Homological properties of abstract and profinite modules and groups

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Abstract

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0. Introduction

In this paper we generalize the main results from [8,12,13] about cohomological dimensions and the homological type $FP_m$ of a normal subgroup of a group $G$ with some finiteness properties (in both discrete and profinite cases) and treat the module versions of the same results. The proofs of the main results of [12,13] use methods from Bieri–Renz–Strebel $\Sigma$-theory and $L^2$-methods for Hilbert modules to resolve a conjecture due to E. Rapaport Strasser about knot-like groups $G$: if the commutator $G'$ is finitely generated then it is free. In [11], Hillman proved that the Rapaport Strasser Conjecture holds under the assumption that the commutator $G'$ is of type $FP_2$. Later on, using techniques from $\Sigma$-theory, it was shown in [12] that if the Novikov ring $\hat{\mathbb{Z}}_G\chi$ associated to a discrete character $\chi : G \to \mathbb{Z}$ is von Neumann finite (i.e. left inverse is right inverse and vice versa) then the commutator $G'$ is of type $FP_2$. The proof of the Rapaport Strasser Conjecture was completed in [13] by showing that the Novikov ring $\hat{\mathbb{Z}}_G\chi$ for a discrete character $\chi$ of an arbitrary finitely generated group $G$ is always von Neumann finite. This was recently extended by R. Bieri to the case of non-discrete characters $\chi$ of $G$ [2].

The results in [12,13] apply to groups $G$ with finite $K(G,1)$ and of Euler characteristic 0. In Theorem 1, Section 2 we show that the above assumptions can be weakened. Furthermore we show a homological version of Theorem 1 that treats more general $\mathbb{Z}G$-modules $A$ not only the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, see Theorem 2. In the specific case $m = 1$ and $A$ the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, Theorem 2 was first proved in [2] and rediscovered (with different proof) in [8].

A pro-$p$ version of the Rapaport Strasser Conjecture was suggested and resolved in [12] and a profinite version of the same conjecture was proved in [8]. The proof in the profinite case given in [8] is based on the proof of [12, Proposition 2]. In Section 3 we follow the same approach and generalize the results from [8,12] to arbitrary finite projective dimensions and general modules not necessarily trivial ones, see Theorem 3.

Finally we show in Section 4 that the Brown criterion for abstract groups of homological type $FP_m$ [6] holds for profinite groups. Various auxiliary results about profinite groups of type $FP_m$ that were used earlier in Section 3 are collected and proved in Section 4.

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1. Preliminaries

1.1. $\Sigma$-invariants and Novikov rings

Let $R$ be an abstract ring. We say that an abstract $R$-module $A$ is of type $\text{FP}_m$ if there is a projective resolution of $A$ with all projectives in dimension at most $m$ finitely generated. An abstract group $G$ is of type $\text{FP}_m$ if the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ is of type $\text{FP}_m$. It is easy to see that $G$ is finitely generated if and only if $G$ is of type $\text{FP}_1$.

For a finitely generated group $G$ the character sphere $S(G)$ is the set of equivalence classes $[\chi]$ of characters $\chi \in \text{Hom}(G, \mathbb{R}) \setminus \{0\}$, where $[\chi] = \mathbb{R}_{\geq 0} \chi$. For a finitely generated (right) $\mathbb{Z}[G]$-module $A$ and $G_x = \{ g \in G \mid \chi(g) \geq 0 \}$ the Bieri–Strebel–Renz invariant $\Sigma^m(G, A)$ is defined by

$$\Sigma^m(G, A) = \{ [\chi] \in S(G) \mid A \text{ is of type } \text{FP}_m \text{ as a } \mathbb{Z}[G_x]\text{-module}\}.$$  

The invariant $\Sigma^m(G, A)$ is responsible for the homological finiteness type of $A$ as a $\mathbb{Z}[G]$-module, where $H$ is a subgroup of $G$ that contains the commutator. More precisely suppose that $A$ is of type $\text{FP}_m$ as a $\mathbb{Z}[G]$-module, then $A$ is of type $\text{FP}_m$ over $\mathbb{Z}[H]$ if and only if $S(G, H) = \{ [\chi] \in S(G) \mid \chi(H) = 0 \} \subseteq \Sigma^m(G, A)$ [4, Thm. B].

There is a strong link between the $\Sigma$-invariants and the Novikov ring associated to a real character, this was first observed in [22]. For non-zero real character $\chi$ of $G$ the Novikov ring $(\mathbb{Z}[G])_\chi$ contains the (possibly infinite) formal sums $\sum_{g \in G} z_g g$ such that for every natural number $j$ the set $\{ g \in G \mid z_g \neq 0, \chi(g) \leq j \}$ is finite. By [3, Appendix B] or by [2, Appendix] $[\chi] \in \Sigma^m(G, A)$ if and only if $\text{Tor}^\mathbb{Z}[G](A, (\mathbb{Z}[G])_\chi) = 0$ for all $i \leq m$. Combining with [4, Thm. B] we get that for a subgroup $H$ of $G$ that contains the commutator and $A$ a $\mathbb{Z}[G]$-module of type $\text{FP}_m$, $A$ is of type $\text{FP}_m$ as a $\mathbb{Z}[H]$-module if and only if $\text{Tor}^\mathbb{Z}[G](A, (\mathbb{Z}[G])_\chi) = 0$ for all $i \leq m$ and every non-zero character $\chi : G \to \mathbb{R}$ such that $\chi(H) = 0$.

1.2. $L^2$-Betti numbers

The $L^2$-Betti numbers of groups and spaces are powerful analytic invariants, see [18]. There is an algebraic way to define the $L^2$-Betti numbers for $i \leq m$ by

$$b^{(2)}_i(G) = \dim H_i(\mathcal{D}^{\text{det}} \otimes_{\mathbb{Z}[G]} \mathbb{Z}(G)) \in [0, \infty),$$

where $\mathcal{D} : \cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$ is a free resolution of the trivial right $\mathbb{Z}[G]$-module $\mathbb{Z}$ with $P_i$ of finite rank for $j \leq m$, $N(G)$ is the von Neumann algebra, in particular $b^{(2)}_i(G) \leq rk(P_i)$ for $i \leq m$ and $\dim$ is the extended dimension function defined in [17].

In [7] it was conjectured that a compact aspherical manifold that is fibered over the circle has trivial $L^2$-Betti numbers and later on this was proved in [16], [18, Thm. 1.39]. Actually the same proof gives that if $G$ is a group of type $\text{FP}_m$ with a normal subgroup $N$ of type $\text{FP}_m$ such that $G/N \simeq \mathbb{Z}$ then the $L^2$-Betti numbers $b^{(2)}_i(G) = 0$ for $i \leq m$. This was already observed in dimension $m = 1$ in [11, Lemma, p.273]. Indeed $\mathbb{Q} = G/N$ is a group with a tower of subgroups of finite indices $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_s \supseteq \cdots$ such that all $Q_i \simeq \mathbb{Z}$ and let $G_i$ be the preimage of $Q_i$ in $G$. If $N$ has a free resolution with free modules of rank $r_i$ in dimension $i$, $r_0 = 1$ and $r_i$ finite for $i \leq m$ then $G_i$ has a free resolution with free modules of rank $r_i + r_{i-1}$ in dimension $i$, where $r_0 = 0$ for $s < 0$. In particular $0 \leq b^{(2)}_i(G) = b^{(2)}_i(G)/|G : G_i| \leq (r_i + r_{i-1})/|G : G_i|$ is arbitrarily small for $i \leq m$. The same works for $Q \simeq \mathbb{Z}^n$ since $G$ has a free resolution with free modules of rank $r_i + \left(\begin{array}{c} n \\ i \end{array}\right) r_{i-1} + \cdots + \left(\begin{array}{c} n \\ 0 \end{array}\right) r_{-n}$ in dimension $i$.

1.3. Profinite modules

Let $S$ be a profinite ring and $A$ be a (right) profinite $S$-module. We say that $A$ is of type $\text{FP}_m$ if $A$ has a profinite projective resolution as a profinite $S$-module with all projectives finitely generated in dimension less than or equal to $m$. A profinite group $G$ is said to be of type $\text{FP}_m$ over a profinite ring $R$ if the trivial profinite $R[[G]]$-module $R$ is of type $\text{FP}_m$. Note that by [15, Lemma 1.1] a profinite group $G$ is of type $\text{FP}_m$ over $R$ if and only if $R$ as an abstract module over the ring $R[[G]]$ is of type $\text{FP}_m$, this is an easy corollary of [25, Lemma 7.2.2].

2. Applications of $L^2$-Betti numbers and Novikov rings

Theorem 1. Let $m$ be a natural number and $G$ be a finitely presented abstract group with a normal subgroup $N$ such that:

1. $Q = G/N$ is infinite, abelian and $N$ has type $\text{FP}_m$;
2. there is a $K(G, 1)$-complex $Y$ such that its $(m + 1)$-skeleton has $\alpha_i$ cells in dimension $i$ for $0 \leq i \leq m + 1$ and

$$\sum_{0 \leq i \leq m + 1} (-1)^{m+1-i} \alpha_i \leq 0.$$  

Then

(a) $G$ has geometric dimension at most $m + 1$ i.e. there is a $K(G, 1)$ complex of dimension at most $m + 1$;
(b) the Euler characteristic $\chi(G) = 0$;
(c) $N$ has type $\text{FP}_{m+1}$, and consequently has type $\text{FP}_\infty$;
(d) $\text{cd}(G) = \text{cd}(N) + \text{vcd}(G/N)$. 


Proof. Let $X$ be the $(m + 1)$-skeleton of $Y$. Then

$$(1)^{m+1} \chi (X) = \sum_{0 \leq i \leq m+1} (1)^{m+1-i} \alpha_i \leq 0.$$  

By going down to a subgroup of finite index we can assume that $Q \simeq \mathbb{Z}^m$. Then as observed in Section 1.2, $b_i^{(2)} (G) = 0$ for $i \leq m$, hence

$$(-1)^{m+1} \chi (X) = (-1)^{m+1} \left( \sum_{0 \leq i \leq m} (-1)^i b_i^{(2)} (G) \right) + b_{m+1}^{(2)} (X) = b_{m+1}^{(2)} (X) \geq 0.$$  

Thus $\chi (X) = 0$ and by [11, Thm. 1] $X$ is aspherical, consequently there is a finite $K(G, 1)$-complex of dimension $m + 1$ and Euler characteristic 0. This implies immediately (a) and (b). Parts (c) and (d) follow immediately from (a), (b) and [13, Thm. 3].  

Theorem 1 has a homological version that treats non-trivial $\mathbb{Z}G$-modules and uses substantially $\Sigma$-theory and Novikov rings. We write $pd_{\mathbb{Z}G} (A)$ for the projective dimension and $\chi_{\mathbb{Z}G} (A)$ for the Euler characteristic of the $\mathbb{Z}G$-module $A$.

**Theorem 2.** Let $G$ be a finitely generated abstract group with a normal subgroup $N$, $m$ a natural number and $A$ a (right) $\mathbb{Z}G$-module such that:

1. $Q = G/N$ is abelian and infinite;
2. the $\mathbb{Z}G$-module $A$ has a free resolution

$$\mathcal{P} : \cdots \rightarrow P_i \xrightarrow{d_i} P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

with $P_i$ free of rank $\alpha_i$, for $i \leq m$, and $\sum_{0 \leq i \leq m+1} (-1)^{m+1-i} \alpha_i \leq 0$;
3. the $\mathbb{Z}N$-module $A$ has type $FP_m$.

Then

(a) $A$ has finite projective dimension $pd_{\mathbb{Z}G} (A) \leq m + 1$;
(b) $A$ is of type $FP_\infty$ over $\mathbb{Z}G$ and over $\mathbb{Z}N$;
(c) $\sum_{0 \leq i \leq m+1} (1)^{m+1-i} \alpha_i = 0$ and $\chi_{\mathbb{Z}G} (A) = 0$.

If in addition $A$ is infinite and $N$ acts trivially on $A$ then

(d) $N$ has type $FP_\infty$ and is of finite cohomological dimension $cd (N) = s \leq m + 1$.

If furthermore $pd_{\mathbb{Z}G} (A) < \infty$ then

(e) $pd_{\mathbb{Z}G} (A) \leq cd (N) + pd_{\mathbb{Z}Q} (A)$ and the equality holds if $Ext^{k}_{\mathbb{Z}Q} (A, ZQ) \otimes H^i (N, ZN) \neq 0$, where $k = pd_{\mathbb{Z}Q} (A)$. In particular the equality holds if $A$ is self dual as a $\mathbb{Z}Q$-module or $N$ is a Poincaré duality group.

Proof. By [3, Appendix B] $\sum_{0 \leq i \leq m+1} (1)^{m+1-i} \alpha_i \geq 0$. By the assumptions of the theorem $\sum_{0 \leq i \leq m+1} (1)^{m+1-i} \alpha_i \leq 0$, so we deduce

$$\sum_{0 \leq i \leq m+1} (1)^{m+1-i} \alpha_i = 0.$$  

Let $\chi : G \to \mathbb{R}$ be a non-zero character such that $\chi (N) = 0$ and consider the Novikov ring $(\mathbb{ZG})_\chi$. As pointed in Section 1.1 since $A$ is of type $FP_m$ over $\mathbb{Z}N$ by [2, Appendix] and [4, Thm. B] we have $Tor^1_{\mathbb{Z}G} (A, (\mathbb{ZG})_\chi) = 0$ for all $i \leq m$ and every non-zero character $\chi : G \to \mathbb{R}$ such that $\chi (N) = 0$. Thus $H_i ((p^{\text{del}} \otimes \mathbb{ZG})_\chi) = 0$ for $i \leq m$ i.e.

$$(\mathcal{P}^{\text{del}})^{(m+1)} \otimes_{\mathbb{ZG}} (\mathbb{ZG})_\chi : 0 \rightarrow (\mathbb{ZG})_{\chi}^{d_{m+1}} \rightarrow (\mathbb{ZG})_{\chi}^{d_m} \rightarrow \cdots \rightarrow (\mathbb{ZG})_{\chi}^{d_1} \rightarrow 0$$

is exact in dimension $\leq m$. Then the short exact sequence $0 \rightarrow \text{Ker}(\partial_i) \rightarrow (\mathbb{ZG})_{\chi}^{d_i} \rightarrow \text{Im}(\partial_i) \rightarrow 0$ splits for $0 \leq i \leq m+1$ and consequently

$$\oplus_{m+1-i \text{ even, } i \leq m+1} (\mathbb{ZG})_{\chi}^{d_i} \simeq (\oplus_{1 \leq j \leq m+1} \text{Im}(\partial_j)) \oplus \text{Ker}(\partial_{m+1})$$  

and

$$\oplus_{m-i \text{ even, } i \leq m} (\mathbb{ZG})_{\chi}^{d_i} \simeq \oplus_{1 \leq j \leq m+1} \text{Im}(\partial_j).$$

By (2) and (3) $\text{Ker}(\partial_{m+1})$ is the kernel of a surjective endomorphism of $(\mathbb{ZG})_{\chi}^{\beta}$, where $\beta = \sum_{m+1-i \text{ even, } i \leq m+1} \alpha_i = \sum_{m-i \text{ even, } i \leq m} \alpha_i$. Since $(\mathbb{ZG})_{\chi}$ is von Neumann finite [13] for any natural number $k$ any epimorphism $(\mathbb{ZG})_{\chi}^{k} \rightarrow (\mathbb{ZG})_{\chi}^{k}$ is an isomorphism, hence $0 = \text{Ker}(\partial_{m+1}) = H_{m+1} ((p^{\text{del}})^{(m+1)} \otimes \mathbb{ZG})_{\chi})$. Using again [2, Appendix] and [4, Thm. B] it follows that $A$ is of homological type $FP_{m+1}$ over $\mathbb{Z}N$.

Since $\partial_{m+1}$ is injective, the differential $P_{m+1} \xrightarrow{d_{m+1}} P_m$ of the complex $\mathcal{P}$ is injective. Then the $(m + 1)$-skeleton $\mathcal{P}^{(m+1)}$ of $\mathcal{P}$ is a free resolution of the trivial $\mathbb{Z}G$-module $A$ of length $m + 1$ and Euler characteristic $0$. This implies (a), (c) and the part
of item (b) that $A$ is $FP_\infty$ over $\mathbb{Z}G$. Furthermore since $pd_{2N}(A) \leq pd_{2C}(A) \leq m + 1$ and $A$ is of type $FP_{m+1}$ over $\mathbb{Z}N$ we get that $A$ is of type $FP_\infty$ over $\mathbb{Z}N$, thus (b) holds.

Now suppose that $N$ acts trivially on $A$. Since $A$ is finitely generated as a $\mathbb{Z}N$-module, $A$ is finitely generated as an abelian group i.e. is a finite direct sum of cyclic groups. Since $A$ is infinite at least one of these cyclic groups is $\mathbb{Z}$. Consequently the projective dimension as a $\mathbb{Z}N$-module of the direct summand $\mathbb{Z}$ (of $A$) is finite and at most $pd_{2N}(A) \leq m + 1$ i.e. $cd(N) \leq m + 1$ and the homological type of the direct summand $\mathbb{Z}$ (of $A$) as a $\mathbb{Z}N$-module is $FP_\infty$ i.e. $N$ is $FP_\infty$. This completes the proof of (d).

Finally we prove item (e). By an obvious module version of [2, Prop. 5.1(a)] for $t = pd_{2N}(A) \leq m + 1$ there is a free $\mathbb{Z}G$-module $F$ such that $\Ext^t_{\mathbb{Z}G}(A, F) = 0$. Since $A$ is $FP_\infty$ over $\mathbb{Z}G$ the functor $\Ext^t_{\mathbb{Z}G}(A, -)$ commutes with direct sums and consequently $\Ext^t_{\mathbb{Z}G}(A, \mathbb{Z}G) = 0$. Similarly $\Ext^t_{\mathbb{Z}N}(\mathbb{Z}, \mathbb{Z}N) = 0$ for $s = cd(N)$. Consider the Grothendieck spectral sequence [21, Thm. 11.38]

$$E^{2,q}_2 = \Ext^{p,q}_{\mathbb{Z}Q}(A, \mathbb{E}^p_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G)) \Rightarrow \mathbb{E}^{p,q}_{\mathbb{Z}G}(A, \mathbb{Z}G).$$

By [2, Prop. 5.4 & Lemma 5.6] we have $\Ext^t_{\mathbb{Z}N}(\mathbb{Z}, \mathbb{Z}G) \simeq \Ext^t_{\mathbb{Z}N}(\mathbb{Z}, \mathbb{Z}N) \otimes_\mathbb{Z} \mathbb{Q}$ as a $\mathbb{Q}$-module and consequently $E^{p,q}_{2} \simeq \Ext^t_{\mathbb{Z}N}(\mathbb{Z}, \mathbb{Z}N) \otimes_\mathbb{Z} \Ext^t_{\mathbb{Z}Q}(A, \mathbb{Q})$. In particular for $p \geq k + 1 = pd_{2Q}(A) + 1$ or $q \geq s + 1 = cd(N) + 1$ we have $E^{p,q}_{2} = 0$.

Consequently $\Ext^t_{\mathbb{Z}G}(A, \mathbb{Z}G) = 0$ for $m \geq k + s + 1$ and

$$pd_{2Q}(A) \leq k + s = pd_{2\mathbb{Z}}(A) + cd(N).$$

Comparing the degrees of the differentials in the spectral sequence we get that $E^{k,k}_{\infty} = E^{k,k}_{2}$ and consequently

$$\Ext^{k+k}_{\infty}(A, \mathbb{Z}G) \simeq \Ext^{k+k}_{\infty}(\mathbb{Z}, \mathbb{Z}G) \otimes_\mathbb{Z} \Ext^{k+k}_{\mathbb{Z}Q}(A, \mathbb{Z}Q).$$

Thus if $\Ext^t_{\mathbb{Z}N}(\mathbb{Z}, \mathbb{Z}N) \otimes_\mathbb{Z} \Ext^t_{\mathbb{Z}Q}(A, \mathbb{Z}Q) \neq 0$ we have $pd_{2\mathbb{Z}}(A) \geq k + s$, hence

$$pd_{2\mathbb{Z}}(A) = k + s.$$

Finally if $N$ is a Poincare duality group $\Ext^t_{\mathbb{Z}Q}(\mathbb{Z}, \mathbb{Z}N) \simeq \mathbb{Z}$ we get $\Ext^t_{\mathbb{Z}Q}(A, \mathbb{Z}Q) \otimes_\mathbb{Z} \Ext^t_{\mathbb{Z}Q}(\mathbb{Z}, \mathbb{Z}N) \simeq \Ext^t_{\mathbb{Z}Q}(A, \mathbb{Z}Q) \neq 0$. If $A$ is self-dual then $\Ext^t_{\mathbb{Z}Q}(A, \mathbb{Z}Q) \simeq A$ has a summand isomorphic to $\mathbb{Z}$, so $\Ext^t_{\mathbb{Z}Q}(A, \mathbb{Z}Q) \otimes_\mathbb{Z} \Ext^t_{\mathbb{Z}Q}(\mathbb{Z}, \mathbb{Z}N) = A \otimes_\mathbb{Z} \Ext^t_{\mathbb{Z}Q}(\mathbb{Z}, \mathbb{Z}N)$ has a summand isomorphic to $\mathbb{Z} \otimes_\mathbb{Z} \Ext^t_{\mathbb{Z}Q}(\mathbb{Z}, \mathbb{Z}N) \simeq \Ext^t_{\mathbb{Z}Q}(\mathbb{Z}, \mathbb{Z}N) \neq 0$. $\square$

3. Modules over profinite groups with $p$-adic analytic quotients

In this section we treat a profinite version of Theorem 2. For a profinite group $G$ and a profinite $\mathbb{Z}_p[[G]]$-module $A$ we write $pd_{\mathbb{Z}_p[[G]]}(A)$ for the projective dimension of $A$ and $vpd_{\mathbb{Z}_p[[G]]}(A)$ for the virtual projective dimension of $A$. If $A$ is of finite projective dimension and type $FP_\infty$ there is a profinite projective resolution $\mathcal{P}$ of finite length and finitely generated projectives and the Euler characteristic of $A$ as a profinite $\mathbb{Z}_p[[G]]$-module is defined by

$$\chi_{\mathbb{Z}_p[[G]]}(A) = \sum_i (-1)^i \dim_{\mathbb{F}_p} \text{Tor}^{\mathbb{Z}_p[[G]]}_i(A, \mathbb{F}_p) = \sum_i (-1)^i \dim_{\mathbb{F}_p} \Ext^i_{\mathbb{Z}_p[[G]]}(A, \mathbb{F}_p).$$

By definition $\chi_p(G) = \chi_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the trivial $\mathbb{Z}_p[[G]]$-module.

**Theorem 3.** Let $G$ be a profinite group with a normal closed subgroup $N$ and $A$ be a profinite (right) $\mathbb{Z}_p[[G]]$-module such that:

1. $G/N$ is an infinite $p$-adic analytic profinite group;
2. the trivial (right) $\mathbb{Z}_p[[G]]$-module $A$ has a free (profinite) resolution $\mathcal{P}$, $P_i \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$ with $P_i$ free of rank $\alpha_i$ such that $\alpha_i$ is finite for $0 \leq i \leq m + 1$ and $\sum_{0 \leq i \leq m+1} (-1)^{m+1-i} \alpha_i 
leq 0$;
3. $A$ as a profinite $\mathbb{Z}_p[[N]]$-module has type $FP_m$.

Then the following hold:

(a) $pd_{\mathbb{Z}_p[[N]]}(A) \leq m$ and $pd_{\mathbb{Z}_p[[G]]}(A) \leq m + 1$;
(b) $\chi_{\mathbb{Z}_p[[G]]}(A) = \sum_{0 \leq i \leq m+1} (-1)^{m+1-i} \alpha_i = 0$;
(c) $A$ has type $FP_\infty$ as a profinite $\mathbb{Z}_p[[N]]$-module and as a profinite $\mathbb{Z}_p[[G]]$-module.

Furthermore if $N$ acts trivially on $A$ and if $H^t(N, \mathbb{F}_p)$ is finite and non-zero for $k = \min \{cd_p(N), pd_{\mathbb{Z}_p[[G]]}(A)\}$ then

(d) $N$ is of type $FP_\infty$ over $\mathbb{Z}_p$, $cd_p(N)$ and $vpd_{\mathbb{Z}_p[[G]]}(A)$ are finite and $pd_{\mathbb{Z}_p[[G]]}(A) = vpd_{\mathbb{Z}_p[[G/N]]}(A) - cd_p(N)$;
(e) if furthermore $pd_{\mathbb{Z}_p[[G/N]]}(A)$ is finite then

$$0 = \chi_{\mathbb{Z}_p[[G]]}(A) = \chi_{\mathbb{Z}_p[[G/N]]}(A) \chi_p(N),$$

in particular if $\chi_p(N) \neq 0$ we have $\chi_{\mathbb{Z}_p[[G/N]]}(A) = 0$. 


Proof. By going down to a subgroup of finite index in $G$ that contains $N$ we can suppose that $G/N$ is a powerful pro-$p$ group.

Consider the complex $\mathcal{R} = \varphi_{p\alpha}^* \otimes_{\mathbb{Z}_p[[N]]} \mathbb{Z}_p[[G]]$ with differentials $\{\partial_i\}_{i \geq 0}$. Since $A$ is $FP_m$ over $\mathbb{Z}_p[[N]]$ the homology groups $Tor_{i}[\mathbb{Z}_p[[N]]](A, \mathbb{Z}_p)$ are finitely generated as profinite $\mathbb{Z}_p$-modules for $i \leq m$ and consequently $H_i(\mathcal{R}) = Tor_{i}[\mathbb{Z}_p[[N]]](A, \mathbb{Z}_p)$ is finitely generated over $\mathbb{Z}_p$ for $i \leq m$. Since $\mathbb{Z}_p[[G/N]]$ is right and left Noetherian and has no zero divisors [9, Cor. 7.25], [19] we can consider the skew ring of fractions $S$ of $\mathbb{Z}_p[[G/N]]$ [10, Chapter 9]. Note that since $H_i(\mathcal{R})$ is finitely generated over $\mathbb{Z}_p$ for $i \leq m$ and $\mathbb{Z}_p[[G/N]]$ is not finitely generated as a $\mathbb{Z}_p$-module (topologically or abstractly is the same) for every element $r \in H_i(\mathcal{R})$, there is a non-zero element $\lambda_i \in \mathbb{Z}_p[[G/N]]$ such that $r \lambda_i = 0$. Then applying the abstract tensor product $\otimes_{\mathbb{Z}_p[[N]]}$ we get $H_i(\mathcal{R}) \otimes_{\mathbb{Z}_p[[G/N]]} S = 0$ for $i \leq m$ and for the complex

$$\mathcal{R} \otimes_{\mathbb{Z}_p[[G/N]]} S : \cdots \to S^{\alpha_{m+1}} \otimes_{\mathbb{Z}_p[[G/N]]} \mathbb{Z}_p \otimes \mathbb{Z}_p \to \cdots \to S^{\alpha_0} \to 0$$

we have $H_i(\mathcal{R}) \otimes_{\mathbb{Z}_p[[G/N]]} S = 0$ for $i \leq m$. Since $S$ is a skew field, counting dimensions over $S$, we get that $\sum_{0 \leq i \leq m+1} (-1)^{m+1-i} \alpha_i \leq 0$, hence

$$\sum_{0 \leq i \leq m+1} (-1)^{m+1-i} \alpha_i = 0.$$ 

Then $\partial_{m+1} \otimes id_S$ is injective and consequently $\partial_{m+1}$ is injective. In particular $\text{Tor}_{m+1}[\mathbb{Z}_p[[N]]](A, \mathbb{Z}_p) = H_{m+1}(\mathcal{R}) = \text{Ker}(\partial_{m+1})/\text{Im}(\partial_{m+2}) = 0$. A similar argument substituting $S$ with the skew ring of fractions of $\mathbb{F}_p[[G/N]]$ shows that $\text{Tor}_{m+1}[\mathbb{Z}_p[[N]]](A, \mathbb{F}_p) = 0$.

By Lemma 3 (from the next section) to show that $pd_{\mathbb{Z}_p[[N]]}(A) \leq m$ it is sufficient to show that $\text{Tor}_{m+1}[\mathbb{Z}_p[[U]]](A, \mathbb{F}_p) = 0$ for every open subgroup $U$ of $N$. Note that $U = N \cap U_i$ for an open subgroup $U_i$ of $G$, hence we can substitute $G$ with $U_i$ and $N$ with $U$ and repeat the argument from the previous paragraph to obtain $\text{Tor}_{m+1}[\mathbb{Z}_p[[U]]](A, \mathbb{F}_p) = 0$.

To prove item (a) it remains to show that $pd_{\mathbb{Z}_p[[G]]}(A) \leq m + 1$. Let $d_{m+1} : \mathbb{Z}_{m+1} \to \mathbb{Z}_m$ be the differential of the projective resolution $P$ of $A$. Note that for any open subgroup $U$ of $N$ the map $d_{m+1} \otimes id_{\mathbb{Z}_p[[U]]} : d_{m+1} \otimes id_{\mathbb{Z}_p[[U]]} : \mathbb{Z}_p \to \mathbb{Z}_p \otimes \mathbb{Z}_p[[U]] \otimes \mathbb{Z}_p$ is injective (i.e. for $U = N$ the map $d_{m+1}$ is injective). Since $d_{m+1} \otimes id_{\mathbb{Z}_p[[U]]}$ we have $\text{Ker}(d_{m+1}) \subseteq (\mathbb{Z}_p[[G]] \otimes \mathbb{Z}_p[[U]]) \otimes_{\mathbb{Z}_p[[U]]} \mathbb{Z}_p$ for every open subgroup $U$ of $N$, so $d_{m+1}$ is injective. Thus $pd_{\mathbb{Z}_p[[G]]}(A) \leq m + 1$.

Note that item (a) implies item (c). Indeed by a profinite version of [5, Ch. 8, Prop. 6.1] since the profinite $\mathbb{Z}_p[[N]]$-module (resp. $\mathbb{Z}_p[[G]]$-module) $A$ has type $FP_m$ (resp. $FP_{m+1}$) and has profinite projective dimension at most $m$ (resp. $m + 1$) there is a profinite projective resolution of $A$ as a profinite $\mathbb{Z}_p[[N]]$-module (resp. $\mathbb{Z}_p[[G]]$-module) of length at most $m$ (resp. $m + 1$) and all projectives finitely generated.

From now on suppose that $N$ acts trivially on $A$. Since $A$ is finitely generated as a profinite $\mathbb{Z}_p[[N]]$-module, $A$ is finitely generated as a $\mathbb{Z}_p$-module, so direct sum of (additive) cyclic abelian pro-$p$ groups. Then either $\mathbb{Z}_p$ or $\mathbb{Z}/p^k \mathbb{Z}$, for some $k > 0$, is a profinite $\mathbb{Z}_p[[N]]$-module of type $FP_\infty$ and projective dimension at most $pd_{\mathbb{Z}_p[[N]]}(A) \leq m$. Then by Lemma 4 (from next section) $N$ is of type $FP_\infty$ over $\mathbb{Z}_p[[N]]$ and of finite cohomological $p$-dimension $cd_p(N) \leq m$. Then (d) follows from [14, Thm. 1].

Finally suppose that $pd_{\mathbb{Z}_p[[G]]}(A) < \infty$. Note that $A$ is finitely generated as a profinite $\mathbb{Z}_p[[G]]$-module, hence is finitely generated as a profinite $\mathbb{Z}_p[[G/N]]$-module. Since $\mathbb{Z}_p[[G/N]]$ is left and right Noetherian it follows that $A$ is $FP_\infty$ as a profinite $\mathbb{Z}_p[[G/N]]$-module and $\text{Ext}_{\mathbb{Z}_p[[N]]}^i(A, \mathbb{F}_p)$ is well-defined. By going down to a subgroup of finite index in $G$ that contains $N$ we can suppose that $G/N$ acts trivially on the finite groups $\text{Ext}_{\mathbb{Z}_p[[G/N]]}^i(Z_p, \mathbb{F}_p)$ for all $i \leq \text{cd}_p(N)$, where $\text{Ext}$ is the derived functor of the continuous Hom i.e. continuous cohomology of $N$ with coefficients in $\mathbb{F}_p$. Consider the Grothendieck spectral sequence [14, Cor. 4]

$$E_2^{i,q} = \text{Ext}_{\mathbb{Z}_p[[G/N]]}^i(A, \text{Ext}_{\mathbb{Z}_p[[N]]}^q(Z_p, \mathbb{F}_p)) \Rightarrow \text{Ext}_{\mathbb{Z}_p[[G]]}^{i+q}(A, \mathbb{F}_p).$$

Then

$$E_2^{i,q} \simeq \text{Ext}_{\mathbb{Z}_p[[G]]}^i(A, \mathbb{F}_p) \otimes_{\mathbb{Z}_p} \text{Ext}_{\mathbb{Z}_p[[N]]}^q(Z_p, \mathbb{F}_p),$$

$$E_2^{i,q} = 0 \text{ for } r \geq pd_{\mathbb{Z}_p[[G]]}(A) + 1 \text{ or } q \geq \text{cd}_p(N) + 1 \text{ and}$$

$$\chi_{\text{ext}_{\mathbb{Z}_p[[G]]}^i}(A) = \sum_i (-1)^i \dim_{\mathbb{Z}_p} \text{Ext}_{\mathbb{Z}_p[[G]]}^i(A, \mathbb{F}_p) = \sum_{r, q} (-1)^{i-r} \dim_{\mathbb{Z}_p} E_2^{r,q}$$

$$= \sum_{r, q} (-1)^{i+q} \dim_{\mathbb{Z}_p} E_2^{r,q} \left( \text{Ext}_{\mathbb{Z}_p[[G/N]]}^{i+r}(A, \mathbb{F}_p) \right).$$

$$\left( \sum_{q} (-1)^{i-q} \dim_{\mathbb{Z}_p} \text{Ext}_{\mathbb{Z}_p[[G/N]]}^q(Z_p, \mathbb{F}_p) \right) = \chi_{\text{ext}_{\mathbb{Z}_p[[G/N]]}^i}(A) \cdot \chi_p(N). \quad \Box$$
4. Extension properties of profinite groups of type $FP_m$: A profinite version of Brown criterion and some auxiliary results on profinite groups

In [6] it was shown that if an abstract group $H$ acts on a CW complex $Y$ with cell stabilizers that fix the cells pointwise, the stabilizer in $H$ of any cell of dimension $i \leq m$ has homological type $FP_{m-i}$, $Y$ is $(m-1)$-acyclic and $H$ acts cocompactly on the $m$-skeleton of $Y$ then $H$ has homological type $FP_m$. A homotopical version of this result can be found in [23] and here we give a homological version for profinite modules over completed group algebras of profinite groups.

**Lemma 1.** Let $S$ be a profinite ring and

$$\mathcal{P} : \cdots \rightarrow P_i \xrightarrow{\partial_i} P_{i-1} \rightarrow \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \rightarrow 0$$

be an exact profinite complex of profinite $S$-modules, where every $P_i$ is a profinite $S$-module of type $FP_{m-i}$ for $i \leq m$. Then $A$ is of type $FP_m$ as a profinite $S$-module.

**Proof.** By [25, Lemma 7.2.2] for every two finitely generated profinite $S$-modules $M_1$ and $M_2$ any homomorphism $\varphi : M_1 \rightarrow M_2$ of abstract $S$-modules is continuous. This implies that $A$ is of type $FP_m$ as a profinite $S$-module if and only if $A$ is of type $FP_m$ as an abstract $S$-module. In particular we can apply dimension shifting for abstract modules [1, Prop. 1.4] i.e. if $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$ is a short exact sequence of profinite $S$-modules with $V$ of type $FP_m$ (as profinite or abstract $S$-module is the same as seen above) for some $m \geq 1$ then $V_2$ is of type $FP_m$ if and only if $V_1$ is of type $FP_{m-1}$. Applying this for the short exact sequence $0 \rightarrow \text{Im}(\partial_{i+1}) = \text{Ker}(\partial_i) \rightarrow P_i \rightarrow \text{Im}(\partial_i) \rightarrow 0$ for $i \leq m-1$ we get that $\text{Im}(\partial_i)$ is of type $FP_{m-i}$ if and only if $\text{Im}(\partial_{i+1})$ is of type $FP_{m-i-1}$. Since $\text{Im}(\partial_m)$ is of type $FP_0$ we get that $A = \text{Im}(\partial_0)$ is of type $FP_m$ as required. □

**Proposition 1.** Let $G$ be a profinite group and $R$ be a profinite ring. Suppose that there exists an exact profinite complex of profinite $R[[G]]$-modules

$$\mathcal{P} : \cdots \rightarrow P_i \xrightarrow{\partial_i} P_{i-1} \rightarrow \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \rightarrow 0$$

with $P_i \cong \bigoplus_{j \in J_i} \hat{\otimes}_{R[[H_i]]} R[[G]]$, $H_i j$ are closed subgroups of $G$ of type $FP_{m-i}$ over $R$ for $j \in J_i$ and $J_i$ finite for all $i \leq m$. Then $A$ has type $FP_m$ as a profinite $R[[G]]$-module.

**Proof.** Let $H$ be a closed subgroup of $G$ of type $FP_{m-i}$ over $R$. Since $R$ is of type $FP_{m-i}$ as profinite $R[[H]]$-module and $\hat{\otimes}_{R[[H]]} R[[G]]$ is an exact functor we get that $\hat{\otimes}_{R[[H]]} R[[G]]$ is of type $FP_{m-i}$ as a $R[[G]]$-module. Applying this for $H = H_{i,j}$ we get that $P_i$ is of type $FP_{m-i}$ for every $i \leq m$. The proof is completed by Lemma 1. □

**Corollary 1.** Let $R$ be a profinite ring, $G$ a profinite group, $N$ a closed normal subgroup of $G$ and $A$ a profinite $R[[G]]$-module on which $N$ acts trivially. If $A$ is of type $FP_m$ as a profinite $R[[G/N]]$-module and $N$ is of type $FP_m$ over $R$ then $A$ is of type $FP_m$ over $R[[G]]$.

**Proof.** Let $\mathcal{P} : \cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots \xrightarrow{\partial_1} P_0 \rightarrow A \rightarrow 0$ be a profinite free resolution of $R[[G/N]]$-modules with all $P_i$ finitely generated for $i \leq m$. Since $P_i$ is a finite direct sum of copies of $\hat{\otimes}_{R[[H]]} R[[G]]$ we can apply Proposition 1. □

**Corollary 2.** Let $G$ be a profinite group with a closed normal subgroup $N$ such that both $G/N$ and $N$ are of type $FP_m$ over a profinite ring $R$. Then $G$ is of type $FP_m$ over $R$.

**Proof.** Apply the previous corollary for the trivial $R[[G]]$-module $A = R$. □

**Lemma 2.** Let $G$ be a profinite group with a normal closed subgroup $N$ and a closed subgroup $H$ such that $G \simeq N \times H$ is of type $FP_m$ over $R$. Then $H$ is of type $FP_m$ over $R$.

**Proof.** As already noted the profinite group $G$ is of type $FP_m$ over $R$ if and only if $R$ as an abstract $R[[G]]$-module is of type $FP_m$. By [1, Thm. 1.3] this is equivalent with $\text{Tor}_i^{R[[G]]}(R, \prod R[[G]]) = 0$ for $1 \leq i \leq m - 1$, where $\text{Tor}_i^{R[[G]]}$ is the derived functor of abstract tensor product $\otimes_{R[[G]]}$ and $R$ is finitely presented as an abstract $R[[G]]$-module (this is equivalent with $G$ topologically finitely generated). Note that since $G$ is topologically finitely generated $H$ is topologically finitely generated too.

Finally note that $\text{Tor}_i^{R[[G]]}(R, \prod R[[G]])$ is functorial on $G$. Then the canonical epimorphism $\pi : G \rightarrow G/N$ induces a homomorphism (of abelian groups) $\hat{\pi}_i : \text{Tor}_i^{R[[G]]}(R, \prod R[[G]]) \rightarrow \text{Tor}_i^{R[[G/N]]}(R, \prod R[[G/N]])$ that splits. In particular $\text{Tor}_i^{R[[G/N]]}(R, \prod R[[G/N]]) = 0$ for $1 \leq i \leq m - 1$ and consequently $H \simeq G/N$ is a profinite group of type $FP_m$ over $R$. □

We continue with a simple lemma that is a module version of [24, Prop. 21’]. For a profinite ring $S$ we write $\hat{\otimes}_S$ and $\hat{\text{Ext}}_S^i$ for the derived functors of $\otimes_S$ and continuous $\text{Hom}_S$. 
Lemma 3. Let $G$ be a profinite group, $p$ a prime number and $A$ a profinite $\mathbb{Z}_p[[G]]$-module. Then the following conditions are equivalent:

1. $pd_{\mathbb{Z}_p[[G]]}(A) \leq n$;
2. $\text{Ext}_{\mathbb{Z}_p[[G]]}^{n+1}(A, \mathbb{F}_p) = 0$ for every closed subgroup $H$ of $G$;
3. $\text{Ext}_{\mathbb{Z}_p[[G]]}^{n+1}(A, \mathbb{F}_p) = 0$ for every open subgroup $U$ of $G$;
4. $\text{Tor}_{n+1}^{\mathbb{Z}_p[[G]]}(A, \mathbb{F}_p) = 0$ for every closed subgroup $H$ of $G$;
5. $\text{Tor}_{n+1}^{\mathbb{Z}_p[[G]]}(A, \mathbb{F}_p) = 0$ for every open subgroup $U$ of $G$.

Proof. Condition 1 implies immediately conditions 2, 3, 4 and 5. Condition 3 implies condition 2 since $\mathbb{Z}_p[[G]]$ commutes with inverse limits. Similarly condition 5 implies condition 4. To show that condition 2 implies condition 1 we recall that by [14, Thm. 2] $pd_{\mathbb{Z}_p[[G]]}(A) = pd_{\mathbb{Z}_p[[G]]}(A)$ where $G_p$ is a Sylow $p$-subgroup of $G$. Furthermore for a pro-$p$ group $H$ and a profinite $\mathbb{Z}_p[[G]]$-module $B$ we have that $B$ has projective $p$-dimension $pd_{\mathbb{Z}_p[[G]]}(B) \leq n$ if and only if $\text{Ext}_{\mathbb{Z}_p[[G]]}^{n+1}(B, \mathbb{F}_p) = 0$ [20, Cor. 7.1.6], [24, Prop. 21]. Then condition 2 applied for a Sylow $p$-subgroup $H$ of $G$ gives that $pd_{\mathbb{Z}_p[[G]]}(A) = pd_{\mathbb{Z}_p[[G]]}(A) \leq n$.

It remains to show that condition 4 implies condition 1. It is sufficient to show that for a pro-$p$ group $H$ and a profinite $\mathbb{Z}_p[[H]]$-module $B$, $\text{Tor}_{n+1}^{\mathbb{Z}_p[[H]]}(B, \mathbb{F}_p) = 0$ implies $pd_{\mathbb{Z}_p[[H]]}(B) < \infty$ and then apply this for $H$ a Sylow $p$-subgroup of $G$ and $B = A$. Following the proof of [20, Prop. 7.1.4], in particular the last half of page 261, it is sufficient to show that if for some profinite $\mathbb{Z}_p[[H]]$-module $M$ we have that the functor $\text{Tor}_{n+1}^{\mathbb{Z}_p[[H]]}(M, \cdot)$ is zero then $M$ is projective. Let $\pi : F \rightarrow M$ be an epimorphism of profinite $\mathbb{Z}_p[[H]]$-modules with $F$ free and of minimal rank. Then $\text{Ker}(\pi) \otimes_{\mathbb{Z}_p[[H]]} \mathbb{F}_p = 0$ and since $\mathbb{Z}_p[[H]]$ is a local ring $\text{Ker}(\pi) = 0$. □

Lemma 4. Let $G$ be a profinite group and $k$ be a positive integer. Then

(a) if the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p/p^k\mathbb{Z}_p$ is of type $F_{\infty}$ then the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ is of type $F_{\infty}$;
(b) if the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p/p^k\mathbb{Z}_p$ has finite projective dimension $s$ then the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ has finite projective dimension at most $s$.

Proof. (a) We show first that the trivial $\mathbb{Z}_p/p^k\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p/p^k\mathbb{Z}_p$ is of type $F_{\infty}$. By assumption the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p/p^k\mathbb{Z}_p$ has a profinite projective resolution $P^\bullet$ with $P_i$ finitely generated profinite $\mathbb{Z}_p[[G]]$-module for every $i$. Then

$$H_i(P \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p)) = \text{Tor}_i^{\mathbb{Z}_p}(\mathbb{Z}_p/p^k\mathbb{Z}_p, \mathbb{Z}_p/p^k\mathbb{Z}_p) = 0 \text{ for } i \geq 2$$

(5)

and

$$H_1(P \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p)) \simeq \text{Tor}_1^{\mathbb{Z}_p}(\mathbb{Z}_p/p^k\mathbb{Z}_p, \mathbb{Z}_p/p^k\mathbb{Z}_p) \simeq \mathbb{Z}_p/p^k\mathbb{Z}_p.$$ 

Let $\{d_i\}$ be the differentials of the complex $P \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p)$. Then there are exact sequences of $\mathbb{Z}_p/p^k\mathbb{Z}_p[[G]]$-modules

$$0 \rightarrow \text{Im}(d_2) \rightarrow \text{Ker}(d_1) \rightarrow H_1(P \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p)) \simeq \mathbb{Z}_p/p^k\mathbb{Z}_p \rightarrow 0$$

(6)

and

$$0 \rightarrow \text{Ker}(d_1) \rightarrow P_1 \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p) \xrightarrow{d_1} P_0 \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p) \xrightarrow{d_0} \mathbb{Z}_p/p^k\mathbb{Z}_p \rightarrow 0.$$ 

(7)

In this paragraph all modules are profinite $\mathbb{Z}_p/p^k\mathbb{Z}_p[[G]]$-modules. Note that by (5) $\text{Im}(d_2)$ is of type $F_{\infty}$, then by dimension shifting argument for (6) $\text{Ker}(d_1)$ is of type $F_{\infty}$ if and only if $\mathbb{Z}_p/p^k\mathbb{Z}_p$ is of type $F_{\infty}$ and by dimension shifting argument for (7) $\text{Ker}(d_1)$ is of type $F_{\infty}$ if and only if $\mathbb{Z}_p/p^k\mathbb{Z}_p$ is of type $F_{\infty}$. In particular if $\mathbb{Z}_p/p^k\mathbb{Z}_p$ is of type $F_{\infty}$ then it is of type $F_{\infty}$. Consequently is of type $F_{\infty}$.

Now assume that the trivial profinite $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ is of type $F_{\infty}$ (note this holds for $m = 0$) and we show that it is of type $F_{m+1}$. Let

$$\mathcal{R} : \cdots \rightarrow P_i \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p) \xrightarrow{\partial_i} P_{i-1} \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p) \rightarrow \cdots \rightarrow P_0 \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p) \rightarrow \mathbb{Z}_p/p^k\mathbb{Z}_p \rightarrow 0$$

be a profinite projective resolution of the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ with $P_i$ finitely generated for $i \leq m$. Since $\text{Tor}_i^{\mathbb{Z}_p}(\mathbb{Z}_p/p^k\mathbb{Z}_p, \mathbb{Z}_p/p^k\mathbb{Z}_p) = 0$ for $i \geq 1$ the complex $\mathcal{R} \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p)$ is exact, hence is a profinite projective resolution of the trivial $\mathbb{Z}_p/p^k\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p/p^k\mathbb{Z}_p$. Since $\mathbb{Z}_p/p^k\mathbb{Z}_p$ is of type $F_{\infty}$ as a $(\mathbb{Z}_p/p^k\mathbb{Z}_p)[[G]]$-module we get that $\text{Ker}(\rho_m)$ is finitely generated as a $(\mathbb{Z}_p/p^k\mathbb{Z}_p)[[G]]$-module, where $\{\rho_i\}$ are the differentials of $\mathcal{R} \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p)$. It is easy to see that $\text{Tor}_i^{\mathbb{Z}_p}(\text{Im}(\partial_m), (\mathbb{Z}_p/p^k\mathbb{Z}_p)) = 0 = \text{Tor}_i^{\mathbb{Z}_p}(\text{Im}(\partial_{m-1}), (\mathbb{Z}_p/p^k\mathbb{Z}_p))$ and this implies

$$\text{Ker}(\rho_m) \simeq \text{Ker}(\rho_m) \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p^k\mathbb{Z}_p).$$
By (8) and the fact that $\text{Ker}(\rho_m)$ is finitely generated as a $(\mathbb{Z}_p/p^k\mathbb{Z}_p)[G]$-module there is a finitely generated $\mathbb{Z}_p[[G]]$-submodule $V$ of $\text{Ker}(\partial_m)$ such that $\text{Ker}(\partial_m) = V + p^k\text{Ker}(\partial_m)$. Consequently for every positive integer $s$

$$\text{Ker}(\partial_m) = V + p^k\text{Ker}(\partial_m) \subseteq R_m.$$  

Then the closure of $V$ in $R_m$ is $\text{Ker}(\partial_m)$ and since $V$ is finitely generated submodule of $R_m$, $V$ is closed. Thus $V = \text{Ker}(\partial_m)$ is finitely generated as a profinite $\mathbb{Z}_p[[G]]$-module, consequently $\mathbb{Z}_p$ is of type $F_{m+1}$ over $\mathbb{Z}_p[[G]]$.  

(b) For $H$ a closed subgroup of $G$ and $s_t = pd_{\mathbb{Z}_p[[H]]}(\mathbb{Z}_p/p^k\mathbb{Z}_p) \leq s = pd_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p/p^k\mathbb{Z}_p) = 0$ for every $p$-primary finite discrete $H$-module $M$ and $i \geq s + 1 \geq s_t + 1$. Then the long exact sequence in $\text{Ext}$ implies that for $i \geq s + 1$ the inclusion of $p^k\mathbb{Z}_p$ in $\mathbb{Z}_p$ induces an isomorphism

$$\varphi_{i,k} : \text{Ext}^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, M) \rightarrow \text{Ext}^i_{\mathbb{Z}_p[[H]]}(p^k\mathbb{Z}_p, M).$$

If $P$ is a profinite projective resolution of $\mathbb{Z}_p$ as a $\mathbb{Z}_p[[H]]$-module then $p^kP$ is a profinite projective resolution of $p^k\mathbb{Z}_p$ as a $\mathbb{Z}_p[[H]]$-module. Then the inclusion of $p^kP$ in $P$ induces the map $\psi_k : \text{Hom}_{\mathbb{Z}_p[[G]]}(P, M) \rightarrow \text{Hom}_{\mathbb{Z}_p[[H]]}(p^kP, M)$ that is trivial if $p^kM = 0$. Since $\varphi_{i,k}$ is induced by $\psi_k$ we get $\text{Ext}^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, M) = 0$ for $p^kM = 0$ and $i \geq s + 1$. In particular this holds for $M$ the trivial $\mathbb{Z}_p[[H]]$-module $\mathbb{F}_p$. This combined with Lemma 3 implies that $pd_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p) \leq s$. □

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