant H_\mu\big(\beta^*|I_\mu\big) = \int_X H_{\mu_x}\big(\beta^*\big) \, d\mu(x) \quad \big(\text{using (3.17)}\big) \\ &= \int_X H_{\mu_x}(\mathcal{V}) \, d\mu(x) \leqslant \inf_{\beta \in \mathcal{P}_X\colon \beta \succcurlyeq \mathcal{V}} \int_X H_{\mu_x}(\beta) \, d\mu(x) \\ &= \inf_{\beta \in \mathcal{P}_X\colon \beta \succcurlyeq \mathcal{V}} H_\mu(\beta|I_\mu) \\ &= H_\mu(\mathcal{V}|I_\mu). \end{aligned}$$

Thus $H_\mu(\mathcal{V}|I_\mu) = \int_X H_{\mu_x}(\mathcal{V}) \, d\mu(x)$. This finishes the proof. □

Then we have

Proposition 3.11. *Let* $\mathcal{U} \in \mathcal{C}_X$ *and* $\mu \in \mathcal{M}(X, G)$. *If* $e_G \in F_1 \subseteq F_2 \subseteq \cdots$ *then*

$$h_\mu^-(G, \mathcal{U}) = \lim_{n \to +\infty} \frac{1}{|F_n|} H_\mu(\mathcal{U}_{F_n} | I_\mu).$$

Proof. It is easy to check that $F \in F(G) \mapsto H_\mu(\mathcal{U}_F | I_\mu)$ is a m.n.i.s.a. function, and so the sequence $\{\frac{1}{|F_n|} H_\mu(\mathcal{U}_{F_n} | I_\mu)\}_{n \in \mathbb{N}}$ converges, say it converges to $f_\mathcal{U}$ (see Lemma 2.4). Clearly $h_\mu^-(G, \mathcal{U}) \geqslant f_\mathcal{U}$.

Now we aim to prove $h_\mu^-(G, \mathcal{U}) \leqslant f_\mathcal{U}$. Let $\epsilon \in (0, \frac{1}{4})$ and $N \in \mathbb{N}$. By Proposition 2.3 there exist integers $n_1, \ldots, n_k$ with $N \leqslant n_1 < \cdots < n_k$ such that if n is large enough then $F_{n_1}, \ldots, F_{n_k}$ ϵ-quasi-tile the set F_n with tiling centers $C_1^n, \ldots, C_k^n$ and so

$$F_n \supseteq \bigcup_{i=1}^k F_{n_i} C_i^n \quad \text{and} \quad \left| \bigcup_{i=1}^k F_{n_i} C_i^n \right| \geqslant \max \left\{ (1-\epsilon)|F_n|, (1-\epsilon) \sum_{i=1}^k |C_i^n| \cdot |F_{n_i}| \right\}. \tag{3.20}$$

Thus if $\alpha \in \mathcal{P}_X$ and n is large enough then

$$\begin{aligned} H_\mu(\mathcal{U}_{F_n} | \alpha_{F_n}) &\leqslant H_\mu(\mathcal{U}_{F_n \setminus \bigcup_{i=1}^k F_{n_i} C_i^n} | \alpha_{F_n}) + \sum_{i=1}^k H_\mu(\mathcal{U}_{F_{n_i} C_i^n} | \alpha_{F_n}) \\ &\leqslant \left| F_n \setminus \bigcup_{i=1}^k F_{n_i} C_i^n \right| \cdot \log N(\mathcal{U}) + \sum_{i=1}^k H_\mu(\mathcal{U}_{F_{n_i} C_i^n} | \alpha_{F_{n_i} C_i^n}). \end{aligned} \tag{3.21}$$

This implies

$$\begin{aligned} &\limsup_{n \to +\infty} \frac{1}{|F_n|} H_\mu(\mathcal{U}_{F_n} | \alpha_{F_n}) \\ &\quad \leqslant \epsilon \log N(\mathcal{U}) + \limsup_{n \to +\infty} \frac{1}{|F_n|} \sum_{i=1}^k \sum_{g \in C_i^n} H_\mu(\mathcal{U}_{F_{n_i} g} | \alpha_{F_{n_i} g}) \quad \big(\text{using (3.20) and (3.21)}\big) \\ &\quad \leqslant \epsilon \log N(\mathcal{U}) + \limsup_{n \to +\infty} \frac{\sum_{i=1}^k |F_{n_i}||C_i^n|}{|F_n|} \max_{1 \leqslant i \leqslant k} \frac{1}{|F_{n_i}|} H_\mu(\mathcal{U}_{F_{n_i}} | \alpha_{F_{n_i}}) \\ &\quad \leqslant \epsilon \log N(\mathcal{U}) + \frac{1}{1-\epsilon} \max_{1 \leqslant i \leqslant k} \frac{1}{|F_{n_i}|} H_\mu(\mathcal{U}_{F_{n_i}} | \alpha_{F_{n_i}}) \quad \big(\text{using (3.20)}\big). \end{aligned}$$

Thus

$$\begin{aligned} h_\mu^-(G, \mathcal{U}) &\leqslant \limsup_{n \to +\infty} \frac{1}{|F_n|} \big(H_\mu(\mathcal{U}_{F_n} | \alpha_{F_n}) + H_\mu(\alpha_{F_n}) \big) \\ &\leqslant h_\mu(G, \alpha) + \epsilon \log N(\mathcal{U}) + \frac{1}{1-\epsilon} \max_{1 \leqslant i \leqslant k} \frac{1}{|F_{n_i}|} H_\mu(\mathcal{U}_{F_{n_i}} | \alpha_{F_{n_i}}). \end{aligned} \tag{3.22}$$

Note that if $\alpha \in \mathcal{P}_X$ satisfies $\alpha \subseteq I_\mu$ then $h_\mu(G,\alpha) = 0$. In particular, in (3.22) we replace α by a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ in $\mathcal{P}_X$ with $\alpha_i \nearrow I_\mu$, then

$$h_\mu^-(G,\mathcal{U}) \leqslant \epsilon \log N(\mathcal{U}) + \frac{1}{1-\epsilon} \sup_{m\geqslant N} \frac{1}{|F_m|} H_\mu(\mathcal{U}_{F_m}|I_\mu).$$

Since the above inequality is true for any $\epsilon \in (0, \frac{1}{4})$ and $N \in \mathbb{N}$, one has $h_\mu^-(G,\mathcal{U}) \leqslant f_{\mathcal{U}}$. □

Lemma 3.12. *Let $\mathcal{U} \in \mathcal{C}_X$ and $\mu \in \mathcal{M}(X,G)$ with $\mu = \int_X \mu_x \, d\mu(x)$ the ergodic decomposition of μ. Then*

$$h_\mu^-(G,\mathcal{U}) = \int_X h_{\mu_x}^-(G,\mathcal{U})\, d\mu(x) \quad \text{and} \quad h_\mu(G,\mathcal{U}) = \int_X h_{\mu_x}(G,\mathcal{U})\, d\mu(x).$$

Proof. With no loss of generality we assume $e_G \in F_1 \subseteq F_2 \subseteq \cdots$ (by Lemma 2.4). Then we have

$$\begin{aligned} h_\mu^-(G,\mathcal{U}) &= \lim_{n\to+\infty} \frac{1}{|F_n|} H_\mu(\mathcal{U}_{F_n}|I_\mu) \quad \text{(by Proposition 3.11)} \\ &= \lim_{n\to+\infty} \frac{1}{|F_n|} \int_X H_{\mu_x}(\mathcal{U}_{F_n})\, d\mu(x) \quad \text{(by Lemma 3.10)} \\ &= \int_X \lim_{n\to+\infty} \frac{1}{|F_n|} H_{\mu_x}(\mathcal{U}_{F_n})\, d\mu(x) \quad \text{(by Dominant Convergence Theorem).} \end{aligned}$$

That is, $h_\mu^-(G,\mathcal{U}) = \int_X h_{\mu_x}^-(G,\mathcal{U})\, d\mu(x)$. In particular, if $\alpha \in \mathcal{P}_X$ then

$$h_\mu(G,\alpha) = \int_X h_{\mu_x}(G,\alpha)\, d\mu(x). \tag{3.23}$$

Next we follow the idea of the proof of [23, Lemma 4.8] to prove $h_\mu(G,\mathcal{U}) = \int_X h_{\mu_x}(G,\mathcal{U})\, d\mu(x)$. Let $\mathcal{U} = \{U_1, \ldots, U_M\}$ and put $\mathcal{U}^* = \{\alpha = \{A_1, \ldots, A_M\} \in \mathcal{P}_X\colon A_m \subseteq U_m,\ m = 1, \ldots, M\}$. As $(X, \mathcal{B}_X)$ is a standard Borel space, there exists a countable algebra $\mathcal{A} \subseteq \mathcal{B}_X$ such that $\mathcal{B}_X$ is the σ-algebra generated by $\mathcal{A}$. It is well known that if $\nu \in \mathcal{M}(X)$ then

$$\mathcal{B}_X = \bigl\{A \in \mathcal{B}_X\colon \forall \epsilon > 0,\ \exists B \in \mathcal{A} \text{ such that } \nu(A \Delta B) < \epsilon\bigr\}. \tag{3.24}$$

Take $\mathcal{C}$ to be the countable algebra generated by $\mathcal{A}$ and $\{U_1, \ldots, U_M\}$, then $\mathcal{F} = \{P \in \mathcal{U}^*\colon P \subseteq \mathcal{C}\}$ is a countable set and for each $\alpha \in \mathcal{U}^*$, $\epsilon > 0$ and $\nu \in \mathcal{M}(X)$ there exists $\beta \in \mathcal{F}$ such that $\nu(\alpha \Delta \beta) < \epsilon$ by (3.24), i.e. $\mathcal{F}$ is $L^1(X, \mathcal{B}_X, \nu)$-dense in $\mathcal{U}^*$. In particular, say $\mathcal{F} = \{\alpha_k\colon k \in \mathbb{N}\}$ (denote $\alpha_k = \{A_1^k, \ldots, A_M^k\}$ for each $k \in \mathbb{N}$), if $\nu \in \mathcal{M}(X,G)$ then

$$h_\nu(G,\mathcal{U}) = \inf_{\alpha\in\mathcal{U}^*} h_\nu(G,\alpha) = \inf_{k\in\mathbb{N}} h_\nu(G,\alpha_k). \tag{3.25}$$

First, for one inequality one has

$$\begin{aligned} h_\mu(G,\mathcal{U}) &= \inf_{k\in\mathbb{N}} h_\mu(G,\alpha_k) = \inf_{k\in\mathbb{N}} \int_X h_{\mu_x}(G,\alpha_k)\,d\mu(x) \quad \big(\text{by (3.23)}\big) \\ &\geqslant \int_X \inf_{k\in\mathbb{N}} h_{\mu_x}(G,\alpha_k)\,d\mu(x) = \int_X h_{\mu_x}(G,\mathcal{U})\,d\mu(x) \quad \big(\text{by (3.25)}\big). \end{aligned}$$

For the other inequality, let $\epsilon > 0$. For each $n \in \mathbb{N}$ define $B_n^\epsilon = \{x \in X\colon h_{\mu_x}(G,\alpha_n) < h_{\mu_x}(G,\mathcal{U}) + \epsilon\}$. Then B_n^ϵ is G-invariant and $\mu(\bigcup_{n\in\mathbb{N}} B_n^\epsilon) = 1$ by (3.25), and so there exists a measurable partition $\{X_n\colon n \in \mathbb{N}\}$ of X with $X_n \in I_\mu$ and $\mu(X_n) > 0$, and a sequence $\{\alpha_{k_n}\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ and μ-a.e. $x \in X_n$ one has $h_{\mu_x}(G,\alpha_{k_n}) < h_{\mu_x}(G,\mathcal{U}) + \epsilon$. For every $n \in \mathbb{N}$ we define $\mu_n(\cdot) = \frac{1}{\mu(X_n)} \int_X \mu_x(\cdot \cap X_n)\,d\mu(x) \in \mathcal{M}(X,G)$. We deduce

$$\begin{aligned} h_{\mu_n}(G,\alpha_{k_n}) &= \frac{1}{\mu(X_n)} \int_{X_n} h_{\mu_x}(G,\alpha_{k_n})\,d\mu(x) \quad \big(\text{by (3.23)}\big) \\ &\leqslant \frac{1}{\mu(X_n)} \int_{X_n} h_{\mu_x}(G,\mathcal{U})\,d\mu(x) + \epsilon. \end{aligned}$$

Note that, by definition, for every $n \in \mathbb{N}$, $\mu_n(X_n) = 1$ and $\mu_n(X_k) = 0$ if $k \neq n$. For $m \in \{1,\ldots,M\}$ define $A_m = \bigcup_{n\in\mathbb{N}} (X_n \cap A_m^{k_n})$, then $\alpha = \{A_1,\ldots,A_M\} \in \mathcal{U}^*$. We get,

$$\begin{aligned} h_\mu(G,\mathcal{U}) \leqslant h_\mu(G,\alpha) &= \sum_{n\in\mathbb{N}} \mu(X_n) h_{\mu_n}(G,\alpha) \quad \big(\text{by (3.23)}\big) \\ &= \sum_{n\in\mathbb{N}} \mu(X_n) h_{\mu_n}(G,\alpha_{k_n}) \leqslant \int_X h_{\mu_x}(G,\mathcal{U})\,d\mu(x) + \epsilon. \end{aligned}$$

Letting $\epsilon \to 0+$ we conclude $h_\mu(G,\mathcal{U}) \leqslant \int_X h_{\mu_x}(G,\mathcal{U})\,d\mu(x)$ and the desired equality holds. □

Denote by $C(X;\mathbb{R})$ the Banach space of the set of all continuous real-valued functions on X equipped with the maximal norm $\|\cdot\|$. Note that the Banach space $C(X;\mathbb{R})$ is separable, let $\{f_n\colon n \in \mathbb{N}\} \subseteq C(X;\mathbb{R}) \setminus \{0\}$ be a countable dense subset, where 0 is the constant 0 function on X, then a compatible metric on $\mathcal{M}(X)$ is given by

$$\rho(\mu,\nu) = \sum_{n\in\mathbb{N}} \frac{|\int_X f_n\,d\mu - \int_X f_n\,d\nu|}{2^n \|f_n\|}, \quad \text{for each } \mu,\nu \in \mathcal{M}(X).$$

Let $\mu \in \mathcal{M}(X,G)$ with $\mu = \int_X \mu_x\,d\mu(x)$ the ergodic decomposition of μ. Then there exists a G-invariant subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that the map $\Phi : X_0 \to \mathcal{M}^e(X,G)$ with $\Phi(x) = \mu_x$ is well defined. We extend Φ to the whole space X such that $\Phi(x) \in \mathcal{M}^e(X,G)$ for each $x \in X$. For any $g_i \in C(X;\mathbb{R})$, $\mu_i \in \mathcal{M}(X,G)$ and $\epsilon_i > 0$, $i = 1,\ldots,k$, note that for any $f \in C(X;\mathbb{R})$, the function $x \in X_0 \mapsto \int_X f\,d\mu_x$ is an element of $L^1(X, I_\mu, \mu)$, we have $\Phi^{-1}(\bigcap_{i=1}^k\{\nu \in \mathcal{M}(X,G)\colon |\int_X g_i\,d\nu - \int_X g_i\,d\mu_i| < \epsilon_i\}) \in I_\mu$. Since all the sets having the form of $\bigcap_{i=1}^k\{\nu \in \mathcal{M}(X,G)\colon |\int_X g_i\,d\nu - \int_X g_i\,d\mu_i| < \epsilon_i\}$ form a topological base of $\mathcal{M}(X,G)$, the map $\Phi : (X, I_\mu) \to (\mathcal{M}(X,G), \mathcal{B}_{\mathcal{M}(X,G)})$ is measurable, i.e. $\Phi^{-1}(A) \in I_\mu$ for any $A \in \mathcal{B}_{\mathcal{M}(X,G)}$. Now we define $m \in \mathcal{M}(\mathcal{M}(X,G))$ as following: $m(A) = \mu(\Phi^{-1}(A))$ for any $A \in \mathcal{B}_{\mathcal{M}(X,G)}$. Then if g is a bounded Borel function on $\mathcal{M}(X,G)$ then $g \circ \Phi \in L^1(X, I_\mu, \mu)$ and

$$\int_X g \circ \Phi(x)\,d\mu(x) = \int_{\mathcal{M}(X,G)} g(\theta)\,dm(\theta). \tag{3.26}$$

Now if $f \in C(X;\mathbb{R})$, let $L_f : \theta \in \mathcal{M}(X,G) \mapsto \int_X f\,d\theta$, then L_f is a continuous function, and so

$$\int_X \left(\int_X f\,d\mu_x\right) d\mu(x) = \int_X L_f \circ \Phi(x)\,d\mu(x) = \int_{\mathcal{M}(X,G)} L_f(\theta)\,dm(\theta) \quad \bigl(\text{using (3.26)}\bigr),$$

moreover,

$$\int_X f(x)\,d\mu(x) = \int_{\mathcal{M}(X,G)} \left(\int_X f(x)\,d\theta(x)\right) dm(\theta) \quad \text{for any } f \in C(X;\mathbb{R}) \quad \bigl(\text{using (3.18)}\bigr). \tag{3.27}$$

Note that $m(\mathcal{M}^e(X,G)) \geqslant \mu(X_0) = 1$, m can be viewed as a Borel probability measure on $\mathcal{M}^e(X,G)$. So (3.27) can also be written as

$$\int_X f(x)\,d\mu(x) = \int_{\mathcal{M}^e(X,G)} \left(\int_X f(x)\,d\theta(x)\right) dm(\theta) \quad \text{for any } f \in C(X;\mathbb{R}), \tag{3.28}$$

which is denoted by $\mu = \int_{\mathcal{M}^e(X,G)} \theta\,dm(\theta)$ (also called the *ergodic decomposition of* μ). Finally, it is not hard to check that if m' is another Borel probability measure on $\mathcal{M}(X,G)$ satisfying $m'(\mathcal{M}^e(X,G)) = 1$ and (3.28) then $m' = m$. That is, for any given $\mu \in \mathcal{M}(X,G)$ there exists uniquely a Borel probability measure m' on $\mathcal{M}(X,G)$ with $m'(\mathcal{M}^e(X,G)) = 1$ satisfying (3.28).

Theorem 3.13. *Let $\mathcal{U} \in \mathcal{C}_X$. Then the function $\eta \in \mathcal{M}(X,G) \mapsto h_\eta(G,\mathcal{U})$ and the function $\eta \in \mathcal{M}(X,G) \mapsto h_\eta^-(G,\mathcal{U})$ are both bounded affine Borel functions on $\mathcal{M}(X,G)$. Moreover, if we let $\mu \in \mathcal{M}(X,G)$ with $\mu = \int_{\mathcal{M}^e(X,G)} \theta\,dm(\theta)$ the ergodic decomposition of μ, then*

$$h_\mu(G,\mathcal{U}) = \int_{\mathcal{M}^e(X,G)} h_\theta(G,\mathcal{U})\,dm(\theta) \quad \textit{and}$$

$$h_\mu^-(G,\mathcal{U}) = \int_{\mathcal{M}^e(X,G)} h_\theta^-(G,\mathcal{U})\,dm(\theta). \tag{3.29}$$

Proof. First we aim to establish (3.29). Similar to the proof of Lemma 3.12, there exists $\{\alpha_k\}_{k\in\mathbb{N}} \subseteq \mathcal{P}_X$ such that $\alpha_k \succcurlyeq \mathcal{U}$ for each $k \in \mathbb{N}$ and $H_\eta(\mathcal{U}) = \inf_{k\in\mathbb{N}} H_\eta(\alpha_k)$, $h_\eta(G,\mathcal{U}) = \inf_{k\in\mathbb{N}} h_\eta(G,\alpha_k)$ for each $\eta \in \mathcal{M}(X,G)$. Note that, for any $A \in \mathcal{B}_X$, the function $\eta \in \mathcal{M}(X,G) \mapsto \eta(A)$ is Borel measurable and hence if $\alpha \in \mathcal{P}_X$ then the function $\eta \in \mathcal{M}(X,G) \mapsto H_\eta(\alpha)$ and the function $\eta \in \mathcal{M}(X,G) \mapsto h_\eta(G,\alpha)$ are both bounded Borel functions. Moreover, the function $\eta \in \mathcal{M}(X,G) \mapsto H_\eta(\mathcal{U})$ is a bounded Borel function. Thus, the function $\eta \in \mathcal{M}(X,G) \mapsto h_\eta(G,\mathcal{U})$ and the function $\eta \in \mathcal{M}(X,G) \mapsto h_\eta^-(G,\mathcal{U})$ are both bounded Borel functions. In particular, (3.29) follows directly from Lemma 3.12 and (3.26).

Now let $\mu_1,\mu_2 \in \mathcal{M}(X,G)$ and $\lambda \in (0,1)$. For $i = 1,2$, let $\mu_i = \int_{\mathcal{M}^e(X,T)} \theta\, dm_i(\theta)$ be the ergodic decomposition of μ_i, where m_i is a Borel probability measure on $\mathcal{M}^e(X,G)$. Consider $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ and $m = \lambda m_1 + (1-\lambda)m_2$. Then m is a Borel probability measure on $\mathcal{M}^e(X,G)$ and $\mu = \int_{\mathcal{M}^e(X,G)} \theta\, dm(\theta)$ is the ergodic decomposition of μ. By (3.29), we have

$$\begin{aligned} h_\mu(G,\mathcal{U}) &= \int\limits_{\mathcal{M}^e(X,G)} h_\theta(G,\mathcal{U})\, dm(\theta) \\ &= \lambda \int\limits_{\mathcal{M}^e(X,G)} h_\theta(G,\mathcal{U})\, dm_1(\theta) + (1-\lambda) \int\limits_{\mathcal{M}^e(X,G)} h_\theta(G,\mathcal{U})\, dm_2(\theta) \\ &= \lambda h_{\mu_1}(G,\mathcal{U}) + (1-\lambda) h_{\mu_2}(G,\mathcal{U}). \end{aligned}$$

This shows the affinity of $h_{\{\cdot\}}(G,\mathcal{U})$. We can obtain similarly the affinity of $h_{\{\cdot\}}^-(G,\mathcal{U})$. □

4. The equivalence of measure-theoretic entropy of covers

In the section, following arguments of Danilenko in [7], we will develop an orbital approach to local entropy theory for actions of an amenable group. Then combining it with the equivalence of measure-theoretic entropy of covers in the case of $G = \mathbb{Z}$, we will establish the equivalence of those two kinds of measure-theoretic entropy of covers for a general G.

4.1. Backgrounds of orbital theory

Let $(X,\mathcal{B}_X,\mu)$ be a Lebesgue space. Denote by $Aut(X,\mu)$ the group of all μ-measure preserving invertible transformations of $(X,\mathcal{B}_X,\mu)$, which is endowed with the weak topology, i.e. the weakest topology which makes continuous the following unitary representation: $Aut(X,\mu) \ni \gamma \mapsto U_\gamma \in \mathcal{U}(L^2(X,\mu))$ with $U_\gamma f = f \circ \gamma^{-1}$, where the unitary group $\mathcal{U}(L^2(X,\mu))$ is the set of all unitary operators on $L^2(X,\mu)$ endowed with the strong operator topology. Let a Borel subset $\mathcal{R} \subseteq X \times X$ be an equivalence relation on X. For each $x \in X$, we denote $\mathcal{R}(x) = \{y \in X\colon (x,y) \in \mathcal{R}\}$. Following [14], $\mathcal{R}$ is called *measure preserving* if it is generated by some countable sub-group $G \subseteq Aut(X,\mu)$, in general, this generating sub-group is highly non-unique; $\mathcal{R}$ is *ergodic* if A belongs to the trivial sub-σ-algebra of $\mathcal{B}_X$ when $A \in \mathcal{B}_X$ is $\mathcal{R}$-invariant (i.e. $A = \bigcup_{x\in A} \mathcal{R}(x)$); $\mathcal{R}$ is *discrete* if $\#\mathcal{R}(x) \leqslant \#\mathbb{Z}$ for μ-a.e. $x \in X$; $\mathcal{R}$ is of *type I* if $\#\mathcal{R}(x) < +\infty$ for μ-a.e. $x \in X$, equivalently, there is a subset $B \in \mathcal{B}_X$ with $\#(B \cap \mathcal{R}(x)) = 1$ for μ-a.e. $x \in X$, such a B is called an *$\mathcal{R}$-fundamental domain*; $\mathcal{R}$ is *countable* if $\#\mathcal{R}(x) = +\infty$ for μ-a.e. $x \in X$, observe that if $\mathcal{R}$ is measure preserving then it is countable iff it is *conservative*, i.e.

$\mathcal{R} \cap (B \times B) \setminus \Delta_2(X) \neq \emptyset$ for each $B \in \mathcal{B}_X$ satisfying $\mu(B) > 0$, where $\Delta_2(X) = \{(x, x): x \in X\}$; $\mathcal{R}$ is *hyperfinite* if there exists a sequence $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \cdots$ of type I sub-relations of $\mathcal{R}$ such that $\bigcup_{n \in \mathbb{N}} \mathcal{R}_n = \mathcal{R}$, the sequence $\{\mathcal{R}_n\}_{n \in \mathbb{N}}$ is called a *filtration* of $\mathcal{R}$. Note that a measure preserving discrete equivalence relation is hyperfinite iff it is generated by a single invertible transformation [12], the orbit equivalence relation of a measure preserving action of a countable discrete amenable group is hyperfinite [6,46], any two ergodic hyperfinite measure preserving countable equivalence relations are isomorphic in the natural sense (i.e. there exists an isomorphism between the Lebesgue spaces which intertwines the corresponding equivalent classes) [12]. Everywhere below $\mathcal{R}$ is a measure preserving discrete equivalence relation on a Lebesgue space $(X, \mathcal{B}_X, \mu)$.

The *full group* $[\mathcal{R}]$ of $\mathcal{R}$ and its *normalizer* $N[\mathcal{R}]$ are defined, respectively, by

$$[\mathcal{R}] = \big\{\gamma \in Aut(X, \mu): (x, \gamma x) \in \mathcal{R} \text{ for } \mu\text{-a.e. } x \in X\big\},$$
$$N[\mathcal{R}] = \big\{\theta \in Aut(X, \mu): \theta\mathcal{R}(x) = \mathcal{R}(\theta x) \text{ for } \mu\text{-a.e. } x \in X\big\}.$$

Let A be a Polish group. A Borel map $\phi : \mathcal{R} \to A$ is called a *cocycle* if

$$\phi(x, z) = \phi(x, y)\phi(y, z) \quad \text{for all } (x, y), (y, z) \in \mathcal{R}.$$

Letting $\theta \in N[\mathcal{R}]$, we define a cocycle $\phi \circ \theta$ by setting $\phi \circ \theta(x, y) = \phi(\theta x, \theta y)$ for all $(x, y) \in \mathcal{R}$. Let $(Y, \mathcal{B}_Y, \nu)$ be another Lebesgue space and A be embedded continuously into $Aut(Y, \nu)$. For each cocycle $\phi : \mathcal{R} \to A$, we associate a measure preserving discrete equivalence relation $\mathcal{R}(\phi)$ on $(X \times Y, \mathcal{B}_X \times \mathcal{B}_Y, \mu \times \nu)$ by setting $(x, y) \sim_{\mathcal{R}(\phi)} (x', y')$ if $(x, x') \in \mathcal{R}$ and $y' = \phi(x', x)y$. Then a one-to-one group homomorphism $[\mathcal{R}] \ni \gamma \mapsto \gamma_\phi \in [\mathcal{R}_\phi]$ is well defined via the formula

$$\gamma_\phi(x, y) = \big(\gamma x, \phi(\gamma x, x)y\big) \quad \text{for each } (x, y) \in X \times Y.$$

The transformation γ_ϕ is called the *ϕ-skew product extension* of γ, and the equivalence relation $\mathcal{R}(\phi)$ is called the *ϕ-skew product extension* of $\mathcal{R}$.

4.2. Local entropy for a cocycle of a discrete measure preserving equivalence relation

Denote by $I(\mathcal{R})$ the set of all type I sub-relations of $\mathcal{R}$. Let $\epsilon > 0$ and $\mathcal{T}, \mathcal{S} \in I(\mathcal{R})$. We write $\mathcal{T} \subseteq_\epsilon \mathcal{S}$ if there is $A \in \mathcal{B}_X$ such that $\mu(A) > 1 - \epsilon$ and

$$\#\big\{y \in \mathcal{S}(x): \mathcal{T}(y) \subseteq \mathcal{S}(x)\big\} > (1 - \epsilon)\#\mathcal{S}(x) \quad \text{for each } x \in A.$$

Replacing, if necessary, A by $\bigcup_{x \in A} \mathcal{S}(x)$ we may (and so shall) assume that A is $\mathcal{S}$-invariant. Let $A_0 = \{x \in A: \mathcal{T}(x) \subseteq \mathcal{S}(x)\}$. The following two lemmas are proved in [7].

Lemma 4.1. *A_0 is $\mathcal{T}$-invariant, $\mu(A_0) > 1 - 2\epsilon$ and $\#(\mathcal{S}(x) \cap A_0) > (1 - \epsilon)\#\mathcal{S}(x)$ for each $x \in A_0$.*

Lemma 4.2. *Let $\epsilon > 0$ and $\mathcal{R}$ be hyperfinite with $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ a filtration of $\mathcal{R}$.*

1. *If $\Gamma \subseteq [\mathcal{R}]$ is a countable subset satisfying $\#(\Gamma x) < +\infty$ for μ-a.e. $x \in X$ then for each sufficiently large n there is an $\mathcal{R}_n$-invariant subset A_n such that $\mu(A_n) > 1-\epsilon$ and*

$$\#\big\{y \in \mathcal{R}_n(x)\colon\ \Gamma y \subseteq \mathcal{R}_n(x)\big\} > (1-\epsilon)\#\mathcal{R}_n(x) \quad \text{for each } x \in A_n.$$

2. *If $\mathcal{S} \in I(\mathcal{R})$ then $\mathcal{S} \subseteq_\epsilon \mathcal{R}_n$ if n is large enough.*

Let $(Y, \mathcal{B}_Y, \nu)$ be a Lebesgue space and $\phi : \mathcal{R} \to Aut(Y, \nu)$ a cocycle. For $\mathcal{U} \in \mathcal{C}_{X\times Y}$, we consider $\mathcal{U}$ as a measurable field $\{\mathcal{U}_x\}_{x\in X} \subseteq \mathcal{C}_Y$, where $\{x\} \times \mathcal{U}_x = \mathcal{U} \cap (\{x\} \times Y)$.

Definition 4.3. For $\mathcal{U} \in \mathcal{C}_{X\times Y}$, we define

$$h_\nu^-(\mathcal{S}, \phi, \mathcal{U}) = \int_X \frac{1}{\#\mathcal{S}(x)} H_\nu\Big(\bigvee_{y\in\mathcal{S}(x)} \phi(x, y)\mathcal{U}_y\Big)\, d\mu(x) \quad \text{and}$$

$$h_\nu(\mathcal{S}, \phi, \mathcal{U}) = \inf_{\alpha\in\mathcal{P}_{X\times Y}\colon \alpha\succcurlyeq\mathcal{U}} h_\nu^-(\mathcal{S}, \phi, \alpha).$$

Then we define the *ν^--entropy* $h_\nu^-(\phi, \mathcal{U})$ and the *ν-entropy* $h_\nu(\phi, \mathcal{U})$ of $(\phi, \mathcal{U})$, respectively, by

$$h_\nu^-(\phi, \mathcal{U}) = \inf_{\mathcal{S}\in I(\mathcal{R})} h_\nu^-(\mathcal{S}, \phi, \mathcal{U}) \quad \text{and} \quad h_\nu(\phi, \mathcal{U}) = \inf_{\mathcal{S}\in I(\mathcal{R})} h_\nu(\mathcal{S}, \phi, \mathcal{U}).$$

It is clear that if $\beta \in \mathcal{P}_{X\times Y}$ and $\mathcal{U} \in \mathcal{C}_{X\times Y}$ then $h_\nu(\mathcal{S}, \phi, \beta) = h_\nu^-(\mathcal{S}, \phi, \beta)$, $h_\nu(\phi, \beta) = h_\nu^-(\phi, \beta)$ and $h_\nu(\phi, \mathcal{U}) = \inf_{\alpha\in\mathcal{P}_{X\times Y}\colon \alpha\succcurlyeq\mathcal{U}} h_\nu^-(\phi, \alpha)$. Moreover, if $\mathcal{U}, \mathcal{V} \in \mathcal{C}_{X\times Y}$ satisfy $\mathcal{U} \succcurlyeq \mathcal{V}$ then $h_\nu(\mathcal{S}, \phi, \mathcal{U}) \geqslant h_\nu(\mathcal{S}, \phi, \mathcal{V})$ and $h_\nu^-(\mathcal{S}, \phi, \mathcal{U}) \geqslant h_\nu^-(\mathcal{S}, \phi, \mathcal{V})$. It's not hard to obtain

Proposition 4.4. *Let $(Z, \mathcal{B}_Z, \kappa)$ be a Lebesgue space, $\mathcal{S} \in I(\mathcal{R})$, $\beta : \mathcal{S} \to Aut(Z, \kappa)$ a cocycle and $\sigma : Z \times X \to X \times Z$, $(z, x) \mapsto (x, z)$ the flip.*

1. *Let $\alpha' : \sigma^{-1}\mathcal{S}(\beta)\sigma \to Aut(Y, \nu)$ and $\alpha : \mathcal{S} \to Aut(Y, \nu)$ be cocycles satisfying $\alpha'((z, x), (z', x')) = \alpha(x, x')$ when $((z, x), (z', x')) \in \sigma^{-1}\mathcal{S}(\beta)\sigma$. Then $h_\nu^-(\sigma^{-1}\mathcal{S}(\beta)\sigma, \alpha', Z \times \mathcal{U}) = h_\nu^-(\mathcal{S}, \alpha, \mathcal{U})$ for any $\mathcal{U} \in \mathcal{C}_{X\times Y}$.*
2. *Let $\alpha'' : \mathcal{S}(\beta) \to Aut(Y, \nu)$ and $\alpha : \mathcal{S} \to Aut(Y, \nu)$ be cocycles satisfying $\alpha''((x, z), (x'', z'')) = \alpha(x, x'')$ when $((x, z), (x'', z'')) \in \mathcal{S}(\beta)$. Then if $\mathcal{U}'' \in \mathcal{C}_{X\times Z\times Y}$ and $\mathcal{U} \in \mathcal{C}_{X\times Y}$ satisfies $\mathcal{U}''_{(x,z)} = \mathcal{U}_x$ for each $(x, z) \in X \times Z$ then $h_\nu^-(\mathcal{S}(\beta), \alpha'', \mathcal{U}'') = h_\nu^-(\mathcal{S}, \alpha, \mathcal{U})$.*

Proof. As the proof is similar, we only present the proof for 1. Let $\mathcal{U} \in \mathcal{C}_{X\times Y}$. Then

$$\begin{aligned}
&h_\nu^-\big(\sigma^{-1}\mathcal{S}(\beta)\sigma, \alpha', Z \times \mathcal{U}\big) \\
&\quad = \int_{Z\times X} \frac{1}{\#\sigma^{-1}\mathcal{S}(\beta)\sigma(z, x)} H_\nu\Big(\bigvee_{(z',x')\in\sigma^{-1}\mathcal{S}(\beta)\sigma(z,x)} \alpha'\big((z, x), (z', x')\big)(Z \times \mathcal{U})_{(z',x')}\Big)\, d\kappa \\
&\qquad \times \mu(z, x)
\end{aligned}$$

$$= \int\limits_{Z\times X} \frac{1}{\#\mathcal{S}(x)} H_\nu\Biggl(\bigvee_{(x',z')\in\mathcal{S}(\beta)(x,z)} \alpha\bigl(x,x'\bigr)\mathcal{U}_{x'}\Biggr)\,d\kappa\times\mu(z,x)$$

$$= \int\limits_X \frac{1}{\#\mathcal{S}(x)} H_\nu\Biggl(\bigvee_{x'\in\mathcal{S}(x)} \alpha\bigl(x,x'\bigr)\mathcal{U}_{x'}\Biggr)\,d\mu(x) = h_\nu^-(\mathcal{S},\alpha,\mathcal{U}). \qquad \square$$

Proposition 4.5. *Let $\epsilon > 0$ and $\mathcal{T},\mathcal{S}\in I(\mathcal{R})$. If $\mathcal{T}\subseteq_\epsilon \mathcal{S}$ then*

$$h_\nu^-(\mathcal{S},\phi,\mathcal{U}) \leqslant h_\nu^-(\mathcal{T},\phi,\mathcal{U}) + 3\epsilon\log N(\mathcal{U}) \quad \textit{and} \quad h_\nu(\mathcal{S},\phi,\mathcal{U}) \leqslant h_\nu(\mathcal{T},\phi,\mathcal{U}) + 3\epsilon\log N(\mathcal{U}).$$

In particular, if $\mathcal{T}\subseteq\mathcal{S}$ then $h_\nu^-(\mathcal{S},\phi,\mathcal{U}) \leqslant h_\nu^-(\mathcal{T},\phi,\mathcal{U})$ and $h_\nu(\mathcal{S},\phi,\mathcal{U}) \leqslant h_\nu(\mathcal{T},\phi,\mathcal{U})$.

Proof. The proof follows the arguments of the proof of [7, Proposition 2.6]. Let $A_0 = \{x\in A\colon \mathcal{T}(x)\subseteq\mathcal{S}(x)\}$. Then $\mu(A_0) > 1-2\epsilon$ by Lemma 4.1. We define the maps $f,g : A_0\to\mathbb{R}$ by

$$f(x) = \frac{1}{\#(\mathcal{S}(x)\cap A_0)} H_\nu\Biggl(\bigvee_{y\in\mathcal{S}(x)\cap A_0} \phi(x,y)\mathcal{U}_y\Biggr) \quad \text{and}$$

$$g(x) = \frac{1}{\#\mathcal{T}(x)} H_\nu\Biggl(\bigvee_{y\in\mathcal{T}(x)} \phi(x,y)\mathcal{U}_y\Biggr).$$

Since A_0 is $\mathcal{T}$-invariant, for each $x\in A_0$ there are $x_1,\ldots,x_k\in X$ such that $\mathcal{S}(x)\cap A_0 = \bigsqcup_{i=1}^k \mathcal{T}(x_i)$, here the sign $\bigsqcup$ denotes the union of disjoint subsets. It follows that

$$\begin{aligned} f(x) &\leqslant \frac{1}{\#(\mathcal{S}(x)\cap A_0)} \sum_{i=1}^k H_\nu\Biggl(\phi(x,x_i)\bigvee_{y\in\mathcal{T}(x_i)} \phi(x_i,y)\mathcal{U}_y\Biggr) \\ &= \frac{1}{\#(\mathcal{S}(x)\cap A_0)} \sum_{i=1}^k \#\mathcal{T}(x_i)\cdot g(x_i) \\ &= \frac{1}{\#(\mathcal{S}(x)\cap A_0)} \sum_{i=1}^k \sum_{y\in\mathcal{T}(x_i)} g(y) \\ &= \frac{1}{\#(\mathcal{S}(x)\cap A_0)} \sum_{z\in\mathcal{S}(x)\cap A_0} g(z) = \mathbb{E}\bigl(g|\mathcal{S}\cap(A_0\times A_0)\bigr)(x), \end{aligned}$$

where $\mathbb{E}(g|\mathcal{S}\cap(A_0\times A_0))$ denotes the conditional expectation of g w.r.t. $\mathcal{S}_{A_0}$, the σ-algebra of all measurable $\mathcal{S}\cap(A_0\times A_0)$-invariant subsets. Hence

$$\begin{aligned} h_\nu^-(\mathcal{S},\phi,\mathcal{U}) &= \int\limits_X \frac{1}{\#\mathcal{S}(x)} H_\nu\Biggl(\bigvee_{y\in\mathcal{S}(x)} \phi(x,y)\mathcal{U}_y\Biggr)\,d\mu(x) \\ &\leqslant \int\limits_{A_0} \frac{1}{\#\mathcal{S}(x)} H_\nu\Biggl(\bigvee_{y\in\mathcal{S}(x)} \phi(x,y)\mathcal{U}_y\Biggr)\,d\mu(x) + \int\limits_{X\setminus A_0} \frac{1}{\#\mathcal{S}(x)} \sum_{y\in\mathcal{S}(x)} H_\nu(\mathcal{U}_y)\,d\mu(x) \end{aligned}$$

$$\leqslant \int\limits_{A_0} \left(f(x) + \frac{1}{\#\mathcal{S}(x)} H_\nu \left(\bigvee_{y \in \mathcal{S}(x) \setminus A_0} \phi(x,y)\mathcal{U}_y \right) \right) d\mu(x) + \int\limits_{X \setminus A_0} \log N(\mathcal{U})\, d\mu(x)$$

$$\leqslant \int\limits_{A_0} \left(\mathbb{E}\big(g|\mathcal{S} \cap (A_0 \times A_0)\big)(x) + \frac{\#(\mathcal{S}(x) \setminus A_0)}{\#\mathcal{S}(x)} \log N(\mathcal{U}) \right) d\mu(x) + 2\epsilon \log N(\mathcal{U})$$

$$\leqslant \int\limits_{A_0} \mathbb{E}\big(g|\mathcal{S} \cap (A_0 \times A_0)\big)(x)\, d\mu(x) + 3\epsilon \log N(\mathcal{U})$$

$$= \int\limits_{A_0} g(x)\, d\mu(x) + 3\epsilon \log N(\mathcal{U}) \leqslant h_\nu^-(\mathcal{T}, \phi, \mathcal{U}) + 3\epsilon \log N(\mathcal{U}).$$

By the same reason, one has $h_\nu(\mathcal{S}, \phi, \alpha) \leqslant h_\nu(\mathcal{T}, \phi, \alpha) + 3\epsilon \log N(\alpha)$ for any $\alpha \in \mathcal{P}_{X \times Y}$. Thus

$$\begin{aligned} h_\nu(\mathcal{S}, \phi, \mathcal{U}) &= \inf\big\{h_\nu(\mathcal{S}, \phi, \alpha)\colon \alpha \in \mathcal{P}_{X\times Y} \text{ with } \alpha \succcurlyeq \mathcal{U},\ N(\alpha) \leqslant N(\mathcal{U})\big\} \\ &\leqslant \inf\big\{h_\nu(\mathcal{T}, \phi, \alpha) + 3\epsilon \log N(\alpha)\colon \alpha \in \mathcal{P}_{X\times Y} \text{ with } \alpha \succcurlyeq \mathcal{U},\ N(\alpha) \leqslant N(\mathcal{U})\big\} \\ &\leqslant \inf\big\{h_\nu(\mathcal{T}, \phi, \alpha) + 3\epsilon \log N(\mathcal{U})\colon \alpha \in \mathcal{P}_{X\times Y} \text{ with } \alpha \succcurlyeq \mathcal{U},\ N(\alpha) \leqslant N(\mathcal{U})\big\} \\ &= h_\nu(\mathcal{T}, \phi, \mathcal{U}) + 3\epsilon \log N(\mathcal{U}). \end{aligned}$$

Now if $\mathcal{T} \subseteq \mathcal{S}$ then $\mathcal{T} \subseteq_\epsilon \mathcal{S}$ for each $\epsilon > 0$, so letting $\epsilon \to 0+$ we have $h_\nu^-(\mathcal{S}, \phi, \mathcal{U}) \leqslant h_\nu^-(\mathcal{T}, \phi, \mathcal{U})$ and $h_\nu(\mathcal{S}, \phi, \mathcal{U}) \leqslant h_\nu(\mathcal{T}, \phi, \mathcal{U})$. This finishes the proof. □

As a direct application of Lemma 4.2(2) and Proposition 4.5 we have

Corollary 4.6. *Let $\mathcal{R}$ be hyperfinite with $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ a filtration of $\mathcal{R}$. Then*

$$\lim_{n\to+\infty} h_\nu(\mathcal{R}_n, \phi, \mathcal{U}) = h_\nu(\phi, \mathcal{U}) \quad \textit{and} \quad \lim_{n\to+\infty} h_\nu^-(\mathcal{R}_n, \phi, \mathcal{U}) = h_\nu^-(\phi, \mathcal{U}).$$

4.3. Two kinds virtual entropy of covers

Everywhere below, $\mathcal{R}$ is generated by a free G-measure preserving system $(X, \mathcal{B}_X, \mu, G)$. Then $\mathcal{R}$ is hyperfinite and conservative. Let $\mathcal{S} \in I(\mathcal{R})$ with $B \subseteq X$ an $\mathcal{S}$-fundamental domain. Then there is a measurable map $B \ni x \mapsto G_x \in F(G)$ with $G_x x = \mathcal{S}(x)$ and hence $X = \bigsqcup_{x\in B} G_x x$. Noting that $F(G)$ is a countable set, we obtain that $X = \bigsqcup_i \bigsqcup_{g\in G_i} gB_i$ for a countable family $\{G_i\}_i \subseteq F(G)$ and a decomposition $B = \bigsqcup_i B_i$ with $G_i x = \mathcal{S}(x)$ for each $x \in B_i$. We shall write it as $\mathcal{S} \sim (B_i, G_i)$. Then

$$\begin{aligned} h_\nu^-(\mathcal{S}, \phi, \mathcal{U}) &= \sum_i \sum_{g\in G_i} \int\limits_{gB_i} \frac{1}{\#\mathcal{S}(x)} H_\nu\left(\bigvee_{y\in\mathcal{S}(x)} \phi(x,y)\mathcal{U}_y \right) d\mu(x) \\ &= \sum_i \sum_{g\in G_i} \int\limits_{B_i} \frac{1}{|G_i|} H_\nu\left(\bigvee_{g'\in G_i} \phi\big(gx, g'x\big)\mathcal{U}_{g'x} \right) d\mu(x) \end{aligned}$$

$$= \sum_i \sum_{g \in G_i} \int_{B_i} \frac{1}{|G_i|} H_\nu\left(\bigvee_{g' \in G_i} \phi\left(x, g'x\right) \mathcal{U}_{g'x} \right) d\mu(x)$$

$$= \sum_i \int_{B_i} H_\nu\left(\bigvee_{g \in G_i} \phi(x, gx) \mathcal{U}_{gx} \right) d\mu(x). \tag{4.1}$$

Definition 4.7. Let $(Y, \mathcal{B}_Y, \nu, G)$ be a G-measure preserving system, $\mathcal{U} \in \mathcal{C}_Y$, $\Pi_g \in Aut(Y, \nu)$ the action of $g \in G$ on $(Y, \mathcal{B}_Y, \nu)$ and $\phi_G : \mathcal{R} \to Aut(Y, \nu)$ a cocycle given by $\phi_G(gx, x) = \Pi_g$ for any $x \in X$, $g \in G$. The ν^--*virtual entropy* and ν-*virtual entropy* of $\mathcal{U}$ are defined respectively by

$$\widehat{h_\nu}^-(G, \mathcal{U}) = h_\nu^-(\phi_G, X \times \mathcal{U}) \quad \text{and} \quad \widehat{h_\nu}(G, \mathcal{U}) = h_\nu(\phi_G, X \times \mathcal{U}).$$

Clearly, if $\alpha \in \mathcal{P}_Y$ then $\widehat{h_\nu}(G, \alpha) = \widehat{h_\nu}^-(G, \alpha)$. Thus, for $\mathcal{U} \in \mathcal{C}_Y$, $\widehat{h_\nu}(G, \mathcal{U}) = \inf_{\alpha \in \mathcal{P}_Y:\, \alpha \succeq \mathcal{U}} \widehat{h_\nu}^-(G, \alpha)$. Note that there may exist plenty of free G-actions generating $\mathcal{R}$, ϕ_G is not determined uniquely by Π_g. Hence, we need to show that $\widehat{h_\nu}^-(G, \mathcal{U})$ and $\widehat{h_\nu}(G, \mathcal{U})$ are well defined.

Proposition 4.8. *Let $\{U_g\}_{g \in G}$ and $\{U'_g\}_{g \in G}$ be two free G-actions on $(X, \mathcal{B}_X, \mu)$ such that*

$$\{U_g x:\ g \in G\} = \left\{U'_g x:\ g \in G\right\} = \mathcal{R}(x)$$

for μ-a.e. $x \in X$. Define cocycles $\phi, \phi' : \mathcal{R} \to Aut(Y, \nu)$ by

$$\phi(U_g x, x) = \phi'\left(U'_g x, x\right) = \Pi_g \quad \textit{for any } g \in G,\ x \in X.$$

Then for any $\mathcal{U} \in \mathcal{C}_Y$, $h_\nu^-(\phi, X \times \mathcal{U}) = h_\nu^-(\phi', X \times \mathcal{U})$ and $h_\nu(\phi, X \times \mathcal{U}) = h_\nu(\phi', X \times \mathcal{U})$.

Proof. Denote by $\mathcal{S}$ the equivalence relation on $X \times X$ generated by the diagonal G-action $\{U_g \times U'_g\}_{g \in G}$. Clearly, $\mathcal{S}$ is measure preserving and hyperfinite. Let $\varphi_U, \varphi_{U'} : \mathcal{R} \to Aut(X, \mu)$ and $\phi_G : \mathcal{S} \to Aut(Y, \nu)$ be cocycles defined by

$$\varphi_U\left(U'_g x, x\right) = U_g, \qquad \varphi_{U'}(U_g x, x) = U'_g \quad \text{and} \quad \phi_G\left(\left(U_g x, U'_g x'\right), \left(x, x'\right)\right) = \Pi_g$$

for any $g \in G$, $x, x' \in X$. Then $\mathcal{S} = \mathcal{R}(\varphi_{U'}) = \sigma^{-1} \mathcal{R}(\varphi_U) \sigma$, where $\sigma : X \times X \to X \times X$ is the flip map, that is, $\sigma(x, x') = (x', x)$. Hence if $\{\mathcal{R}_n\}_{n \in \mathbb{N}}$ is a filtration of $\mathcal{R}$ then $\{\mathcal{R}_n(\varphi_{U'})\}_{n \in \mathbb{N}}$ and $\{\sigma^{-1} \mathcal{R}_n(\varphi_U) \sigma\}_{n \in \mathbb{N}}$ are both filtrations of $\mathcal{S}$.

For each $n \in \mathbb{N}$, one has $\phi_G((x, z), (x'', z'')) = \phi(x, x'')$ if $((x, z), (x'', z'')) \in \mathcal{R}_n(\varphi_{U'})$ and $\phi_G((z, x), (z', x')) = \phi'(x, x')$ if $((z, x), (z', x')) \in \sigma^{-1} \mathcal{R}_n(\varphi_U) \sigma$. Then by Proposition 4.4, for any $\mathcal{U} \in \mathcal{C}_Y$ one has

$$h_\nu^-\left(\mathcal{R}_n(\varphi_{U'}), \phi_G, X \times X \times \mathcal{U}\right) = h_\nu^-(\mathcal{R}_n, \phi, X \times \mathcal{U}),$$

$$h_\nu^-\left(\sigma^{-1} \mathcal{R}_n(\varphi_U) \sigma, \phi_G, X \times X \times \mathcal{U}\right) = h_\nu^-\left(\mathcal{R}_n, \phi', X \times \mathcal{U}\right).$$

Letting $n \to +\infty$ we obtain $h_\nu^-(\phi_G, X \times X \times \mathcal{U}) = h_\nu^-(\phi, X \times \mathcal{U})$ and $h_\nu^-(\phi_G, X \times X \times \mathcal{U}) = h_\nu^-(\phi', X \times \mathcal{U})$ for any $\mathcal{U} \in \mathcal{C}_Y$ (see Corollary 4.6). This implies that $h_\nu^-(\phi, X \times \mathcal{U}) = h_\nu^-(\phi', X \times \mathcal{U})$ for any $\mathcal{U} \in \mathcal{C}_Y$. Moreover, for $\mathcal{U} \in \mathcal{C}_Y$ we have

$$\begin{aligned} h_\nu(\phi, X \times \mathcal{U}) &= \inf_{\alpha \in \mathcal{P}_{X \times Y}:\ \alpha \succcurlyeq X \times \mathcal{U}} h_\nu^-(\phi, \alpha) = \inf_{\beta \in \mathcal{P}_Y:\ \beta \succcurlyeq \mathcal{U}} h_\nu^-(\phi, X \times \beta) \\ &= \inf_{\beta \in \mathcal{P}_Y:\ \beta \succcurlyeq \mathcal{U}} h_\nu^-\big(\phi', X \times \beta\big) = h_\nu\big(\phi', X \times \mathcal{U}\big). \end{aligned}$$

This finishes the proof of the proposition. □

Before proceeding, we need the following result. Let $K \in F(G)$ and $\epsilon > 0$. $F \in F(G)$ is called $[K, \epsilon]$-*invariant* if $|\{g \in F \mid Kg \subseteq F\}| > (1 - \epsilon)|F|$.

Lemma 4.9. *Let $(Y, \mathcal{B}_Y, \nu, G)$ be a G-measure preserving system, $\mathcal{U} \in \mathcal{C}_Y$ and $\epsilon > 0$. Then there exist $K \in F(G)$ and $0 < \epsilon' < \epsilon$ such that if $F \in F(G)$ is $[K, \epsilon']$-invariant then*

$$\left| \frac{1}{|F|} H_\nu(\mathcal{U}_F) - h_\nu(G, \mathcal{U}) \right| < \epsilon.$$

Proof. Choose $e_G \in K_1 \subseteq K_2 \subseteq \cdots$ with $\bigcup_{i \in \mathbb{N}} K_i = G$. For each $i \in \mathbb{N}$ set $\delta_i = \frac{1}{2^i(|K_i|+1)}$. Now if the lemma is not true then there exists $\epsilon > 0$ such that for each $i \in \mathbb{N}$ there exists $F_i \in F(G)$ such that it is $[K_i^{-1} K_i, \delta_i]$-invariant and

$$\left| \frac{1}{|F_i|} H_\nu(\mathcal{U}_{F_i}) - h_\nu(G, \mathcal{U}) \right| \geqslant \epsilon. \tag{4.2}$$

Let $K \in F(G)$ with $e_G \in K$ and $\delta > 0$. If $F \in F(G)$ is $[K^{-1}K, \delta]$-invariant then

$$\begin{aligned} B(F, K) &= \big\{g \in G:\ Kg \cap F \neq \emptyset \text{ and } Kg \cap (G \setminus F) \neq \emptyset\big\} \\ &= K^{-1}F \setminus \{g \in F:\ Kg \subseteq F\} = \big(K^{-1}F \setminus F\big) \cup \big(F \setminus \{g \in F:\ Kg \subseteq F\}\big) \\ &\subseteq K^{-1}\big(F \setminus \big\{g \in F:\ K^{-1}g \subseteq F\big\}\big) \cup \big(F \setminus \{g \in F:\ Kg \subseteq F\}\big) \\ &\subseteq K^{-1}\big(F \setminus \big\{g \in F:\ K^{-1}Kg \subseteq F\big\}\big) \cup \big(F \setminus \big\{g \in F:\ K^{-1}Kg \subseteq F\big\}\big), \end{aligned}$$

hence $|B(F, K)| \leqslant (|K| + 1) \cdot |F \setminus \{g \in F:\ K^{-1}Kg \subseteq F\}| \leqslant \delta(|K| + 1)|F|$ (as $F \in F(G)$ is $[K^{-1}K, \delta]$-invariant), i.e. F is a $(K, (|K| + 1)\delta)$-invariant set. Particularly, we have that F_i is $(K_i, \frac{1}{2^i})$-invariant for each $i \in \mathbb{N}$. Moreover, since $e_G \in K_1 \subseteq K_2 \subseteq \cdots$ and $\bigcup_{i \in \mathbb{N}} K_i = G$, we have that $\{F_i\}_{i \in \mathbb{N}}$ is a Følner sequence of G. Hence $\lim_{i \to +\infty} \frac{1}{|F_i|} H_\nu(\mathcal{U}_{F_i}) = h_\nu^-(G, \mathcal{U})$, a contradiction with (4.2). □

Theorem 4.10. *Let $(Y, \mathcal{B}_Y, \nu, G)$ be a G-measure preserving system and $\mathcal{U} \in \mathcal{C}_Y$. Then*

$$h_\nu^-(G, \mathcal{U}) = \widehat{h_\nu}^-(G, \mathcal{U}) \quad \text{and} \quad h_\nu(G, \mathcal{U}) = \widehat{h_\nu}(G, \mathcal{U}).$$

Proof. By Lemma 4.9 for each $\epsilon > 0$ there exist $K \in F(G)$ and $0 < \epsilon' < \epsilon$ such that if $F \in F(G)$ is $[K, \epsilon']$-invariant then $|\frac{1}{|F|} H_\nu(\mathcal{U}_F) - h_\nu(G, \mathcal{U})| < \epsilon$. Let $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ be a filtration of $\mathcal{R}$ with $\mathcal{R}_n \sim (B_i^{(n)}, G_i^{(n)})$ for each $n \in \mathbb{N}$. Thus by Lemma 4.2(1), for each sufficiently large n there is a measurable $\mathcal{R}_n$-invariant subset $A_n \subseteq X$ such that $\mu(A_n) > 1 - \epsilon'$ and

$$\#\{x' \in \mathcal{R}_n(x)\colon\ Kx' \subseteq \mathcal{R}_n(x)\} > (1-\epsilon')\#\mathcal{R}_n(x) \quad \text{for each } x \in A_n. \tag{4.3}$$

Since A_n is $\mathcal{R}_n$-invariant, $A_n = \bigsqcup_{i\in J} G_i^{(n)} C_i^{(n)}$ for some subset $J \subseteq \mathbb{N}$ and a family of measurable subsets $C_i^{(n)} \subseteq B_i^{(n)}$ with $\mu(C_i^{(n)}) > 0$, $i \in J$. By (4.3), if $i \in J$, $x \in C_i^{(n)}$ and $g' \in G_i^{(n)}$ then

$$(1-\epsilon')\#\mathcal{R}_n(g'x) < \#\{x' \in \mathcal{R}_n(g'x)\colon\ Kx' \subseteq \mathcal{R}_n(g'x)\} = \#\{x' \in \mathcal{R}_n(x)\colon\ Kx' \subseteq \mathcal{R}_n(x)\}.$$

That is, $(1-\epsilon')|G_i^{(n)}| < |\{g \in G_i^{(n)}\colon\ Kg \subseteq G_i^{(n)}\}|$, i.e. $G_i^{(n)}$ is $[K, \epsilon']$-invariant. Set

$$f(x) = \frac{1}{\#\mathcal{R}_n(x)} H_\nu\Big(\bigvee_{y\in\mathcal{R}_n(x)} \phi_G(x, y)\mathcal{U}\Big) \leqslant \log N(\mathcal{U}) \quad \text{for each } x \in X.$$

Then by similar reasoning of (4.1), one has

$$\int_{A_n} f(x)\,d\mu(x) = \sum_{j\in J} \int_{C_j^{(n)}} H_\nu\Big(\bigvee_{g\in G_j^{(n)}} \Pi_g^{-1}\mathcal{U}\Big)\,d\mu(x).$$

Hence

$$\begin{aligned}
&\big|h_\nu^-(\mathcal{R}_n, \phi_G, X \times \mathcal{U}) - \mu(A_n) h_\nu^-(G, \mathcal{U})\big| \\
&\quad\leqslant \bigg|\int_{A_n} \big(f(x) - h_\nu^-(G,\mathcal{U})\big)\,d\mu(x)\bigg| + \bigg|\int_{X\setminus A_n} f(x)\,d\mu(x)\bigg| \\
&\quad\leqslant \bigg|\sum_{j\in J} \int_{C_j^{(n)}} |G_j^{(n)}| \bigg(\frac{1}{|G_j^{(n)}|} H_\nu\Big(\bigvee_{g\in G_j^{(n)}} \Pi_g^{-1}\mathcal{U}\Big) - h_\nu^-(G,\mathcal{U})\bigg)\,d\mu(x)\bigg| + \big(1-\mu(A_n)\big)\log N(\mathcal{U}) \\
&\quad\leqslant \Big(\sum_{j\in J} |G_j^{(n)}| \mu\big(C_j^{(n)}\big)\Big)\epsilon + \big(1-\mu(A_n)\big)\log N(\mathcal{U}) \quad \big(\text{by the selection of } K \text{ and } \epsilon'\big).
\end{aligned}$$

Noting that $A_n = \bigsqcup_{i\in J} G_i^{(n)} C_i^{(n)}$ and $\mu(A_n) > 1 - \epsilon'$ where $0 < \epsilon' < \epsilon$, first let $n \to +\infty$ and then let $\epsilon \to 0+$, thus we have $\widehat{h_\nu}^-(G,\mathcal{U}) = h_\nu^-(\phi_G, X \times \mathcal{U}) = h_\nu^-(G,\mathcal{U})$ (see Corollary 4.6). Moreover,

$$\widehat{h_\nu}(G,\mathcal{U}) = \inf_{\alpha\in\mathcal{P}_X\colon \alpha\succcurlyeq\mathcal{U}} \widehat{h_\nu}^-(G,\alpha) = \inf_{\alpha\in\mathcal{P}_X\colon \alpha\succcurlyeq\mathcal{U}} h_\nu(G,\alpha) = h_\nu(G,\mathcal{U}).$$

This finishes the proof. □

Let $(Z, \mathcal{B}_Z, \kappa)$ be a Lebesgue space with T an invertible measure-preserving transformation, $\mathcal{W} \in \mathcal{C}_Z$ and $\mathcal{D} \subseteq \mathcal{B}_Z$ a T-invariant sub-σ-algebra, i.e. $T^{-1}\mathcal{D} = \mathcal{D}$. Set $\mathcal{W}_0^{n-1} = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{W}$ for each $n \in \mathbb{N}$. It is clear that the sequence $\{H_\kappa(\mathcal{W}_0^{n-1}|\mathcal{D})\}_{n\in\mathbb{N}}$ is non-negative and sub-additive. So we may define

$$h_\kappa(T, \mathcal{W}|\mathcal{D}) = \inf_{\gamma \in \mathcal{P}_Z:\, \gamma \succcurlyeq \mathcal{W}} h_\kappa^-(T, \gamma|\mathcal{D}),$$

$$h_\kappa^-(T, \mathcal{W}|\mathcal{D}) = \lim_{n\to+\infty} \frac{1}{n} H_\kappa\big(\mathcal{W}_0^{n-1}|\mathcal{D}\big) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\kappa\big(\mathcal{W}_0^{n-1}|\mathcal{D}\big).$$

Clearly $h_\kappa^-(T, \mathcal{W}|\mathcal{D}) = h_\kappa(T, \mathcal{W}|\mathcal{D})$ when $\mathcal{W} \in \mathcal{P}_Z$. We shall write simply

$$h_\kappa^-(T, \mathcal{W}) = h_\kappa^-\big(T, \mathcal{W}|\{\emptyset, Z\}\big) \quad \text{and} \quad h_\kappa(T, \mathcal{W}) = h_\kappa\big(T, \mathcal{W}|\{\emptyset, Z\}\big).$$

Theorem 4.11. *Let γ be an invertible measure-preserving transformation on $(X, \mathcal{B}_X, \mu)$ generating $\mathcal{R}$, $\phi : \mathcal{R} \to Aut(Y, \nu)$ a cocycle and γ_ϕ stand for the ϕ-skew product extension of γ. Then for each $\mathcal{U} \in \mathcal{C}_{X\times Y}$, one has*

$$h_\nu^-(\phi, \mathcal{U}) = h_{\mu\times\nu}^-\big(\gamma_\phi, \mathcal{U}|\mathcal{B}_X \otimes \{\emptyset, Y\}\big) \quad \textit{and} \quad h_\nu(\phi, \mathcal{U}) = h_{\mu\times\nu}\big(\gamma_\phi, \mathcal{U}|\mathcal{B}_X \otimes \{\emptyset, Y\}\big).$$

Proof. Let $\Sigma = \prod_{i=1}^{+\infty}\{0, 1\}$ be the product space of the discrete space $\{0, 1\}$. If $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in \Sigma$ then the sum $x \oplus y = (z_1, z_2, \ldots)$ is defined as follows. If $x_1 + y_1 < 2$ then $z_1 = x_1 + y_1$, if $x_1 + y_1 \geqslant 2$ then $z_1 = x_1 + y_1 - 2$ and we carry 1 to the next position. The other terms $z_2, \ldots$ are successively determined in the same fashion. Let $\delta : \Sigma \to \Sigma$, $z \mapsto z \oplus 1$ with $1 = (1, 0, 0, \ldots)$. It is known that (Σ, δ) is minimal, which is called an *adding machine*. Let λ be the Haar measure on $(\Sigma, \oplus)$. Denote by $\mathcal{S}$ the $\delta \times \gamma$-orbit equivalence relation on $\Sigma \times X$. Let $\sigma : \Sigma \times X \to X \times \Sigma$ be the flip map. We have $\mathcal{S} = \sigma^{-1}\mathcal{R}(\varphi)\sigma$ for the cocycle $\varphi : \mathcal{R} \to Aut(\Sigma, \lambda)$ given by $(\gamma^n x, x) \mapsto \delta^n$, $n \in \mathbb{Z}$ (as $\mathcal{R}$ is conservative, γ is aperiodic and so φ is well defined).

Now we define a cocycle $1 \oplus \phi : \mathcal{S} \to Aut(Y, \nu)$ by setting $((z, x), (z', x')) \mapsto \phi(x, x')$. Let $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ be a filtration of $\mathcal{R}$. Then $\{\sigma^{-1}\mathcal{R}_n(\varphi)\sigma\}_{n\in\mathbb{N}}$ is a filtration of $\mathcal{S}$ and so for each $\mathcal{U} \in \mathcal{C}_{X\times Y}$

$$\begin{aligned} h_\nu^-(1 \oplus \phi, \Sigma \times \mathcal{U}) &= \lim_{n\to+\infty} h_\nu^-\big(\sigma^{-1}\mathcal{R}_n(\varphi)\sigma, 1 \oplus \phi, \Sigma \times \mathcal{U}\big) \quad \text{(by Corollary 4.6)} \\ &= \lim_{n\to+\infty} h_\nu^-(\mathcal{R}_n, \phi, \mathcal{U}) \quad \big(\text{by Proposition 4.4(1)}\big) \\ &= h_\nu^-(\phi, \mathcal{U}) \quad \text{(by Corollary 4.6)}. \end{aligned} \tag{4.4}$$

On the other hand, for each $n \in \mathbb{N}$ we let $A_n = \{z \in \Sigma\colon\ z_i = 0 \text{ for } 1 \leqslant i \leqslant n\}$. Then $A_1 \supseteq A_2 \supseteq \cdots$ is a sequence of measurable subsets of Σ such that $\Sigma = \bigsqcup_{i=0}^{2^n-1} \delta^i A_n$ and so $\Sigma \times X = \bigsqcup_{i=0}^{2^n-1} (\delta \times \gamma)^i (A_n \times X)$ for each $n \in \mathbb{N}$. Let $\mathcal{S}_n \in I(\mathcal{S})$ with $\mathcal{S}_n \sim (A_n \times X, \{(\delta \times \gamma)^i\colon\ i = 0, 1, \ldots, 2^n - 1\})$. By (4.1) we obtain that

$$
\begin{aligned}
&h^-_\nu(\mathcal{S}_n, 1\oplus\phi, \Sigma\times\mathcal{U})\\
&\quad= \int\limits_{A_n\times X} H_\nu\left(\bigvee_{i=0}^{2^n-1}\phi\big(x,\gamma^i x\big)\mathcal{U}_{\gamma^i x}\right) d\lambda\times\mu(z,x)\\
&\quad= \frac{1}{2^n}\int\limits_X H_\nu\left(\bigvee_{i=0}^{2^n-1}\phi\big(x,\gamma^i x\big)\mathcal{U}_{\gamma^i x}\right) d\mu(x) \quad \left(\text{as } \lambda(A_n)=\frac{1}{2^n}\right)\\
&\quad= \frac{1}{2^n}\int\limits_X H_\nu\left(\left(\bigvee_{i=0}^{2^n-1}\gamma_\phi^{-i}\mathcal{U}\right)_x\right) d\mu(x) \quad \left(\text{as } \bigvee_{i=0}^{2^n-1}\phi\big(x,\gamma^i x\big)\mathcal{U}_{\gamma^i x} = \left(\bigvee_{i=0}^{2^n-1}\gamma_\phi^{-i}\mathcal{U}\right)_x\right)\\
&\quad= \frac{1}{2^n}\int\limits_X H_{\delta_x\times\nu}\left(\bigvee_{i=0}^{2^n-1}\gamma_\phi^{-i}\mathcal{U}\right) d\mu(x) = \frac{1}{2^n}H_{\mu\times\nu}\left(\bigvee_{i=0}^{2^n-1}\gamma_\phi^{-i}\mathcal{U}|\mathcal{B}_X\otimes\{\emptyset,Y\}\right).
\end{aligned}
$$

Note that $\mathcal{S}_1\subseteq\mathcal{S}_2\subseteq\cdots$ and $\bigcup_{n\in\mathbb{N}}\mathcal{S}_n=\mathcal{S}$, then

$$
\begin{aligned}
h^-_\nu(1\oplus\phi,\Sigma\times\mathcal{U}) &= \lim_{n\to+\infty} h^-_\nu(\mathcal{S}_n, 1\oplus\phi, \Sigma\times\mathcal{U}) \quad \text{(by Corollary 4.6)}\\
&= \lim_{n\to+\infty}\frac{1}{2^n}H_{\mu\times\nu}\left(\bigvee_{i=0}^{2^n-1}\gamma_\phi^{-i}\mathcal{U}|\mathcal{B}_X\otimes\{\emptyset,Y\}\right)\\
&= h^-_{\mu\times\nu}\big(\gamma_\phi,\mathcal{U}|\mathcal{B}_X\otimes\{\emptyset,Y\}\big)
\end{aligned}
$$

and so $h^-_\nu(\phi,\mathcal{U}) = h^-_{\mu\times\nu}(\gamma_\phi,\mathcal{U}|\mathcal{B}_X\otimes\{\emptyset,Y\})$ for each $\mathcal{U}\in\mathcal{C}_{X\times Y}$ by (4.4). Finally,

$$
\begin{aligned}
h_\nu(\phi,\mathcal{U}) &= \inf_{\alpha\in\mathcal{P}_{X\times Y}:\,\alpha\succcurlyeq\mathcal{U}} h^-_\nu(\phi,\alpha) = \inf_{\alpha\in\mathcal{P}_{X\times Y}:\,\alpha\succcurlyeq\mathcal{U}} h^-_{\mu\times\nu}\big(\gamma_\phi,\alpha|\mathcal{B}_X\otimes\{\emptyset,Y\}\big)\\
&= h_{\mu\times\nu}\big(\gamma_\phi,\mathcal{U}|\mathcal{B}_X\otimes\{\emptyset,Y\}\big)
\end{aligned}
$$

for each $\mathcal{U}\in\mathcal{C}_{X\times Y}$. This finishes the proof of the theorem. □

4.4. The proof of the equivalence of measure-theoretic entropy of covers

The following result was proved by the same authors [24, Theorem 6.4] (see also [19,21]).

Lemma 4.12. *Let (X,T) be a TDS with $\mathcal{U}\in\mathcal{C}_X$ and $\mu\in\mathcal{M}(X,T)$. Then $h^-_\mu(T,\mathcal{U}) = h_\mu(T,\mathcal{U})$.*

Lemma 4.13. *Let $(Z,\mathcal{B}_Z,\kappa)$ be a Lebesgue space with T an invertible measure-preserving transformation, $\mathcal{W}\in\mathcal{C}_Z$ and $\mathcal{D}\subseteq\mathcal{B}_Z$ a T-invariant sub-σ-algebra. Then $h^-_\kappa(T,\mathcal{W}|\mathcal{D}) = h_\kappa(T,\mathcal{W}|\mathcal{D})$.*

Proof. First we claim the conclusion for the case $\mathcal{D}=\{\emptyset,Z\}$. By the ergodic decomposition of $h^-_\kappa(T,\mathcal{W})$ and $h_\kappa(T,\mathcal{W})$ (see (3.29) in the case of $G=\mathbb{Z}$), it suffices to prove it when κ is ergodic. By the Jewett–Krieger Theorem (see for example [8]), (Z,κ,T) is measure theoretical isomorphic to a uniquely ergodic zero-dimensional topological dynamical system $(\widehat{Z}, \widehat{\kappa}, \widehat{T})$. Let $\pi : (\widehat{Z}, \widehat{\kappa}, \widehat{T}) \to (Z, \kappa, T)$ be such an isomorphism. Then using Lemma 4.12 we have

$$h_\kappa^-(T, \mathcal{W}) = h_{\widehat{\kappa}}^-\big(\widehat{T}, \pi^{-1}\mathcal{W}\big) = h_{\widehat{\kappa}}\big(\widehat{T}, \pi^{-1}\mathcal{W}\big) = h_\kappa(T, \mathcal{W}).$$

In general case, let $\{\beta_j\}_{j\in\mathbb{N}} \subseteq \mathcal{P}_Z$ with $\beta_j \nearrow \mathcal{D}$ (mod μ). For simplicity, we write $\mathcal{P}(\mathcal{V}) = \{\alpha \in \mathcal{P}_Z\colon \alpha \succcurlyeq \mathcal{V}\}$ for $\mathcal{V} \in \mathcal{C}_X$. Then

$$\begin{aligned}
h_\kappa^-(T, \mathcal{W}|\mathcal{D}) &= \inf_{n\geqslant 1} \frac{1}{n} H_\kappa\big(\mathcal{W}_0^{n-1}|\mathcal{D}\big) = \inf_{n\geqslant 1} \frac{1}{n}\Big(\inf_{\alpha\in\mathcal{P}(\mathcal{W}_0^{n-1})} H_\kappa(\alpha|\mathcal{D})\Big) \\
&= \inf_{n\geqslant 1} \frac{1}{n}\Big(\inf_{\alpha\in\mathcal{P}(\mathcal{W}_0^{n-1})} \inf_{j\geqslant 1} H_\kappa\big(\alpha|(\beta_j)_0^{n-1}\big)\Big) \quad \big(\text{as } \beta_j \nearrow \mathcal{D} \text{ (mod } \mu)\big) \\
&= \inf_{j\geqslant 1} \inf_{n\geqslant 1} \frac{1}{n}\Big(\inf_{\alpha\in\mathcal{P}(\mathcal{W}_0^{n-1})} H_\kappa\big(\alpha|(\beta_j)_0^{n-1}\big)\Big) \\
&= \inf_{j\geqslant 1} \inf_{n\geqslant 1} \frac{1}{n} H_\kappa\big(\mathcal{W}_0^{n-1}|(\beta_j)_0^{n-1}\big).
\end{aligned} \tag{4.5}$$

Let $j \in \mathbb{N}$. Since for any $n, m \in \mathbb{N}$ and $\mathcal{V} \in \mathcal{C}_X$ one has

$$\begin{aligned}
H_\kappa\big(\mathcal{V}_0^{n+m-1}|(\beta_j)_0^{n+m-1}\big) &\leqslant H_\kappa\big(\mathcal{V}_0^{n-1}|(\beta_j)_0^{n+m-1}\big) + H_\kappa\big(T^{-n}\mathcal{V}_0^{m-1}|(\beta_j)_0^{n+m-1}\big) \\
&\leqslant H_\kappa\big(\mathcal{V}_0^{n-1}|(\beta_j)_0^{n-1}\big) + H_\kappa\big(T^{-n}\mathcal{V}_0^{m-1}|T^{-n}(\beta_j)_0^{m-1}\big) \\
&= H_\kappa\big(\mathcal{V}_0^{n-1}|(\beta_j)_0^{n-1}\big) + H_\kappa\big(\mathcal{V}_0^{m-1}|(\beta_j)_0^{m-1}\big),
\end{aligned}$$

hence

$$\inf_{n\geqslant 1} \frac{1}{n} H_\kappa\big(\mathcal{V}_0^{n-1}|(\beta_j)_0^{n-1}\big) = \lim_{n\to+\infty} \frac{1}{n} H_\kappa\big(\mathcal{V}_0^{n-1}|(\beta_j)_0^{n-1}\big). \tag{4.6}$$

Combining (4.6) for $\mathcal{V} = \mathcal{W}$ with (4.5), one has

$$\begin{aligned}
h_\kappa^-(T, \mathcal{W}|\mathcal{D}) &= \inf_{j\geqslant 1} \lim_{n\to+\infty} \frac{1}{n} H_\kappa\big(\mathcal{W}_0^{n-1}|(\beta_j)_0^{n-1}\big) \\
&= \inf_{j\geqslant 1} \lim_{n\to+\infty} \frac{1}{n} \inf_{\alpha\in\mathcal{P}(\mathcal{W}_0^{n-1})} H_\kappa\big(\alpha|(\beta_j)_0^{n-1}\big) \\
&= \inf_{j\geqslant 1} \lim_{n\to+\infty} \frac{1}{n}\Big(\inf_{\alpha\in\mathcal{P}(\mathcal{W}_0^{n-1})} H_\kappa\big(\alpha \vee (\beta_j)_0^{n-1}\big) - H_\kappa\big((\beta_j)_0^{n-1}\big)\Big) \\
&\geqslant \inf_{j\geqslant 1} \lim_{n\to+\infty} \frac{1}{n}\big(H_\kappa\big(\mathcal{W}_0^{n-1} \vee (\beta_j)_0^{n-1}\big) - H_\kappa\big((\beta_j)_0^{n-1}\big)\big) \\
&= \inf_{j\geqslant 1}\big(h_\kappa^-(T, \mathcal{W} \vee \beta_j) - h_\kappa^-(T, \beta_j)\big) \\
&= \inf_{j\geqslant 1}\big(h_\kappa(T, \mathcal{W} \vee \beta_j) - h_\kappa(T, \beta_j)\big) \quad \text{(by the first part)}
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{j\geqslant 1}\Big(\inf_{\alpha\in\mathcal{P}(\mathcal{W})} h_\kappa(T,\alpha\vee\beta_j)-h_\kappa(T,\beta_j)\Big)\\
&= \inf_{j\geqslant 1}\inf_{\alpha\in\mathcal{P}(\mathcal{W})}\lim_{n\to+\infty}\frac{1}{n}\big(H_\kappa\big((\alpha\vee\beta_j)_0^{n-1}\big)-H_\kappa\big((\beta_j)_0^{n-1}\big)\big)\\
&\geqslant \inf_{j\geqslant 1}\inf_{\alpha\in\mathcal{P}(\mathcal{W})}\inf_{n\geqslant 1}\frac{1}{n}H_\kappa\big(\alpha_0^{n-1}|(\beta_j)_0^{n-1}\big)\quad\big(\text{by (4.6) for } \mathcal{V}=\alpha\big)\\
&= \inf_{\alpha\in\mathcal{P}(\mathcal{W})}\inf_{n\geqslant 1}\inf_{j\geqslant 1}\frac{1}{n}H_\kappa\big(\alpha_0^{n-1}|(\beta_j)_0^{n-1}\big)\\
&= \inf_{\alpha\in\mathcal{P}(\mathcal{W})}\inf_{n\geqslant 1}\frac{1}{n}H_\kappa\big(\alpha_0^{n-1}|\mathcal{D}\big)\quad\big(\text{as } \beta_j\nearrow\mathcal{D}\ (\mathrm{mod}\ \mu)\big)\\
&= h_\kappa(T,\mathcal{W}|\mathcal{D}).
\end{aligned}
$$

As the inequality of $h_\kappa^-(T,\mathcal{W}|\mathcal{D})\leqslant h_\kappa(T,\mathcal{W}|\mathcal{D})$ is straightforward, this finishes the proof. □

The following result is our main result in the section.

Theorem 4.14. *Let $(Y,\mathcal{B}_Y,\nu,G)$ be a G-measure preserving system with $(Y,\mathcal{B}_Y,\nu)$ a Lebesgue space and $\mathcal{U}\in\mathcal{C}_Y$. Then $h_\nu(G,\mathcal{U})=h_\nu^-(G,\mathcal{U})$.*

Proof. Let $(X,\mathcal{B}_X,\mu,G)$ be a free G-measure preserving system with $\mathcal{R}\subseteq X\times X$ the G-orbit equivalence relation and γ an invertible measure-preserving transformation on $(X,\mathcal{B}_X,\mu)$ generating $\mathcal{R}$. The cocycle $\phi_G:\mathcal{R}\to Aut(Y,\nu)$ is given by $\phi_G(gx,x)=\Pi_g$, where $\Pi_g\in Aut(Y,\nu)$ is the action of $g\in G$ on $(Y,\mathcal{B}_Y,\nu)$. By Definition 4.7 of virtual entropy and Theorem 4.10, we have

$$
h_\nu^-(G,\mathcal{U})=h_\nu^-(\phi_G,X\times\mathcal{U})\quad\text{and}\quad h_\nu(G,\mathcal{U})=h_\nu(\phi_G,X\times\mathcal{U}). \tag{4.7}
$$

Let $T=\gamma_{\phi_G}$ be the ϕ-skew production extension of γ. Using Theorem 4.11 one has

$$
\begin{aligned}
&h_\nu^-(\phi_G,X\times\mathcal{U})=h_{\mu\times\nu}^-\big(T,\mathcal{U}|\mathcal{B}_X\times\{\emptyset,Y\}\big)\quad\text{and}\\
&h_\nu(\phi_G,X\times\mathcal{U})=h_{\mu\times\nu}\big(T,\mathcal{U}|\mathcal{B}_X\times\{\emptyset,Y\}\big).
\end{aligned}\tag{4.8}
$$

As $\mathcal{B}_X\times\{\emptyset,Y\}$ is T-invariant, $h_{\mu\times\nu}^-(T,\mathcal{U}|\mathcal{B}_X\times\{\emptyset,Y\})=h_{\mu\times\nu}(T,\mathcal{U}|\mathcal{B}_X\times\{\emptyset,Y\})$ by Lemma 4.13. Combining this fact with (4.7) and (4.8), we get $h_\nu^-(G,\mathcal{U})=h_\nu(G,\mathcal{U})$. This finishes the proof. □

4.5. A local version of Katok's result

At the end of this section, we shall give a local version of a well-known result of Katok [26, Theorem I.I] for a G-action. Let (X,G) be a G-system, $\mu\in\mathcal{M}(X,G)$ and $\mathcal{U}\in\mathcal{C}_X$. Let $a\in(0,1)$ and $F\in F(G)$. Set

$$
b(F,a,\mathcal{U})=\min\Big\{\#(\mathcal{C}):\ \mathcal{C}\subseteq\mathcal{U}_F \text{ and } \mu\Big(\bigcup\mathcal{C}\Big)\geqslant a\Big\}.
$$

The following simple fact is inspired by [44, Lemma 5.11].

Lemma 4.15. $H_\mu(\mathcal{U}_F) \leqslant \log b(F,a,\mathcal{U}) + (1-a)|F|\log N(\mathcal{U}) + \log 2$.

Proof. Let $\mathcal{C} = \{C_1,\dots,C_\ell\} \subseteq \mathcal{U}_F$ such that $\mu(\bigcup \mathcal{C}) \geqslant a$ and $\ell = b(F,a,\mathcal{U})$. Let $\alpha_1 = \{C_1, C_2 \setminus C_1, \dots, C_\ell \setminus \bigcup_{j=1}^{\ell-1} C_j\}$. Then α_1 is a partition of $\bigcup_{i=1}^{\ell} C_i$ and $\#\alpha_1 = b(F,a,\mathcal{U})$. Similarly, we take $\alpha_2' \in \mathcal{P}_X$ satisfying $\#\alpha_2' = N(\mathcal{U}_F)$. Then let $\alpha_2 = \{A \cap (X \setminus \bigcup_{i=1}^{\ell} C_i)\colon A \in \alpha_2'\}$. Then $\#\alpha_2 \leqslant N(\mathcal{U}_F)$. Set $\alpha = \alpha_1 \cup \alpha_2$. Then $\alpha \in \mathcal{P}_X$ and $\alpha \succcurlyeq \mathcal{U}_F$. Note that if $x_1,\dots,x_m \geqslant 0$ then

$$\sum_{i=1}^{m} \phi(x_i) \leqslant \left(\sum_{i=1}^{m} x_i\right)\log m + \phi\left(\sum_{i=1}^{m} x_i\right), \tag{4.9}$$

thus one has

$$\begin{aligned}
H_\mu(\mathcal{U}_F) &\leqslant H_\mu(\alpha)\\
&\leqslant \mu\left(\bigcup_{i=1}^{\ell} C_i\right)\left(\log\#\alpha_1 - \log\mu\left(\bigcup_{i=1}^{\ell} C_i\right)\right)\\
&\quad + \left(1-\mu\left(\bigcup_{i=1}^{\ell} C_i\right)\right)\left(\log\#\alpha_2 - \log\left(1-\mu\left(\bigcup_{i=1}^{\ell} C_i\right)\right)\right) \quad \big(\text{by (4.9)}\big)\\
&\leqslant \log b(F,a,\mathcal{U}) + (1-a)\log N(\mathcal{U}_F) - \mu\left(\bigcup_{i=1}^{\ell} C_i\right)\log\mu\left(\bigcup_{i=1}^{\ell} C_i\right)\\
&\quad - \left(1-\mu\left(\bigcup_{i=1}^{\ell} C_i\right)\right)\log\left(1-\mu\left(\bigcup_{i=1}^{\ell} C_i\right)\right)\\
&\leqslant \log b(F,a,\mathcal{U}) + (1-a)|F|\log N(\mathcal{U}) + \log 2. \qquad \square
\end{aligned}$$

As a direct application of Lemma 4.15 by letting $a \to 1-$ we have

Proposition 4.16. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a Følner sequence of G. Then*

$$h_\mu^-(G,\mathcal{U}) \leqslant \lim_{\epsilon\to 0+}\liminf_{n\to+\infty}\frac{1}{|F_n|}\log b(F_n, 1-\epsilon, \mathcal{U}).$$

The following result is [30, Theorem 1.3].

Lemma 4.17. *Let $\alpha \in \mathcal{P}_X$ and $\{F_n\}_{n\in\mathbb{N}}$ be a Følner sequence of G such that $\lim_{n\to+\infty}\frac{|F_n|}{\log n} = +\infty$ and for some constant $C > 0$ one has $|\bigcup_{k=1}^{n-1} F_k^{-1}F_n| \leqslant C|F_n|$ for each $n \in \mathbb{N}$. If μ is ergodic then for μ-a.e. $x \in X$ and in the sense of $L^1(X,\mathcal{B}_X,\mu)$-norm one has*

$$\lim_{n\to+\infty} -\frac{\log\mu(\alpha_{F_n}(x))}{|F_n|} = h_\mu(G,\alpha).$$

Proposition 4.18. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a Følner sequence of G. If $\mu \in \mathcal{M}^e(X, G)$ then*

$$h_\mu(G,\mathcal{U}) \geqslant \lim_{\epsilon\to 0+} \limsup_{n\to+\infty} \frac{1}{|F_n|} \log b(F_n, 1-\epsilon, \mathcal{U}).$$

Proof. First for any $\mathcal{P} \in \mathcal{P}_X$ we claim the conclusion by proving

$$h_\mu(G,\alpha) \geqslant \limsup_{n\to+\infty} \frac{1}{|F_n|} \log b(F_n, 1-\epsilon, \alpha) \quad \text{for each } \epsilon \in (0,1). \tag{4.10}$$

Proof of the claim. Fix $\epsilon \in (0, 1)$. In $\{F_n\}_{n\in\mathbb{N}}$ we can select a sub-sequence $\{E_n\}_{n\in\mathbb{N}}$ satisfying

$$\limsup_{n\to+\infty} \frac{1}{|F_n|} \log b(F_n, 1-\epsilon, \alpha) = \lim_{n\to+\infty} \frac{1}{|E_n|} \log b(E_n, 1-\epsilon, \alpha),$$

$\lim_{n\to+\infty} \frac{|E_n|}{\log n} = +\infty$ and for some constant $C > 0$ one has $|\bigcup_{k=1}^{n-1} E_k^{-1} E_n| \leqslant C|E_n|$ for each $n \in \mathbb{N}$. Now applying Lemma 4.17 to $\{E_n\}_{n\in\mathbb{N}}$, for each $\delta > 0$ there exists $N \in \mathbb{N}$ such that for each $n \geqslant N$, $\mu(A_n) \geqslant 1-\epsilon$ where

$$\begin{aligned} A_n &= \left\{x \in X\colon -\frac{\log \mu(\alpha_{E_n}(x))}{|E_n|} \leqslant h_\mu(G,\alpha)+\delta\right\} \\ &\supseteq \left\{x \in X\colon -\frac{\log \mu(\alpha_{E_m}(x))}{|E_m|} \leqslant h_\mu(G,\alpha)+\delta \text{ if } m \geqslant n\right\}. \end{aligned}$$

Note that A_n must be a union of some atoms in α_{E_n}, where each atom has measure at least $e^{-|E_n|(h_\mu(G,\alpha)+\delta)}$, which implies $b(E_n, 1-\epsilon, \alpha) \leqslant (1-\epsilon)e^{|E_n|(h_\mu(G,\alpha)+\delta)}$ when $n \geqslant N$. So

$$\limsup_{n\to+\infty} \frac{1}{|F_n|} \log b(F_n, 1-\epsilon, \alpha) = \lim_{n\to+\infty} \frac{1}{|E_n|} \log b(E_n, 1-\epsilon, \alpha) \leqslant h_\mu(G,\alpha)+\delta.$$

Since $\delta > 0$ is arbitrary, one claims (4.10). □

Now for general case, by the above discussions we have

$$\begin{aligned} h_\mu(G,\mathcal{U}) &= \inf_{\alpha\in\mathcal{P}_X\colon \alpha\succeq\mathcal{U}} h_\mu(G,\alpha) \\ &\geqslant \inf_{\alpha\in\mathcal{P}_X\colon \alpha\succeq\mathcal{U}} \lim_{\epsilon\to 0+} \limsup_{n\to+\infty} \frac{1}{|F_n|} \log b(F_n, 1-\epsilon, \alpha) \quad \big(\text{by (4.10)}\big) \\ &\geqslant \lim_{\epsilon\to 0+} \limsup_{n\to+\infty} \frac{1}{|F_n|} \log b(F_n, 1-\epsilon, \mathcal{U}). \qquad \Box \end{aligned}$$

Now combining Theorem 4.14 with Propositions 4.16 and 4.18 we obtain (when $G = \mathbb{Z}$, it can be viewed as a local version of Katok's result [26, Theorem I.I]):

Theorem 4.19. *Let* $\{F_n\}_{n\in\mathbb{N}}$ *be a Følner sequence of* G. *If* $\mu \in \mathcal{M}^e(X,G)$ *then*

$$h_\mu(G,\mathcal{U}) = \lim_{\epsilon\to 0+} \limsup_{n\to+\infty} \frac{1}{|F_n|} \log b(F_n, 1-\epsilon, \mathcal{U})$$
$$= \lim_{\epsilon\to 0+} \liminf_{n\to+\infty} \frac{1}{|F_n|} \log b(F_n, 1-\epsilon, \mathcal{U}).$$

5. A local variational principle of topological entropy

The main result of this section is

Theorem 5.1 *(Local variational principle of topological entropy). Let* $\mathcal{U} \in \mathcal{C}_X^o$. *Then*

$$h_{\text{top}}(G,\mathcal{U}) = \max_{\mu\in\mathcal{M}(X,G)} h_\mu(G,\mathcal{U}) = \max_{\mu\in\mathcal{M}^e(X,G)} h_\mu(G,\mathcal{U}).$$

We remark that Theorem 5.1 generalizes the results in [33,41]:

Theorem 5.2 *(Variational principle of topological entropy). (See [33,41].)*

$$h_{\text{top}}(G,X) = \sup_{\mu\in\mathcal{M}(X,G)} h_\mu(G,X) = \sup_{\mu\in\mathcal{M}^e(X,G)} h_\mu(G,X).$$

Proof. It is a direct corollary of Lemma 3.4(3), Theorems 3.5 and 5.1. □

Before proving Theorem 5.1, we need a key lemma.

Lemma 5.3. *Let* $\mathcal{U} \in \mathcal{C}_X^o$ *and* $\alpha_l \in \mathcal{P}_X$ *with* $\alpha_l \succcurlyeq \mathcal{U}$, $1 \leqslant l \leqslant K$. *Then for each* $F \in F(G)$ *there exists a finite subset* $B_F \subseteq X$ *such that each atom of* $(\alpha_l)_F$ *contains at most one point of* B_F, $l = 1, \ldots, K$ *and* $\#B_F \geqslant \frac{N(\mathcal{U}_F)}{K}$.

Proof. We follow the arguments in the proof of [24, Lemma 3.5]. Let $F \in F(G)$. For each $l = 1, \ldots, K$ and $x \in X$, let $A_l(x)$ be the atom of $(\alpha_l)_F$ containing x, then for $x_1, x_2 \in X$, x_1 and x_2 are contained in the same atom of $(\alpha_l)_F$ iff $A_l(x_1) = A_l(x_2)$.

To construct the subset B_F we first take any $x_1 \in X$. If $\bigcup_{l=1}^K A_l(x_1) = X$, then we take $B_F = \{x_1\}$. Otherwise, we take $X_1 = X \setminus \bigcup_{l=1}^K A_l(x_1) \neq \emptyset$ and take any $x_2 \in X_1$. If $\bigcup_{l=1}^K A_l(x_2) \supseteq X_1$, then we take $B_F = \{x_1, x_2\}$. Otherwise, we take $X_2 = X_1 \setminus \bigcup_{l=1}^K A_l(x_2) \neq \emptyset$. Since $\{A_l(x)\colon 1 \leqslant l \leqslant K,\ x \in X\}$ is a finite cover of X, we can continue the above procedure inductively to obtain a finite subset $B_F = \{x_1, \ldots x_m\}$ and non-empty subsets X_j, $j = 1, \ldots, m-1$ such that

(1) $X_1 = X \setminus \bigcup_{l=1}^K A_l(x_1)$,
(2) $X_{j+1} = X_j \setminus \bigcup_{l=1}^K A_l(x_{j+1})$ for $j = 1, \ldots, m-1$,
(3) $\bigcup_{j=1}^m \bigcup_{l=1}^K A_l(x_j) = X$.

From the construction of B_F, clearly each atom of $(\alpha_l)_F$, $l = 1, \ldots, K$, contains at most one point of B_F. Since for any $1 \leqslant i \leqslant m$ and $1 \leqslant l \leqslant K$, $A_l(x_i)$ is an atom of $(\alpha_l)_F$, and thus is contained in some element of $\mathcal{U}_F$, so $mK \geqslant N(\mathcal{U}_F)$ (using (3)), that is, $\#B_F = m \geqslant \frac{N(\mathcal{U}_F)}{K}$. □

Proposition 5.4. *Let $\mathcal{U} \in \mathcal{C}_X^o$. If X is zero-dimensional then there exists $\mu \in \mathcal{M}(X, G)$ satisfying*

$$h_\mu(G,\mathcal{U}) \geqslant h_{\mathrm{top}}(G,\mathcal{U}). \tag{5.1}$$

Proof. Let $\mathcal{U} = \{U_1, \dots, U_d\}$ and $\mathcal{U}^* = \{\alpha = \{A_1, \dots, A_d\} \in \mathcal{P}_X\colon\ A_m \subseteq U_m,\ m = 1, \dots, d\}$. Since X is zero-dimensional, the family of partitions in $\mathcal{U}^*$ consisting of clopen (closed and open) subsets, which are finer than $\mathcal{U}$, is countable. We let $\{\alpha_l\colon\ l \geqslant 1\}$ denote an enumeration of this family. Then $h_\nu(G,\mathcal{U}) = \inf_{l\in\mathbb{N}} h_\nu(G,\alpha_l)$ for each $\nu \in \mathcal{M}(X,G)$ by Lemma 3.7.

Let $\{F_n\}_{n\in\mathbb{N}}$ be a Følner sequence of G satisfying $|F_n| \geqslant n$ for each $n \in \mathbb{N}$ (obviously, such a sequence exists since $|G| = +\infty$). By Lemma 5.3, for each $n \in \mathbb{N}$ there exists a finite subset $B_n \subseteq X$ such that

$$\#B_n \geqslant \frac{N(\mathcal{U}_{F_n})}{n}, \tag{5.2}$$

and each atom of $(\alpha_l)_{F_n}$ contains at most one point of B_n, for each $l = 1, \dots, n$. Let

$$\nu_n = \frac{1}{\#B_n}\sum_{x\in B_n}\delta_x \quad \text{and} \quad \mu_n = \frac{1}{|F_n|}\sum_{g\in F_n} g\nu_n.$$

We can choose a sub-sequence $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\mu_{n_j} \to \mu$ in the weak*-topology of $\mathcal{M}(X)$ as $j \to +\infty$. It is not hard to check the invariance of μ, i.e. $\mu \in \mathcal{M}(X,G)$. Now we aim to show that μ satisfies (5.1). It suffices to show that $h_{\mathrm{top}}(G,\mathcal{U}) \leqslant h_\mu(G,\alpha_l)$ for each $l \in \mathbb{N}$.

Fix an $l \in \mathbb{N}$ and each $n > l$. Using (5.2) we know from the construction of B_n that

$$\log N(\mathcal{U}_{F_n}) - \log n \leqslant \log(\#B_n) = \sum_{x\in B_n} -\nu_n\big(\{x\}\big)\log\nu_n\big(\{x\}\big) = H_{\nu_n}\big((\alpha_l)_{F_n}\big). \tag{5.3}$$

On the other hand, for each $B \in F(G)$, using Lemma 3.1(3) one has

$$\begin{aligned}
\frac{1}{|F_n|}H_{\nu_n}\big((\alpha_l)_{F_n}\big) &\leqslant \frac{1}{|F_n|}\sum_{g\in F_n}\frac{1}{|B|}H_{\nu_n}\big((\alpha_l)_{Bg}\big) + \frac{|F_n\setminus\{g\in G\colon\ B^{-1}g\subseteq F_n\}|}{|F_n|}\cdot\log\#\alpha_l \\
&= \frac{1}{|B|\cdot|F_n|}\sum_{g\in F_n}H_{g\nu_n}\big((\alpha_l)_B\big) + \frac{|F_n\setminus\{g\in G\colon\ B^{-1}g\subseteq F_n\}|}{|F_n|}\cdot\log d \\
&\leqslant \frac{1}{|B|}H_{\mu_n}\big((\alpha_l)_B\big) + \frac{|F_n\setminus\{g\in G\colon\ B^{-1}g\subseteq F_n\}|}{|F_n|}\cdot\log d.
\end{aligned} \tag{5.4}$$

Now by dividing (5.3) on both sides by $|F_n|$, then combining it with (5.4) we obtain

$$\frac{1}{|F_n|}\log N(\mathcal{U}_{F_n}) \leqslant \frac{1}{|B|}H_{\mu_n}\big((\alpha_l)_B\big) + \frac{\log n}{|F_n|} + \frac{|F_n\setminus\{g\in G\colon\ B^{-1}g\subseteq F_n\}|}{|F_n|}\cdot\log d. \tag{5.5}$$

Noting that $\lim_{j\to+\infty} H_{\mu_{n_j}}((\alpha_l)_B) = H_\mu((\alpha_l)_B)$, by substituting n with n_j in (5.5) one has

$$h_{\mathrm{top}}(G,\mathcal{U}) \leqslant \frac{1}{|B|}H_\mu\big((\alpha_l)_B\big) \quad \big(\text{using (3.6)}\big).$$

Now, taking the infimum over $B \in F(G)$, we get $h_{\mathrm{top}}(G,\mathcal{U}) \leqslant h_\mu(G,\alpha_l)$. This ends the proof. □

A continuous map $\pi : (X, G) \to (Y, G)$ is called a *homomorphism* or a *factor map* if it is onto and $\pi \circ g = g \circ \pi$ for each $g \in G$. In this case, (X, G) is called an *extension of* (Y, G) and (Y, G) is called a *factor of* (X, G). If π is also injective then it is called an *isomorphism*.

Proof of Theorem 5.1. First, by Lemma 3.4(1) and Theorem 4.14, it suffices to prove $h_\theta(G,\mathcal{U}) \geqslant h_{\mathrm{top}}(G,\mathcal{U})$ for some $\theta \in \mathcal{M}^e(X, G)$. It is well known that there exists a surjective continuous map $\phi_1 : C \to X$, where C is a cantor set. Let C^G be the product space equipped with the G-shift $G \times C^G \to C^G$, $(g', (z_g)_{g\in G}) \mapsto (z'_g)_{g\in G}$ where $z'_g = z_{g'g}$, $g', g \in G$. Define

$$Z = \bigl\{\bar{z} = (z_g)_{g\in G} \in C^G\colon\ \phi_1(z_{g_1 g_2}) = g_1\phi_1(z_{g_2}) \text{ for each } g_1, g_2 \in G\bigr\},$$

and $\varphi : Z \to X$, $(z_g)_{g\in G} \mapsto \phi_1(z_{e_G})$. It's not hard to check that $Z \subseteq C^G$ is a closed invariant subset under the G-shift. Moreover, $\varphi : (Z, G) \to (X, G)$ becomes a factor map between G-systems. Applying Proposition 5.4 to the G-system (Z, G), there exists $\nu \in \mathcal{M}(Z, G)$ with $h_\nu(G, \varphi^{-1}(\mathcal{U})) \geqslant h_{\mathrm{top}}(G, \varphi^{-1}(\mathcal{U})) = h_{\mathrm{top}}(G,\mathcal{U})$. Let $\mu = \varphi\nu \in \mathcal{M}(X, G)$. Then

$$\begin{aligned} h_\mu(G,\mathcal{U}) &= \inf_{\alpha\in\mathcal{P}_X\colon \alpha\succeq\mathcal{U}} h_\mu(G,\alpha) \\ &= \inf_{\alpha\in\mathcal{P}_X\colon \alpha\succeq\mathcal{U}} h_\nu\bigl(G,\varphi^{-1}(\alpha)\bigr) \geqslant h_\nu\bigl(G,\varphi^{-1}(\mathcal{U})\bigr) \geqslant h_{\mathrm{top}}(G,\mathcal{U}). \end{aligned}$$

Let $\mu = \int_{\mathcal{M}^e(X,T)} \theta\, dm(\theta)$ be the ergodic decomposition of μ. Then by Theorem 3.13 one has

$$\int_{\mathcal{M}^e(X,T)} h_\theta(G,\mathcal{U})\, dm(\theta) = h_\mu(G,\mathcal{U}).$$

Hence, $h_\theta(G,\mathcal{U}) \geqslant h_{\mathrm{top}}(G,\mathcal{U})$ for some $\theta \in \mathcal{M}^e(X, G)$. This ends the proof. □

At last, we ask an open question.

Question 5.5. *In the proof of* [19, Proposition 7.10] (*or its relative version* [24, Theorem A.3]), *a universal version of the well-known Rohlin Lemma* [19, Proposition 7.9] *plays a key role. Thus, a natural open question arises*: *for actions of a countable discrete amenable group, are there a universal version of Rohlin Lemma and a similar result to* [19, Proposition 7.10] *or* [24, Theorem A.3]? *Whereas, up to now they still stand as open questions.*

6. Entropy tuples

In this section we will firstly introduce entropy tuples in both topological and measure-theoretic settings. Then we characterize the set of entropy tuples for an invariant measure as the support of some specific relative product measure. Finally by the lift property of entropy tuples, we will establish the variational relation of entropy tuples. At the same time, we also discuss entropy tuples of a finite product. We need to mention that the proof of those results in this section is similar to the proof of corresponding results in [23,25] for the case $G=\mathbb{Z}$, but for completion we provide the detailed proof.

6.1. Topological entropy tuples

First we are going to define the topological entropy tuples.

Let $n \geqslant 2$. Set $X^{(n)} = X \times \cdots \times X$ (n-times); $\Delta_n(X) = \{(x_i)_1^n \in X^{(n)} \mid x_1 = \cdots = x_n\}$, the *$n$-th diagonal* of X. Let $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$. We say $\mathcal{U} \in \mathcal{C}_X$ *admissible w.r.t.* $(x_i)_1^n$, if for any $U \in \mathcal{U}$, $\overline{U} \not\supseteq \{x_1, \ldots, x_n\}$.

Definition 6.1. Let $n \geqslant 2$. $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$ is called a *topological entropy n-tuple* if $h_{\mathrm{top}}(G,\mathcal{U}) > 0$ when $\mathcal{U} \in \mathcal{C}_X$ is admissible w.r.t. $(x_i)_1^n$.

Remark 6.2. We may replace all admissible finite covers by admissible finite open or closed covers in the definition. Moreover, we can choose all covers to be of the forms $\mathcal{U} = \{U_1, \ldots, U_n\}$, where U_i^c is a neighborhood of x_i, $1 \leqslant i \leqslant n$ such that if $x_i \neq x_j$, $1 \leqslant i < j \leqslant n$ then $U_i^c \cap U_j^c = \emptyset$. Thus, our definition of topological entropy n-tuples is the same as the one defined by Kerr and Li in [27].

For each $n \geqslant 2$, denote by $E_n(X,G)$ the set of all topological entropy n-tuples. Then following the ideas of [2] we obtain directly

Proposition 6.3. *Let $n \geqslant 2$.*

1. *If $\mathcal{U} = \{U_1, \ldots, U_n\} \in \mathcal{C}_X^o$ satisfies $h_{\mathrm{top}}(G,\mathcal{U}) > 0$, then $E_n(X,G) \cap \bigcap_{i=1}^n U_i^c \neq \emptyset$.*
2. *If $h_{\mathrm{top}}(G,X) > 0$, then $\emptyset \neq \overline{E_n(X,G)} \subseteq X^{(n)}$ is G-invariant. Moreover, $\overline{E_n(X,G)} \setminus \Delta_n(X) = E_n(X,G)$.*
3. *Let $\pi : (Z,G) \to (X,G)$ be a factor map between G-systems. Then*

$$E_n(X,G) \subseteq (\pi \times \cdots \times \pi) E_n(Z,G) \subseteq E_n(X,G) \cup \Delta_n(X).$$

4. *Let (W,G) be a sub-G-system of (X,G). Then $E_n(W,G) \subseteq E_n(X,G)$.*

The notion of disjointness of two TDSs was introduced in [15]. Blanchard proved that any u.p.e. TDS was disjoint from all minimal TDSs with zero topological entropy (see [2, Proposition 6]). This is also true for actions of a countable discrete amenable group. First we introduce

Definition 6.4. Let $n \geqslant 2$. We say that

(1) (X,G) has *u.p.e. of order n*, if any cover of X by n non-dense open sets has positive topological entropy. When $n = 2$, we say simply that (X,G) has *u.p.e.*;

(2) (X,G) has *u.p.e. of all orders* or *topological K* if any cover of X by finite non-dense open sets has positive topological entropy, equivalently, it has u.p.e. of order m for any $m \geqslant 2$.

Thus, for each $n \geqslant 2$, (X,G) has u.p.e. of order n iff $E_n(X,G) = X^{(n)} \setminus \Delta_n(X)$.

We say (X, G) is *minimal* if it contains properly no other sub-G-systems. Let (X, G) and (Y, G) be two G-systems and $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ the natural projections. $J \subseteq X \times Y$ is called a *joining of* (X, G) *and* (Y, G) if J is a G-invariant closed subset satisfying $\pi_X(J) = X$ and $\pi_Y(J) = Y$. Clearly, $X \times Y$ is always a joining of (X, G) and (Y, G). We say that (X, G) and (Y, G) are *disjoint* if $X \times Y$ is the unique joining of (X, G) and (Y, G). The proof of the following theorem is similar to that of [2, Proposition 6] or [25, Theorem 2.5].

Theorem 6.5. *Let (X, G) be a G-system having u.p.e. and (Y, G) a minimal G-system with zero topological entropy. Then (X, G) and (Y, G) are disjoint.*

6.2. Measure-theoretic entropy tuples

Now we aim to define the measure-theoretic entropy tuples for an invariant Borel probability measure.

Let $\mu \in \mathcal{M}(X, G)$. $A \subseteq X$ is called a *μ-set* if $A \in \mathcal{B}_X^\mu$. If $\alpha = \{A_1, \dots, A_k\} \subseteq \mathcal{B}_X^\mu$ satisfies $\bigcup_{i=1}^k A_i = X$ and $A_i \cap A_j = \emptyset$ when $1 \leqslant i < j \leqslant k$ then we say α is a *finite μ-measurable partition of* X. Denote by $\mathcal{P}_X^\mu$ the set of all finite μ-measurable partitions of X. Similarly, we can introduce $\mathcal{C}_X^\mu$ and define $\alpha_1 \succcurlyeq \alpha_2$ for $\alpha_1, \alpha_2 \in \mathcal{C}_X^\mu$ and so on.

Definition 6.6. Let $n \geqslant 2$. $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$ is called a *measure-theoretic entropy n-tuple for* μ if $h_\mu(G, \alpha) > 0$ for any admissible $\alpha \in \mathcal{P}_X$ w.r.t. $(x_i)_1^n$.

Remark 6.7. We may replace all admissible $\alpha \in \mathcal{P}_X$ by all admissible $\alpha \in \mathcal{P}_X^\mu$ in the definition.

For each $n \geqslant 2$, denote by $E_n^\mu(X, G)$ the set of all measure-theoretic entropy n-tuples for $\mu \in \mathcal{M}(X, G)$. In the following, we shall investigate the structure of $E_n^\mu(X, G)$. To this purpose, let P_μ be the Pinsker σ-algebra of $(X, \mathcal{B}_X^\mu, \mu, G)$, i.e. $P_\mu = \{A \in \mathcal{B}_X^\mu\colon\ h_\mu(G, \{A, A^c\}) = 0\}$. We define a measure $\lambda_n(\mu)$ on $(X^{(n)}, (\mathcal{B}_X^\mu)^{(n)}, G)$ by letting

$$\lambda_n(\mu)\left(\prod_{i=1}^n A_i\right) = \int_X \prod_{i=1}^n \mathbb{E}(1_{A_i} | P_\mu)\, d\mu,$$

where $(\mathcal{B}_X^\mu)^{(n)} = \mathcal{B}_X^\mu \times \cdots \times \mathcal{B}_X^\mu$ (n times) and $A_i \in \mathcal{B}_X^\mu$, $i = 1, \dots, n$. First we need

Lemma 6.8. *Let $\mathcal{U} = \{U_1, \dots, U_n\} \in \mathcal{C}_X$. Then $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) > 0$ iff for any finite* (*or n-set*) *μ-measurable partition α, finer than $\mathcal{U}$ as a cover, one has $h_\mu(G, \alpha) > 0$.*

Proof. First we assume that for any finite (or n-set) μ-measurable partition α, finer than $\mathcal{U}$ as a cover, one has $h_\mu(G, \alpha) > 0$ and $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) = 0$. For $i = 1, \dots, n$, let $C_i = \{x \in X\colon\ \mathbb{E}(1_{U_i^c} | P_\mu)(x) > 0\} \in P_\mu$, and put $D_i = C_i \cup (U_i^c \setminus C_i)$, $D_i(0) = D_i$ and $D_i(1) = D_i^c$, as

$$0 = \int_{X \setminus C_i} \mathbb{E}(1_{U_i^c} | P_\mu)(x)\, d\mu = \mu\big(U_i^c \cap (X \setminus C_i)\big) = \mu\big(U_i^c \setminus C_i\big),$$

then $D_i^c \subseteq U_i$ and $D_i(0), D_i(1) \in P_\mu$. For any $s = (s(1), \ldots, s(n)) \in \{0,1\}^n$, let $D_s = \bigcap_{i=1}^n D_i(s(i))$ and set $D_0^j = (\bigcap_{i=1}^n D_i) \cap (U_j \setminus \bigcup_{k=1}^{j-1} U_k)$ for $j = 1, \ldots, n$. We consider

$$\alpha = \{D_s\colon s \in \{0,1\}^n \text{ and } s \neq (0, \ldots, 0)\} \cup \{D_0^1, \ldots, D_0^n\}.$$

On one hand, for any $s \in \{0,1\}^n$ with $s \neq (0, \ldots, 0)$, one has $s(i) = 1$ for some $1 \leqslant i \leqslant n$, then $D_s \subseteq D_i^c \subseteq U_i$. Note that $D_0^j \subseteq U_j$, $j = 1, \ldots, n$, thus $\alpha \succcurlyeq \mathcal{U}$ and so $h_\mu(G, \alpha) > 0$. On the other hand, obviously $\mu(\bigcap_{i=1}^n D_i) = \mu(\bigcap_{i=1}^n C_i)$ and

$$0 = \lambda_n(\mu)\left(\prod_{i=1}^n U_i^c\right) = \int_{\bigcap_{i=1}^n C_i} \prod_{i=1}^n \mathbb{E}(1_{U_i^c}|P_\mu)(x)\,d\mu(x),$$

then $\mu(\bigcap_{i=1}^n C_i) = 0$, and so $D_0^1, \ldots, D_0^n \in P_\mu$. As $D_1, \ldots, D_n \in P_\mu$, $D_s \in P_\mu$ for each $s \in \{0,1\}^n$, thus $\alpha \subseteq P_\mu$, one gets $h_\mu(G, \alpha) = 0$, a contradiction.

Now we assume $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) > 0$. For any finite (or n-set) μ-measurable partition α which is finer than $\mathcal{U}$, with no loss of generality we assume $\alpha = \{A_1, \ldots, A_n\}$ with $A_i \subseteq U_i$, $i = 1, \ldots, n$. As

$$\int_X \prod_{i=1}^n \mathbb{E}(1_{X\setminus A_i}|P_\mu)(x)\,d\mu(x) \geqslant \int_X \prod_{i=1}^n \mathbb{E}(1_{U_i^c}|P_\mu)(x)\,d\mu(x) = \lambda_n(\mu)\left(\prod_{i=1}^n U_i^c\right) > 0,$$

therefore $A_j \notin P_\mu$ for every $1 \leqslant j \leqslant n$, and so $h_\mu(G, \alpha) > 0$. This finishes the proof. □

Then we have (we remark that the case of $G = \mathbb{Z}$ is proved in [16] and [23]):

Theorem 6.9. *Let $n \geqslant 2$ and $\mu \in \mathcal{M}(X, G)$. Then $E_n^\mu(X, G) = \operatorname{supp}(\lambda_n(\mu)) \setminus \Delta_n(X)$.*

Proof. Let $(x_i)_1^n \in E_n^\mu(X, G)$. To show $(x_i)_1^n \in \operatorname{supp}(\lambda_n(\mu)) \setminus \Delta_n(X)$, it remains to prove that for any Borel neighborhood $\prod_{i=1}^n U_i$ of $(x_i)_1^n$ in $X^{(n)}$, $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$. Set $\mathcal{U} = \{U_1^c, \ldots, U_n^c\}$. With no loss of generality we assume $\mathcal{U} \in \mathcal{C}_X$ (if necessary we replace U_i by a smaller Borel neighborhood of x_i, $1 \leqslant i \leqslant n$). It is clear that if $\alpha \in \mathcal{P}_X^\mu$ is finer than $\mathcal{U}$ then α is admissible w.r.t. $(x_i)_1^n$, and so $h_\mu(G, \alpha) > 0$. Using Lemma 6.8 one has $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$.

Now let $(x_i)_1^n \in \operatorname{supp}(\lambda_n(\mu)) \setminus \Delta_n(X)$. We shall show that $h_\mu(G, \alpha) > 0$ for any admissible $\alpha = \{A_1, \ldots, A_k\} \in \mathcal{P}_X$ w.r.t. $(x_i)_1^n$. In fact, let α be such a partition. Then there exists a neighborhood U_l of x_l, $1 \leqslant l \leqslant n$ such that for each $i \in \{1, \ldots, k\}$ we find $j_i \in \{1, \ldots, n\}$ with $A_i \subseteq U_{j_i}^c$, i.e. $\alpha \succcurlyeq \mathcal{U} = \{U_1^c, \ldots, U_n^c\}$. As $(x_i)_1^n \in \operatorname{supp}(\lambda_n(\mu)) \setminus \Delta_n(X)$, $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$ and so $h_\mu(G, \alpha) > 0$ (see Lemma 6.8). This ends the proof. □

Before proceeding we also need

Theorem 6.10. *(See [7, Theorem 0.1].) Let $\mu \in \mathcal{M}(X, G)$, $\alpha \in \mathcal{P}_X^\mu$ and $\epsilon > 0$. Then there exists $K \in F(G)$ such that if $F \in F(G)$ satisfies $(FF^{-1} \setminus \{e_G\}) \cap K = \emptyset$ then*

$$\left|\frac{1}{|F|} H_\mu(\alpha_F|P_\mu) - H_\mu(\alpha|P_\mu)\right| < \epsilon.$$

The following theorem is crucial for this section of our paper, and the methods of proving it may be useful in other settings as well.

Theorem 6.11. *Let $\mu \in \mathcal{M}(X,G)$ and $\mathcal{U} = \{U_1, \dots, U_n\} \in \mathcal{C}_X^\mu$, $n \geqslant 2$. If $h_\mu(G,\alpha) > 0$ for any finite (or n-set) μ-measurable partition α, finer than $\mathcal{U}$, then $h_\mu^-(G,\mathcal{U}) > 0$.*

Proof. For any $s = (s(1), \dots, s(n)) \in \{0,1\}^n$, set $A_s = \bigcap_{i=1}^n U_i(s(i))$, where $U_i(0) = U_i$ and $U_i(1) = U_i^c$. Let $\alpha = \{A_s: s \in \{0,1\}^n\}$. Note that $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) = \int_X \prod_{i=1}^n \mathbb{E}(1_{U_i^c}|P_\mu)\,d\mu > 0$ (Lemma 6.8), hence there exists $M \in \mathbb{N}$ such that $\mu(D) > 0$, where

$$D = \left\{x \in X\colon \min_{1\leqslant i\leqslant n} \mathbb{E}(1_{U_i^c}|P_\mu)(x) \geqslant \frac{1}{M}\right\}.$$

Claim. *If $\beta \in \mathcal{P}_X^\mu$ is finer than $\mathcal{U}$ then $H_\mu(\alpha|\beta \vee P_\mu) \leqslant H_\mu(\alpha|P_\mu) - \frac{\mu(D)}{M}\log(\frac{n}{n-1})$.*

Proof. With no loss of generality we assume $\beta = \{B_1, \dots, B_n\}$ with $B_i \subseteq U_i$, $i = 1, \dots, n$. Then

$$\begin{aligned}
H_\mu(\alpha|\beta \vee P_\mu) &= H_\mu(\alpha \vee \beta|P_\mu) - H_\mu(\beta|P_\mu) \\
&= \int_X \sum_{s\in\{0,1\}^n} \sum_{i=1}^n \mathbb{E}(1_{B_i}|P_\mu)\phi\left(\frac{\mathbb{E}(1_{A_s\cap B_i}|P_\mu)}{\mathbb{E}(1_{B_i}|P_\mu)}\right)d\mu \\
&= \sum_{s\in\{0,1\}^n} \int_X \sum_{1\leqslant i\leqslant n,\, s(i)=0} \mathbb{E}(1_{B_i}|P_\mu)\phi\left(\frac{\mathbb{E}(1_{A_s\cap B_i}|P_\mu)}{\mathbb{E}(1_{B_i}|P_\mu)}\right)d\mu,
\end{aligned} \tag{6.1}$$

where the last equality comes from the fact that, for any $s \in \{0,1\}^n$ and $1 \leqslant i \leqslant n$, if $s(i) = 1$ then $A_s \cap B_i = \emptyset$ and so $\frac{\mathbb{E}(1_{A_s\cap B_i}|P_\mu)}{\mathbb{E}(1_{B_i}|P_\mu)}(x) = 0$ for μ-a.e. $x \in X$. Put $c_s = \sum_{1\leqslant k\leqslant n, s(k)=0} \mathbb{E}(1_{B_k}|P_\mu)$. As ϕ is a concave function,

$$\begin{aligned}
(6.1) &\leqslant \sum_{s\in\{0,1\}^n} \int_X c_s \cdot \phi\left(\sum_{1\leqslant i\leqslant n,\, s(i)=0} \frac{\mathbb{E}(1_{B_i}|P_\mu)}{c_s} \cdot \frac{\mathbb{E}(1_{A_s\cap B_i}|P_\mu)}{\mathbb{E}(1_{B_i}|P_\mu)}\right)d\mu \\
&= \sum_{s\in\{0,1\}^n} \int_X c_s \cdot \phi\left(\frac{\mathbb{E}(1_{A_s}|P_\mu)}{c_s}\right)d\mu \\
&= \sum_{s\in\{0,1\}^n} \left(\int_X \phi\big(\mathbb{E}(1_{A_s}|P_\mu)\big)\,d\mu - \int_X \mathbb{E}(1_{A_s}|P_\mu)\log\frac{1}{c_s}\,d\mu\right) \\
&= H_\mu(\alpha|P_\mu) - \sum_{s\in\{0,1\}^n} \int_X \mathbb{E}(1_{A_s}|P_\mu)\log\frac{1}{c_s}\,d\mu.
\end{aligned} \tag{6.2}$$

Note that if $s(i) = 1$, $1 \leqslant i \leqslant n$, then $\sum_{1\leqslant k\leqslant n,\, s(k)=0} \mathbb{E}(1_{B_k}|P_\mu) \leqslant \mathbb{E}(1_{X\setminus B_i}|P_\mu)$; moreover, $(\frac{b_1+\dots+b_n}{n})^n \geqslant b_1 \cdots b_n$ and $\sum_{i=1}^n b_i = \sum_{i=1}^n \sum_{1\leqslant j\leqslant n,\, j\neq i} \mathbb{E}(1_{B_j}|P_\mu) = (n-1)\sum_{i=1}^n \mathbb{E}(1_{B_i}|P_\mu) = n-1$, here $b_i = \mathbb{E}(1_{X\setminus B_i}|P_\mu)$, $i = 1, \dots, n$. Then we have

$$\sum_{s\in\{0,1\}^n}\int_X \mathbb{E}(1_{A_s}|P_\mu)\log\left(\frac{1}{\sum_{1\leqslant k\leqslant n,\, s(k)=0}\mathbb{E}(1_{B_k}|P_\mu)}\right)d\mu$$

$$\geqslant \frac{1}{n}\sum_{i=1}^n\int_X\left(\sum_{s\in\{0,1\}^n,\, s(i)=1}\mathbb{E}(1_{A_s}|P_\mu)\right)\log\frac{1}{b_i}\,d\mu$$

$$=\frac{1}{n}\sum_{i=1}^n\int_X \mathbb{E}(1_{U_i^c}|P_\mu)\log\frac{1}{b_i}\,d\mu \geqslant \frac{1}{nM}\sum_{i=1}^n\int_D\log\frac{1}{b_i}\,d\mu$$

$$=\frac{1}{nM}\int_D\log\frac{1}{\prod_{i=1}^n b_i}\,d\mu \geqslant \frac{1}{M}\int_D\log\frac{n}{\sum_{i=1}^n b_i}\,d\mu=\frac{\mu(D)}{M}\log\left(\frac{n}{n-1}\right). \tag{6.3}$$

Hence, $H_\mu(\alpha|\beta\vee P_\mu)\leqslant H_\mu(\alpha|P_\mu)-\frac{\mu(D)}{M}\log(\frac{n}{n-1})$ (using (6.2) and (6.3)). □

Set $\epsilon=\frac{\mu(D)}{M}\log(\frac{n}{n-1})>0$. By Theorem 6.10, there exists $K\in F(G)$ such that

$$\left|\frac{1}{|F|}H_\mu(\alpha_F|P_\mu)-H_\mu(\alpha|P_\mu)\right|<\frac{\epsilon}{2} \tag{6.4}$$

when $F\in F(G)$ satisfies $(FF^{-1}\setminus\{e_G\})\cap K=\emptyset$. Let $\{F_m\}_{m\in\mathbb{N}}$ be a Følner sequence of G. For each $m\in\mathbb{N}$, we can take $E_m\subseteq F_m$ such that $(E_mE_m^{-1}\setminus\{e_G\})\cap K=\emptyset$ and $|E_m|\geqslant\frac{|F_m|}{2|K|+1}$. Now if $\beta_m\in\mathcal{C}_X^\mu$ is finer than $\mathcal{U}_{F_m}$ then $g\beta_m\succcurlyeq\mathcal{U}$ for each $g\in F_m$, and so

$$\begin{aligned}H_\mu(\beta_m)&\geqslant H_\mu(\beta_m\vee\alpha_{E_m}|P_\mu)-H_\mu(\alpha_{E_m}|\beta_m\vee P_\mu)\\&\geqslant H_\mu(\alpha_{E_m}|P_\mu)-\sum_{g\in E_m}H_\mu(\alpha|g\beta_m\vee P_\mu)\\&\geqslant H_\mu(\alpha_{E_m}|P_\mu)-|E_m|\big(H_\mu(\alpha|P_\mu)-\epsilon\big)\quad\text{(by Claim)}\\&\geqslant|E_m|\frac{\epsilon}{2}\quad\big(\text{by the selection of }E_m\text{ and applying (6.4) to }E_m\big).\end{aligned}$$

Hence, $H_\mu(\mathcal{U}_{F_m})\geqslant|E_m|\frac{\epsilon}{2}$ and so $h_\mu^-(G,\mathcal{U})\geqslant\frac{\epsilon}{2(2|K|+1)}$. This finishes the proof of the theorem. □

An immediate consequence of Lemma 6.8 and Theorem 6.11 is

Corollary 6.12. *Let $\mu\in\mathcal{M}(X,G)$ and $\mathcal{U}=\{U_1,\ldots,U_n\}\in\mathcal{C}_X^\mu$. Then the following statements are equivalent*:

1. $h_\mu^-(G,\mathcal{U})>0$, *equivalently*, $h_\mu(G,\mathcal{U})>0$;
2. $h_\mu(G,\alpha)>0$ *if* $\alpha\in\mathcal{C}_X^\mu$ *is finer than* $\mathcal{U}$;
3. $\lambda_n(\mu)(\prod_{i=1}^n U_i^c)>0$.

Now with the help of Theorem 3.13 and Corollary 6.12 we can obtain Theorem 6.13 which discloses the relation of entropy tuples for an invariant measure and entropy tuples for ergodic measures in its ergodic decomposition, generalizing [3, Theorem 4] and [23, Theorem 4.9].

Theorem 6.13. *Let $\mu \in \mathcal{M}(X,G)$ with $\mu = \int_\Omega \mu_\omega \, dm(\omega)$ the ergodic decomposition of μ. Then*

1. *for m-a.e. $\omega \in \Omega$, $E_n^{\mu_\omega}(X,G) \subseteq E_n^\mu(X,G)$ for each $n \geqslant 2$;*
2. *if $(x_i)_1^n \in E_n^\mu(X,G)$, then for every measurable neighborhood V of $(x_i)_1^n$, $m(\{\omega \in \Omega\colon V \cap E_n^{\mu_\omega}(X,G) \neq \emptyset\}) > 0$. Thus for an appropriate choice of Ω, we can require*

$$\overline{\bigcup\left\{E_n^{\mu_\omega}(X,G)\colon \omega \in \Omega\right\}} \setminus \Delta_n(X) = E_n^\mu(X,G).$$

Proof. 1. It suffices to prove the conclusion for each given $n \geqslant 2$. Let $n \geqslant 2$ be fixed.

Let U_i, $i = 1, \ldots, n$ be open subsets of X with $\bigcap_{i=1}^n \overline{U_i} = \emptyset$ and $(\prod_{i=1}^n \overline{U_i}) \cap E_n^\mu(X,G) = \emptyset$. Then $\lambda_n(\mu)(\prod_{i=1}^n \overline{U_i}) = 0$ by Theorem 6.9, and so $h_\mu(G,\mathcal{U}) = 0$ by Corollary 6.12, where $\mathcal{U} = \{U_1^c, \ldots, U_n^c\}$. As $\int_\Omega h_{\mu_\omega}(G,\mathcal{U})\, dm(\omega) = h_\mu(G,\mathcal{U}) = 0$ (see (3.29)), for m-a.e. $\omega \in \Omega$, $h_{\mu_\omega}(G,\mathcal{U}) = 0$ and so $\lambda_n(\mu_\omega)(\prod_{i=1}^n U_i) = 0$ by Corollary 6.12, hence $(\prod_{i=1}^n U_i) \cap E_n^{\mu_\omega}(X, G) = \emptyset$ (using Theorem 6.9 and the assumption of $\bigcap_{i=1}^n \overline{U_i} = \emptyset$).

Since $E_n^\mu(X,G) \cup \Delta_n(X) \subseteq X^{(n)}$ is closed, its complement can be written as a union of countable sets of the form $\prod_{i=1}^n U_i$ with U_i, $i = 1, \ldots, n$ open subsets satisfying $\bigcap_{i=1}^n \overline{U_i} = \emptyset$. Then applying the above procedure to each such a subset $\prod_{i=1}^n U_i$ one has that for m-a.e. $\omega \in \Omega$, $E_n^{\mu_\omega}(X,G) \cap (E_n^\mu(X,T))^c = \emptyset$, equivalently, $E_n^{\mu_\omega}(X,G) \subseteq E_n^\mu(X,T)$.

2. With no loss of generality we assume $V = \prod_{i=1}^n A_i$, where A_i is a closed neighborhood of x_i, $1 \leqslant i \leqslant n$ and $\bigcap_{i=1}^n A_i = \emptyset$. As $\lambda_n(\mu)(\prod_{i=1}^n A_i) > 0$ by Theorem 6.9, one has

$$\begin{aligned} \int_\Omega h_{\mu_\omega}\left(T, \left\{A_1^c, \ldots, A_n^c\right\}\right) dm(\omega) &= h_\mu\left(T, \left\{A_1^c, \ldots, A_n^c\right\}\right) \\ &> 0 \quad \left(\text{using (3.29) and Corollary 6.12}\right), \end{aligned}$$

there exists $\Omega' \subseteq \Omega$ with $m(\Omega') > 0$ such that if $\omega \in \Omega'$ then

$$h_{\mu_\omega}\left(G, \left\{A_1^c, \ldots, A_n^c\right\}\right) > 0, \quad \text{i.e.} \quad \lambda_n(\mu_\omega)\left(\prod_{i=1}^n A_i\right) > 0 \quad \text{(see Corollary 6.12)},$$

and so $(\prod_{i=1}^n A_i) \cap E_n^{\mu_\omega}(X,G) \neq \emptyset$ (see Theorem 6.9), i.e. $m(\{\omega \in \Omega\colon V \cap E_n^{\mu_\omega}(X,G) \neq \emptyset\}) > 0$. □

Lemma 6.14. *Let $\pi : (X,G) \to (Y,G)$ be a factor map between G-systems, $\mathcal{U} \in \mathcal{C}_Y$ and $\mu \in \mathcal{M}(X,G)$. Then $h_\mu^-(G,\pi^{-1}\mathcal{U}) = h_{\pi\mu}^-(G,\mathcal{U})$.*

Proof. Note that, for each $F \in F(G)$, $P((\pi^{-1}\mathcal{U})_F) = \pi^{-1}P(\mathcal{U}_F)$, using (3.19) we have

$$\begin{aligned} H_{\pi\mu}(\mathcal{U}_F) &= \inf_{\beta \in P(\mathcal{U}_F)} H_{\pi\mu}(\beta) = \inf_{\beta \in P(\mathcal{U}_F)} H_\mu\left(\pi^{-1}\beta\right) \\ &= \inf_{\beta' \in P((\pi^{-1}\mathcal{U})_F)} H_\mu\left(\beta'\right) = H_\mu\left(\left(\pi^{-1}\mathcal{U}\right)_F\right). \end{aligned} \tag{6.5}$$

Then the lemma immediately follows when divide $|F|$ on both sides of (6.5) and then let F range over a fixed Følner sequence of G. □

Then we have

Theorem 6.15. *Let $\pi : (X, G) \to (Y, G)$ be a factor map between G-systems, $\mu \in \mathcal{M}(X, G)$. Then*

$$E_n^{\pi\mu}(Y, G) \subseteq (\pi \times \cdots \times \pi)E_n^{\mu}(X, G) \subseteq E_n^{\pi\mu}(Y, G) \cup \Delta_n(Y) \quad \text{for each } n \geqslant 2.$$

Proof. The second inclusion follows directly from the definition. For the first inclusion, we assume $(y_1, \ldots, y_n) \in E_n^{\pi\mu}(Y, G)$. For $m \in \mathbb{N}$, take a closed neighborhood V_i^m of y_i, $i = 1, \ldots, n$ with diameter at most $\frac{1}{m}$ such that $\bigcap_{i=1}^n V_i^m = \emptyset$. Consider $\mathcal{U}_m = \{(V_1^m)^c, \ldots, (V_n^m)^c\} \in \mathcal{C}_Y^o$, then $h_\mu^-(G, \pi^{-1}\mathcal{U}_m) = h_{\pi\mu}^-(G, \mathcal{U}_m) > 0$ and so $\lambda_n(\mu)(\prod_{i=1}^n \pi^{-1} V_i^m) > 0$ by Corollary 6.12 and Lemma 6.14. Hence $\prod_{i=1}^n \pi^{-1} V_i^m \cap (\operatorname{supp}(\lambda_n(\mu)) \setminus \Delta_n(X)) \neq \emptyset$. Moreover, there exists $(x_i^m)_1^n \in \prod_{i=1}^n \pi^{-1} V_i^m \cap E_n^{\mu}(X, G)$ by Theorem 6.9. We may assume $(x_1^m, \ldots, x_n^m) \to (x_1, \ldots, x_n)$ (if necessary we take a sub-sequence). Clearly, $x_i \in \pi^{-1}(y_i)$, $i = 1, \ldots, n$ and $(x_1, \ldots, x_n) \in E_n^{\mu}(X, G)$ by Proposition 6.3(2). This finishes the proof of the theorem. □

6.3. A variational relation of entropy tuples

Now we are ready to show the variational relation of topological and measure-theoretic entropy tuples.

Theorem 6.16. *Let (X, G) be a G-system. Then*

1. *for each $\mu \in \mathcal{M}(X, G)$ and each $n \geqslant 2$, $E_n(X, G) \supseteq E_n^{\mu}(X, G) = \operatorname{supp}(\lambda_n(\mu)) \setminus \Delta_n(X)$;*
2. *there exists $\mu \in \mathcal{M}(X, G)$ such that $E_n(X, G) = E_n^{\mu}(X, G)$ for each $n \geqslant 2$.*

Proof. 1. Let $(x_i)_{i=1}^n \in E_n^{\mu}(X, G)$ and $\mathcal{U} \in \mathcal{C}_X^o$ admissible w.r.t. $(x_i)_{i=1}^n$. Then if $\alpha \in \mathcal{P}_X$ is finer than $\mathcal{U}$ then it is also admissible w.r.t. $(x_i)_{i=1}^n$, and so $h_\mu(G, \alpha) > 0$ (as $(x_i)_1^n \in E_n^{\mu}(X, G)$), thus $h_\mu^-(G, \mathcal{U}) > 0$ by Theorem 6.11. Moreover, $h_{\mathrm{top}}(G, \mathcal{U}) \geqslant h_\mu^-(G, \mathcal{U}) > 0$. That is, $(x_i)_{i=1}^n \in E_n(X, G)$, as $\mathcal{U}$ is arbitrary.

2. Let $n \geqslant 2$. First we have

Claim. *If $(x_i)_1^n \in E_n(X, G)$ and $\prod_{i=1}^n U_i$ is a neighborhood of $(x_i)_1^n$ in $X^{(n)}$ then $E_n^{\nu}(X, G) \cap \prod_{i=1}^n U_i \neq \emptyset$ for some $\nu \in \mathcal{M}(X, G)$.*

Proof. With no loss of generality we assume that U_i is a closed neighborhood of x_i, $1 \leqslant i \leqslant n$ such that $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$ and $U_i = U_j$ if $x_i = x_j$, $1 \leqslant i < j \leqslant n$. Let $\mathcal{U} = \{U_1^c, \ldots, U_n^c\}$. Then $h_{\mathrm{top}}(G, \mathcal{U}) > 0$ (as $(x_i)_1^n \in E_n(X, G)$). By Theorem 5.1, there exists $\nu \in M(X, G)$ such that $h_\nu(G, \mathcal{U}) = h_{\mathrm{top}}(G, \mathcal{U})$, then $\lambda_n(\nu)(\prod_{i=1}^n U_i) > 0$ by Corollary 6.12, i.e. $\operatorname{supp}(\lambda_n(\nu)) \cap \prod_{i=1}^n U_i \neq \emptyset$. As $\prod_{i=1}^n U_i \cap \Delta_n(X) = \emptyset$, one has $E_n^{\nu}(X, G) \cap \prod_{i=1}^n U_i \neq \emptyset$ by Theorem 6.9. This ends the proof. □

By claim, for each $n \geqslant 2$, we can choose a dense sequence of points $\{(x_1^m, \ldots, x_n^m)\}_{m\in\mathbb{N}} \subseteq E_n(X, G)$ with $(x_1^m, \ldots, x_n^m) \in E_n^{\nu_n^m}(X, G)$ for some $\nu_n^m \in \mathcal{M}(X, G)$. Let

$$\mu = \sum_{n\geqslant 2} \frac{1}{2^{n-1}} \left(\sum_{m\geqslant 1} \frac{1}{2^m} \nu_n^m \right).$$

As if $\alpha \in \mathcal{P}_X$ then

$$h_\mu(G, \alpha) \geqslant \frac{1}{2^{m+n-1}} h_{\nu_n^m}(G, \alpha) \quad \big(\text{using (3.29)}\big)$$

and so $E_n^{\nu_n^m}(X, G) \subseteq E_n^\mu(X, G)$ for all $n \geqslant 2$ and $m \in \mathbb{N}$. Thus $(x_1^m, \ldots, x_n^m) \in E_n^\mu(X, G)$. Hence

$$E_n^\mu(X, G) \supseteq \overline{\big\{\big(x_1^m, \ldots, x_n^m\big)\colon m \in \mathbb{N}\big\}} \setminus \Delta_n(X) = E_n(X, G),$$

moreover, $E_n^\mu(X, G) = E_n(X, G)$ (using 1) for each $n \geqslant 2$. □

6.4. Entropy tuples of a finite production

At the end of this section, we shall provide a result about topological entropy tuples of a finite product.

We say that G-measure preserving system $(X, \mathcal{B}, \mu, G)$ is *free*, if $g = e_G$ when $g \in G$ satisfies $gx = x$ for μ-a.e. $x \in X$, equivalently, for μ-a.e. $x \in X$, the mapping $G \to Gx$, $g \mapsto gx$ is one-to-one. The following is proved in [18, Theorem 4].

Lemma 6.17. *Let $(X, \mathcal{B}, \mu, G)$ and $(Y, \mathcal{D}, \nu, G)$ both be a free ergodic G-measure preserving system with a Lebesgue space as its base space, with P_μ and P_ν Pinsker σ-algebras, respectively. Then $P_\mu \times P_\nu$ is the Pinsker σ-algebra of the product G-measure preserving system $(X \times Y, \mathcal{B} \times \mathcal{D}, \mu \times \nu, G)$.*

We say that (X, G) is *free* if $g = e_G$ when $g \in G$ satisfies $gx = x$ for each $x \in X$. Let $n \geqslant 2$. Denote by $\mathrm{supp}(X, G)$ the *support of* (X, G), i.e. $\mathrm{supp}(X, G) = \bigcup_{\mu\in\mathcal{M}(X,G)} \mathrm{supp}(\mu)$. (X, G) is called *fully supported* if there is an invariant measure $\mu \in \mathcal{M}(X, G)$ with full support (i.e. $\mathrm{supp}(\mu) = X$), equivalently, $\mathrm{supp}(X, G) = X$. Set $\Delta_n^S(X) = \Delta_n(X) \cap (\mathrm{supp}(X, G))^{(n)}$. Then:

Theorem 6.18. *Let (X_i, G), $i = 1, 2$ be two G-systems and $n \geqslant 2$. Then*

$$E_n(X_1 \times X_2, G) = E_n(X_1, G) \times \big(E_n(X_2, G) \cup \Delta_n^S(X_2)\big) \cup \Delta_n^S(X_1) \times E_n(X_2, G). \tag{6.6}$$

Proof. Obviously, $E_n(X_1 \times X_2, G) \subseteq (\mathrm{supp}(X_1, G) \times \mathrm{supp}(X_2, G))^{(n)}$ by Theorem 6.16(2), and so the inclusion of "$\subseteq$" follows directly from Proposition 6.3(3). Now let's turn to the proof of "$\supseteq$".

First we claim this direction if the actions are both free. Let

$$\big(\big(x_i^1, x_i^2\big)\big)_1^n \in E_n(X_1, G) \times \big(E_n(X_2, G) \cup \Delta_n^S(X_2)\big) \cup \Delta_n^S(X_1) \times E_n(X_2, G)$$

and let U_1 (resp. U_2) be any open neighborhood of $(x_i^1)_1^n$ in $X_1^{(n)}$ (resp. $(x_i^2)_1^n$ in $X_2^{(n)}$). With no loss of generality we assume $(x_i^1)_1^n \in E_n(X_1, G)$ and $U_1 \cap \Delta_n(X_1) = \emptyset$. Note that $\mathrm{supp}(\lambda_n(\mu)) \supseteq (\mathrm{supp}(\mu))^{(n)} \cap \Delta_n(X_2)$ for each $\mu \in \mathcal{M}(X_2, G)$, by Theorems 6.9 and 6.13 we can choose $\mu_i \in \mathcal{M}^e(X_i, G)$ such that $U_i \cap (\mathrm{supp}(\mu_i))^{(n)} \neq \emptyset$, $i = 1, 2$. As the actions are both free, we have

Claim. *$U_1 \times U_2 \cap E_n^{\mu_1\times\mu_2}(X_1 \times X_2, G) \neq \emptyset$, and so $U_1 \times U_2 \cap E_n(X_1 \times X_2, G) \neq \emptyset$, which implies $((x_i^1, x_i^2))_1^n \in E_n(X_1 \times X_2, G)$ from the arbitrariness of U_1 and U_2 (using Proposition* 6.3(2)).

Proof. Let P_{μ_i} be the Pinsker σ-algebra of $(X_i, \mathcal{B}_{X_i}, \mu_i, G)$, $i = 1, 2$. Then $P_{\mu_1} \times P_{\mu_2}$ forms the Pinsker σ-algebra of $(X_1 \times X_2, \mathcal{B}_{X_1} \times \mathcal{B}_{X_2}, \mu_1 \times \mu_2, G)$ by Lemma 6.17. Say $\mu_i = \int_{X_i} \mu_{i,x_i}\, d\mu_i(x)$ to be the disintegration of μ_i over P_{μ_i}, $i = 1, 2$. Then the disintegration of $\mu_1 \times \mu_2$ over $P_{\mu_1} \times P_{\mu_2}$ is

$$\mu_1 \times \mu_2 = \int_{X_1\times X_2} \mu_{1,x_1} \times \mu_{2,x_2}\, d\mu_1 \times \mu_2(x_1, x_2).$$

Moreover, $\lambda_n(\mu_i) = \int_{X_i} \mu_{i,x_i}^{(n)}\, d\mu_i(x_i)$, $i = 1, 2$, which implies

$$\lambda_n(\mu_1 \times \mu_2) = \int_{X_1\times X_2} \mu_{1,x_1}^{(n)} \times \mu_{2,x_2}^{(n)}\, d\mu_1 \times \mu_2(x_1, x_2) = \lambda_n(\mu_1) \times \lambda_n(\mu_2).$$

Then $\mathrm{supp}(\lambda_n(\mu_1 \times \mu_2)) = \mathrm{supp}(\lambda_n(\mu_1)) \times \mathrm{supp}(\lambda_n(\mu_2))$. So $U_1 \times U_2 \cap \mathrm{supp}(\lambda_n(\mu_1 \times \mu_2)) \neq \emptyset$ and $U_1 \times U_2 \cap E_n^{\mu_1\times\mu_2}(X_1 \times X_2, G) \neq \emptyset$ (as $U_1 \cap \Delta_n(X_1) = \emptyset$). This ends the proof of the claim. $\square$

Now let's turn to the proof of general case. Let (Z, G) be any free G-system. Then G-systems $(X_i', G) \doteq (X_i \times Z, G)$, $i = 1, 2$ are both free. Applying the first part to (X_i', G), $i = 1, 2$ we obtain

$$E_n\big(X_1' \times X_2', G\big) = E_n\big(X_1', G\big) \times \big(E_n\big(X_2', G\big) \cup \Delta_n^S\big(X_2'\big)\big) \cup \Delta_n^S\big(X_1'\big) \times E_n\big(X_2', G\big). \quad (6.7)$$

Then applying Proposition 6.3(3) to the projection factor maps $(X_1' \times X_2', G) \to (X_1 \times X_2, G)$, $(X_1', G) \to (X_1, G)$ and $(X_2', G) \to (X_2, G)$ respectively we claim the relation (6.6). $\square$

7. An amenable group action with u.p.e. and c.p.e.

In this section, we discuss two special classes of an amenable group action with u.p.e. and c.p.e. We will show that both u.p.e. and c.p.e. are preserved under a finite product; u.p.e. implies c.p.e. and actions with c.p.e. are fully supported; u.p.e. implies mild mixing; minimal topological K implies strong mixing if the group considered is commutative.

Let (X, G) be a G-system and $\alpha \in \mathcal{P}_X$. We say that α is *topological non-trivial* if $\overline{A} \subsetneq X$ for each $A \in \alpha$. It is easy to obtain

Lemma 7.1. *Let $n \geqslant 2$ and $\mu \in \mathcal{M}(X,G)$. Then $E_n^\mu(X,G) = X^{(n)} \setminus \Delta_n(X)$ iff $h_\mu(G,\alpha) > 0$ for any topological non-trivial $\alpha = \{A_1, \ldots, A_n\} \in \mathcal{P}_X$.*

Proof. First assume $E_n^\mu(X,G) = X^{(n)} \setminus \Delta_n(X)$. If $\alpha = \{A_1, \ldots, A_n\} \in \mathcal{P}_X$ is topological non-trivial, we choose $x_i \in X \setminus \overline{A_i}$, $i = 1, \ldots, n$, then $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$ and α is admissible w.r.t. $(x_i)_1^n$. Thus $h_\mu(G,\alpha) > 0$.

Conversely, we assume $h_\mu(G,\alpha) > 0$ for any topological non-trivial $\alpha = \{A_1, \ldots, A_n\} \in \mathcal{P}_X$. Let $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$. If $\alpha = \{A_1, \ldots, A_n\} \in \mathcal{P}_X$ is admissible w.r.t. $(x_i)_1^n$, then it is topological non-trivial and so $h_\mu(G,\alpha) > 0$. Thus $(x_i)_1^n \in E_n^\mu(X,G)$. This completes the proof. □

As a direct consequence of Theorem 6.16 and Lemma 7.1 one has

Theorem 7.2. *Let $n \geqslant 2$. Then*

1. *(X,G) has u.p.e. of order n iff there exists $\mu \in \mathcal{M}(X,G)$ such that $h_\mu(G,\alpha) > 0$ for any topological non-trivial $\alpha = \{A_1, \ldots, A_n\} \in \mathcal{P}_X$;*
2. *(X,G) has topological K iff there is $\mu \in \mathcal{M}(X,G)$ such that $h_\mu(G,\alpha) > 0$ for any topological non-trivial $\alpha \in \mathcal{P}_X$.*

Definition 7.3. We say that (X,G) has *c.p.e.* if any non-trivial topological factor of (X,G) has positive topological entropy.

Blanchard proved that any c.p.e. TDS is fully supported [1, Corollary 7]. As an application of Proposition 6.3(3) and Theorem 6.16 we have a similar result.

Proposition 7.4. *(X,G) has c.p.e. iff $X^{(2)}$ is the closed invariant equivalence relation generated by $E_2(X,G)$. Moreover, each c.p.e. G-system is fully supported and each u.p.e. G-system has c.p.e. (hence is also fully supported).*

Proof. It is easy to complete the proof of the first part. Moreover, note that $(\mathrm{supp}(X,G))^{(2)} \cup \Delta_2(X)$ is a closed invariant equivalence relation containing $E_2(X,G)$ (Theorem 6.16). In particular, if (X,G) has c.p.e. then it is fully supported. Now assume that (X,G) has u.p.e., thus $E_2(X,G) = X^{(2)} \setminus \Delta_2(X)$ and so $X^{(2)}$ is the closed invariant equivalence relation generated by $E_2(X,G)$, particularly, (X,G) has c.p.e. This finishes our proof. □

The following lemma is well known, in the case of $\mathbb{Z}$ see for example [36, Lemma 1].

Lemma 7.5. *Let (X_i,G) be a G-system and $\Delta_2(X_i) \subseteq A_i \subseteq X_i \times X_i$ with $\langle A_i \rangle$ the closed invariant equivalence relation generated by A_i, $i = 1, 2$. Then $\langle A_1 \rangle \times \langle A_2 \rangle$ is the closed invariant equivalence relation generated by $A_1 \times A_2$.*

Thus we have

Corollary 7.6. *Let (X_1,G) and (X_2,G) be two G-systems and $n \geqslant 2$.*

(1) *If (X_1,G) and (X_2,G) both have u.p.e. of order n then so does $(X_1 \times X_2, G)$.*
(2) *If (X_1,G) and (X_2,G) both have topological K then so does $(X_1 \times X_2, G)$.*
(3) *If (X_1,G) and (X_2,G) both have c.p.e. then so does $(X_1 \times X_2, G)$.*

Proof. By Proposition 7.4, any G-system having u.p.e. is full supported, then (1) and (2) follow from Theorem 6.18 directly. Using Theorem 6.18 and Lemma 7.5, we can obtain (3) similarly. □

In the following several sub-sections, we shall discuss more properties of an amenable group action with u.p.e.

7.1. U.p.e. implies weak mixing of all orders

Following the idea of the proof of [1, Proposition 2], it is easy to obtain the following result.

Lemma 7.7. *Let* $\{U_1^c, U_2^c\} \in \mathcal{C}_X$. *If*

$$\limsup_{n\to+\infty} \frac{1}{n} \log N\left(\bigvee_{i=1}^{n} g_i^{-1}\{U_1^c, U_2^c\}\right) > 0 \tag{7.1}$$

for some sequence $\{g_i\colon i \in \mathbb{N}\} \subseteq G$ *then there exist* $1 \leqslant j_1 < j_2$ *with* $U_1 \cap g_{j_1} g_{j_2}^{-1} U_2 \neq \emptyset$.

Proof. Assume the contrary that for each $1 \leqslant j_1 < j_2$, $U_1 \cap g_{j_1} g_{j_2}^{-1} U_2 = \emptyset$ and so $g_{j_1}^{-1} U_1 \subseteq g_{j_2}^{-1} U_2^c$. That is, for each $i \in \mathbb{N}$ one has $g_i^{-1} U_1 \subseteq \bigcap_{j\geqslant i} g_j^{-1} U_2^c$.

Let $n \in \mathbb{N}$. Now for each $x \in X$ consider the first $i \in \{1, \ldots, n\}$ such that $g_i x \in U_1$, when there exists such an i. We get that the Borel cover $\bigvee_{j=1}^n g_j^{-1}\{U_1^c, U_2^c\}$ admits a sub-cover

$$\left\{\bigcap_{s=1}^{i-1} g_s^{-1} U_1^c \cap \bigcap_{t=i}^{n} g_t^{-1} U_2^c\colon i = 1, \ldots, n\right\} \cup \left\{\bigcap_{s=1}^{n} g_s^{-1} U_1^c\right\}.$$

Moreover, $N(\bigvee_{j=1}^n g_j^{-1}\{U_1^c, U_2^c\}) \leqslant n + 1$, a contradiction with the assumption. □

We say that (X, G) is *transitive* if for each non-empty open subsets U and V, the *return time set*, $N(U, V) \doteq \{g \in G\colon U \cap g^{-1} V \neq \emptyset\}$, is non-empty. It is not hard to see that if X has no isolated point then the transitivity of (X, G) is equivalent to that $N(U, V)$ is infinite for each non-empty open subsets U and V. Let $n \geqslant 2$. We say that (X, G) is *weakly mixing of order n* if the product G-system $(X^{(n)}, G)$ is transitive; if $n = 2$ we call it simply *weakly mixing*. We say that (X, G) is called *weakly mixing of all orders* if for each $n \geqslant 2$ it is weakly mixing of order n, equivalently, the product G-system $(X^{\mathbb{N}}, G)$ is transitive. It's well known that for $\mathbb{Z}$-actions u.p.e. implies weakly mixing of all orders [1]. In fact, this result holds for a general countable discrete amenable group action by applying Corollary 7.6 and Lemma 7.7 to a u.p.e. G-system as many times as required.

Theorem 7.8. *Each u.p.e. G-system is weakly mixing of all orders.*

7.2. U.p.e. implies mild mixing

We say that (X, G) is *mildly mixing* if the product G-system $(X \times Y, G)$ is transitive for each transitive G-system (Y, G) containing no isolated points. We shall prove that each u.p.e.

G-system is mildly mixing. Note that similarly to the proof of Lemma 7.7, it is easy to show that each non-trivial u.p.e. G-system contains no any isolated point, thus the result in this sub-section strengthens Theorem 7.8. Before proceeding first we need

Lemma 7.9. *Let $\mu \in \mathcal{M}(X, G)$, $\mathcal{U} = \{U_1, \dots, U_n\} \in \mathcal{C}_X^o$, $\alpha \in \mathcal{P}_X$ and $\{g_i\}_{i\in\mathbb{N}} \subseteq G$ be a sequence of pairwise distinct elements. Then*

1. $\limsup_{n\to+\infty} \frac{1}{n} \log N(\bigvee_{i=1}^n g_i^{-1}\alpha) \geqslant h_\mu(G, \alpha)$;
2. *if* $h_{\text{top}}(G, \mathcal{U}) > 0$ *then* $\limsup_{n\to+\infty} \frac{1}{n} \log N(\bigvee_{i=1}^n g_i^{-1}\mathcal{U}) > 0$.

Proof. 1 follows directly from Lemma 3.1(4). Now let's turn to the proof of 2.

By Theorem 5.1 there exists $\mu \in \mathcal{M}^e(X, G)$ such that $h_\mu(G, \mathcal{U}) = h_{\text{top}}(G, \mathcal{U}) > 0$. Let P_μ be the Pinsker σ-algebra of $(X, \mathcal{B}_X^\mu, \mu, G)$. As $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) = \int_X \prod_{i=1}^n \mathbb{E}(1_{U_i^c}|P_\mu)\,d\mu > 0$ (see Corollary 6.12), repeating the same procedure of the proof of Theorem 6.11 we can obtain some $M \in \mathbb{N}$, $D \in P_\mu$ and $\alpha \in \mathcal{P}_X$ such that $\mu(D) > 0$ and if $\beta \in \mathcal{P}_X^\mu$ is finer than $\mathcal{U}$ then $H_\mu(\alpha|\beta \vee P_\mu) \leqslant H_\mu(\alpha|P_\mu) - \epsilon$, here $\epsilon = \frac{\mu(D)}{M} \log(\frac{n}{n-1}) > 0$. Note that there exists $K \in F(G)$ such that if $F \in F(G)$ satisfies $(FF^{-1} \setminus \{e_G\}) \cap K = \emptyset$ then $|\frac{1}{|F|} H_\mu(\alpha_F|P_\mu) - H_\mu(\alpha|P_\mu)| < \frac{\epsilon}{2}$ (see Theorem 6.10). Obviously, there exists a sub-sequence $\{s_1 < s_2 < \cdots\} \subseteq \mathbb{N}$ such that $\frac{i}{s_i} \geqslant \frac{1}{2|K|+1}$ for each $i \in \mathbb{N}$ and $g_{s_i} g_{s_j}^{-1} \notin K$ when $i \neq j$. Then for each $n \in \mathbb{N}$ one has

$$\left|\frac{1}{n} H_\mu\left(\bigvee_{i=1}^n g_{s_i}^{-1}\alpha|P_\mu\right) - H_\mu(\alpha|P_\mu)\right| < \frac{\epsilon}{2}. \tag{7.2}$$

Now let $n \in \mathbb{N}$. If $\beta_n \in \mathcal{P}_X^\mu$ is finer than $\bigvee_{i=1}^n g_{s_i}^{-1}\mathcal{U}$, then $g_{s_i}\beta_n \succcurlyeq \mathcal{U}$ for each $i = 1, \dots, n$, and so

$$\begin{aligned}
H_\mu(\beta_n) &\geqslant H_\mu\left(\beta_n \vee \bigvee_{i=1}^n g_{s_i}^{-1}\alpha|P_\mu\right) - H_\mu\left(\bigvee_{i=1}^n g_{s_i}^{-1}\alpha|\beta_n \vee P_\mu\right) \\
&\geqslant H_\mu\left(\bigvee_{i=1}^n g_{s_i}^{-1}\alpha|P_\mu\right) - \sum_{i=1}^n H_\mu(\alpha|g_{s_i}\beta_n \vee P_\mu) \\
&\geqslant H_\mu\left(\bigvee_{i=1}^n g_{s_i}^{-1}\alpha|P_\mu\right) - n\big(H_\mu(\alpha|P_\mu) - \epsilon\big) \\
&\geqslant n\left(H_\mu(\alpha|P_\mu) - \frac{\epsilon}{2}\right) - n\big(H_\mu(\alpha|P_\mu) - \epsilon\big) \quad \big(\text{by (7.2)}\big) \\
&= \frac{n\epsilon}{2}.
\end{aligned}$$

Hence, $\frac{1}{n} H_\mu(\bigvee_{i=1}^n g_{s_i}^{-1}\mathcal{U}) \geqslant \frac{\epsilon}{2}$, which implies

$$\limsup_{n\to+\infty}\frac{1}{n}\log N\left(\bigvee_{i=1}^{n} g_i^{-1}\mathcal{U}\right) \geqslant \limsup_{m\to+\infty}\frac{1}{s_m}H_\mu\left(\bigvee_{i=1}^{s_m} g_i^{-1}\mathcal{U}\right)$$
$$\geqslant \limsup_{m\to+\infty}\frac{m}{s_m}\cdot\frac{1}{m}H_\mu\left(\bigvee_{i=1}^{m} g_{s_i}^{-1}\mathcal{U}\right) \geqslant \frac{\epsilon}{2(2|K|+1)} > 0.$$

This ends the proof of the lemma. □

Now we claim that u.p.e. implies mild mixing.

Theorem 7.10. *Let (X, G) be a u.p.e. G-system. Then (X, G) is mildly mixing.*

Proof. Let (Y, G) be any transitive G-system containing no isolated points and (U_Y, V_Y) any pair of non-empty open subsets of Y. It remains to show that $N(\overline{U_X} \times U_Y, \overline{V_X} \times V_Y) \neq \emptyset$ for each pair of non-empty open subsets (U_X, V_X) of X. As (Y, G) is transitive, there is $g \in G$ with $U_Y \cap g^{-1}V_Y \neq \emptyset$. Set $W_Y = U_Y \cap g^{-1}V_Y$. Then

$$N(\overline{U_X} \times U_Y, \overline{V_X} \times V_Y) \supseteq gN\big(\overline{U_X} \times W_Y, \overline{g^{-1}V_X} \times W_Y\big).$$

Now it suffices to show that $N(\overline{U_X} \times W_Y, \overline{g^{-1}V_X} \times W_Y) \neq \emptyset$.

If $\overline{U_X} \cap \overline{g^{-1}V_X} \neq \emptyset$ then the proof is finished, so we assume $\overline{U_X} \cap \overline{g^{-1}V_X} = \emptyset$. As (Y, G) is a transitive G-system containing no isolated points, there exists $g_1' \in G \setminus \{e_G\}$ with $g_1'W_Y \cap W_Y \neq \emptyset$. Now find $g_2' \in G \setminus \{e_G, (g_1')^{-1}\}$ with $g_2'(g_1'W_Y \cap W_Y) \cap (g_1'W_Y \cap W_Y) \neq \emptyset$. By induction, similarly there exists a sequence $\{g_n'\}_{n\geqslant 1} \subseteq G$ such that for each $j \geqslant 1$ one has $g_j' \in G \setminus \{e_G, (g_{j-1}')^{-1}, (g_{j-1}'g_{j-2}')^{-1}, \ldots, (g_{j-1}'g_{j-2}'\cdots g_1')^{-1}\}$ and for each $n \in \mathbb{N}$ it holds that

$$W_Y \cap \bigcap_{1\leqslant i\leqslant j\leqslant n} \big(g_j'g_{j-1}'\cdots g_i'W_Y\big) \neq \emptyset. \tag{7.3}$$

Set $g_n = g_n'g_{n-1}'\cdots g_1'$ for each $n \in \mathbb{N}$. Then $g_i \neq g_j$ if $1 \leqslant i \neq j$. Note that $\overline{U_X} \cap \overline{g^{-1}V_X} = \emptyset$ and (X, G) is u.p.e., then $h_{\text{top}}(G, \{\overline{U_X}^c, \overline{g^{-1}V_X}^c\}) > 0$ and so by Lemma 7.9 one has

$$\limsup_{n\to+\infty}\frac{1}{n}\log N\left(\bigvee_{i=1}^{n} g_i^{-1}\big\{\overline{U_X}^c, \overline{g^{-1}V_X}^c\big\}\right) > 0.$$

Then by Lemma 7.7, there exists $1 \leqslant i < j$ such that

$$\emptyset \neq \overline{U_X} \cap g_ig_j^{-1}\overline{g^{-1}V_X} = \overline{U_X} \cap \big(g_j'g_{j-1}'\cdots g_{i+1}'\big)^{-1}\overline{g^{-1}V_X},$$

which implies (using (7.3))

$$g_j'g_{j-1}'\cdots g_{i+1}' \in N\big(\overline{U_X}, \overline{g^{-1}V_X}\big) \cap N(W_Y, W_Y) = N\big(\overline{U_X} \times W_Y, \overline{g^{-1}V_X} \times W_Y\big) \neq \emptyset.$$

This finishes the proof of the theorem. □

7.3. Minimal topological K-actions of an amenable group

We say that (X, G) is *strongly mixing* if $N(U, V)$ is cofinite (i.e. $G \setminus N(U, V)$ is finite) for each pair of non-empty open subsets (U, V) of X. It's proved in [22] that any topological K minimal $\mathbb{Z}$-system is strongly mixing. In fact, this result holds again in general case of considering a commutative countable discrete amenable group. In the remaining part of this sub-section we are ready to show it.

Denote by $\mathcal{F}_{inf}(G)$ the family of all infinite subsets of G. Let d be the compatible metric on (X, G), $S = \{g_1, g_2, \ldots\} \in \mathcal{F}_{inf}(G)$ and $n \geqslant 2$. $RP^n_S(X, G) \subseteq X^{(n)}$ is defined by $(x_i)_1^n \in RP^n_S(X, G)$ iff for each neighborhood U_{x_i} of x_i, $1 \leqslant i \leqslant n$ and $\epsilon > 0$ there exist $x_i' \in U_{x_i}$, $1 \leqslant i \leqslant n$ and $m \in \mathbb{N}$ with $\max_{1 \leqslant k,l \leqslant n} d(g_m^{-1} x_k', g_m^{-1} x_l') \leqslant \epsilon$. Obviously, the definition of $RP^n_S(X, G)$ is independent of the selection of compatible metrics. As a direct corollary of Lemma 7.9 we have

Lemma 7.11. *Let $n \geqslant 2$ and $S \in \mathcal{F}_{inf}(G)$. If (X, G) is u.p.e. of order n then $RP^n_S(X, G) = X^{(n)}$.*

Proof. Assume the contrary that there is $S = \{g_1, g_2, \ldots\} \in \mathcal{F}_{inf}(G)$ such that $RP^n_S(X, G) \subsetneq X^{(n)}$. Fix such an S and take $(x_i)_1^n \in X^{(n)} \setminus RP^n_S(X, G)$. Then we can find a closed neighborhood U_i of x_i, $1 \leqslant i \leqslant n$ and $\epsilon > 0$ such that if $x_i' \in U_i$, $1 \leqslant i \leqslant n$ and $m \in \mathbb{N}$ then $\max_{1 \leqslant k,l \leqslant n} d(g_m^{-1} x_k', g_m^{-1} x_l') > \epsilon$. Now let $\{C_1, \ldots, C_k\}$ $(k \geqslant n)$ be a closed cover of X such that the diameter of each C_i, $1 \leqslant i \leqslant k$, is at most ϵ and if $i \in \{1, \ldots, n\}$ then $x_i \in (C_i)^{\mathrm{o}} \subseteq C_i \subseteq U_i$. Clearly $(x_i)_1^n \notin \Delta_n(X)$, we may assume that $\{C_1^c, \ldots, C_n^c\}$ forms an admissible open cover of X w.r.t. $(x_i)_1^n$, and so $h_{\mathrm{top}}(G, \{C_1^c, \ldots, C_n^c\}) > 0$. Moreover,

$$\limsup_{m \to +\infty} \frac{1}{m} \log N\left(\bigvee_{i=1}^{m} g_i^{-1}\{C_1^c, \ldots, C_n^c\}\right) > 0 \quad \text{(by Lemma 7.9)}. \tag{7.4}$$

Whereas, it's not hard to claim that for each $i \in \{1, \ldots, k\}$ and $m \in \mathbb{N}$ there exists $j_i^m \in \{1, \ldots, n\}$ such that $g_m C_i \cap C_{j_i^m} = \emptyset$. Otherwise, for some $i_0 \in \{1, \ldots, k\}$ and $m_0 \in \mathbb{N}$, it holds that for each $i \in \{1, \ldots, n\}$, $g_{m_0} C_{i_0} \cap C_i \neq \emptyset$, let $y_i \in g_{m_0} C_{i_0} \cap C_i \subseteq U_i$. Thus $\max_{1 \leqslant k,l \leqslant n} d(g_{m_0}^{-1} y_k, g_{m_0}^{-1} y_l)$ is at most the diameter of C_{i_0}, which is at most ϵ, a contradiction with the selection of $y_1, \ldots, y_n$. Therefore, $C_i \subseteq \bigcap_{m \in \mathbb{N}} g_m^{-1} C_{j_i^m}^c$ for each $i \in \{1, \ldots, k\}$, which implies $N(\bigvee_{i=1}^m g_i^{-1}\{C_1^c, \ldots, C_n^c\}) \leqslant k$ for each $m \in \mathbb{N}$, a contradiction with (7.4). This finishes the proof of the lemma. □

Then we have

Theorem 7.12. *Let U and V be non-empty open subsets of X. If (X, G) is minimal and topological K then there exists $g_1, \ldots, g_l \in G$ $(l \in \mathbb{N})$ such that $\bigcup_{i=1}^l g_i N(U, V) g_i^{-1} \subseteq G$ is cofinite. In particular, if G is commutative then (X, G) is strongly mixing.*

Proof. As (X, G) is a minimal G-system, there exist distinct elements $g_1, \ldots, g_N \in G$ such that $\bigcup_{i=1}^N g_i U = X$. Let $\delta > 0$ be a Lebesgue number of $\{g_1 U, \ldots, g_N U\} \in \mathcal{C}_X^o$ and set

$$B = \left\{g \in G\colon \exists x_i \in g_i V \ (1 \leqslant i \leqslant N) \text{ s.t. } \max_{1 \leqslant k,l \leqslant N} d\big(g^{-1} x_k, g^{-1} x_l\big) < \frac{\delta}{2}\right\}.$$

As (X, G) is topological K, $(g_i x)_1^N \in RP_S^n(X, G)$ for each $S \in \mathcal{F}_{inf}(G)$ and $x \in X$ by Lemma 7.11. This implies $B \cap S \neq \emptyset$ for each $S \in \mathcal{F}_{inf}(G)$. Hence, $G \setminus B$ is a finite subset, i.e. $B \subseteq G$ is cofinite. Now if $g \in B$ then for each $i \in \{1, \ldots, N\}$ there exists $x_i \in g_i V$ such that $\max_{1 \leqslant k,l \leqslant N} d(g^{-1}x_k, g^{-1}x_l) < \frac{\delta}{2}$. Moreover, the diameter of $\{g^1 x_1, \ldots, g^{-1}x_N\}$ is less than δ. So by the selection of δ, for some $1 \leqslant k \leqslant N$, $g^{-1}x_1, \ldots, g^{-1}x_N \in g_k U$, in particular, $g_k U \cap g^{-1} g_k V \neq \emptyset$. That is, for each $g \in B$ there exists $k \in \{1, \ldots, N\}$ such that $g_k^{-1} g g_k \in N(U, V)$, i.e. $B \subseteq \bigcup_{k=1}^N g_k N(U, V) g_k^{-1}$. □

Acknowledgments

The first author is supported by NSFC (10911120388, 11071231), Fok Ying Tung Education Foundation and the Fundamental Research Funds for the Central Universities; the second author is supported by NSFC (11071231); and the third author is supported by NSFC (10801035), FANEDD (201018) and a grant from Chinese Ministry of Education (200802461004).

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