Solution of Nonlinear Partial Differential Equations from Base Equations

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1. Introduction

It is well known that certain nonlinear ordinary differential equations can be solved in terms of the solution of a related linear ordinary differential equation. Although Kamke [1] lists at least twenty such nonlinear differential equations, a note by Pinney [2] seems to be the starting point of recent interest [3–10] in this method of solution. Ames [11] gives a summary of results and references through 1965. Solutions of specific nonlinear differential equations [10] have also served as the basis of solution for nonlinear equations. The differential equation whose solutions are used to solve another differential equations will be called the base equation.

Compared to ordinary differential equations, the use of base equations for solving nonlinear partial differential equations is not as extensive. Ames [12] has discussed certain equations equivalent to linear forms. Along the same lines, Dasarathy [13] has shown through a transformation technique the equivalence of a class of second-order nonlinear equations to third-order linear equations. Montroll [14] has used the idea of a base equation in solving nonlinear equations describing population growth and diffusion. Recently, Reid and Pritchard [15] have extended a nonlinear differential equation due to Herbst [16, Eq. IV] to one of $n$-dimensions, for which initial value problems may be posed if certain conditions are met by the solution of the base equation. Without being so designated, the base equation technique has been used in constructing nonlinear quantum field theories in which the Klein–Gordon equation becomes the base equation (see, e.g. [17–19]). We mention that Chambers [20] employed solutions of the scalar wave equation to form solutions of the Klein–Gordon equation.

In this paper we apply the base equation method to obtain a class of nonlinear partial differential equations of the second order. Rather elaborate analyses [10–11] have been used to establish forms of nonlinear ordinary differential equations. However, to derive a number of important equations it
is sufficient to use a direct substitution technique [2–5], which we extend to \( n \) independent variables. We first assume the form of a linear differential operator \( L \) and the form of a function \( \Phi(x) \) and then determine the nonlinear terms \( R \) from the relation

\[
L \Phi(x) = R(x, \Phi, \Phi_x), \quad \Phi_x = \frac{\partial \Phi}{\partial x}, \tag{1.1}
\]

where

\[
x = (x_1, \ldots, x_n) \tag{1.2}
\]
denotes a set of \( n \) independent variables. Performing the operations on the left side of (1.1) and invoking the use of the base equation provide an identity for \( R \). The function \( \Phi \) will certainly satisfy the resulting nonlinear equation. We will find that the nonlinear terms have the same form as those found for the ordinary differential equations, provided we introduce a formal generalization of the Wronskian. Hence, it is necessary only to demonstrate the calculation for a single choice of \( \Phi \). For other choices of \( \Phi \) the nonlinear partial differential equation will follow by inspection from the analogous ordinary equation.

We define \( L \) and \( \Phi \) in Section 2 and perform the calculation indicated by (1.1) to obtain explicitly the nonlinear terms. A generalized Wronskian is defined. In Section 3, we examine a special case to illustrate the generation of specific nonlinear equations, and we suggest an algorithm for analyzing a given nonlinear equation. The possibility of describing initial value problems is considered in Section 4. We determine a sufficient condition for constant coefficients of the nonlinear terms in Section 5. Applications to a nonlinear differential equation encountered in quantum field theory are discussed in Section 6. Finally, in Section 7 we obtain a result for a nonlinear base equation.

### 2. Linear Base Equation

We define a linear operator \( L_k \) as

\[
L_k = \sum_{i,j=1}^{n} f_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} g_i(x) \frac{\partial}{\partial x_i} + kh(x), \tag{2.1}
\]

and for \( \Phi \) we construct the function

\[
\Phi = (au^m + bv^m)^{k/m}, \tag{2.2}
\]

where \( k \) and \( m \) are assumed real and nonzero. The constants \( a \) and \( b \) are arbitrary, and the functions \( u(x) \) and \( v(x) \) are assumed to satisfy the linear base equation

\[
L_k \Phi = 0, \tag{2.3}
\]
where $L_1$ is (2.1) with $k = 1$. The function $\Phi$ satisfies the nonlinear partial differential equation

$$L_k \Phi = (1 - l) \Phi^{-1} \sum_{i,j=1}^n f_{ij} \Phi_{x_i} \Phi_{x_j} + k(m - 1) ab(uv)^{m-2} W^2 \Phi^{1-2m}, \quad kl = 1,$$

provided that $W^2$ is defined by

$$W^2 = \sum_{i,j=1}^n f_{ij} [u^{b} v_{x_i} v_{x_j} - uv(u_{x_i} v_{x_j} + v_{x_i} u_{x_j}) + u^{b} u_{x_i} u_{x_j}].$$

To prove this statement we substitute (2.1) and (2.2) into (1.1) and calculate the left side. The calculation is readily made in terms of the quantity

$$A_i = au^{m-1} u_{x_i} + b v^{m-1} v_{x_i}$$

defined such that

$$\Phi_{x_i} = k\Phi^{1-ml} A_i.$$ Using (2.6) and (2.7), we express (1.1) as

$$k^2(1 - ml) \sum_{i,j=1}^n f_{ij} A_i + k \Phi^{ml} \left[ \sum_{i,j=1}^n f_{ij}(A_i)_{x_i} + \sum_{i=1}^n g_i A_i + h \Phi^{ml} \right] = \Phi^{2ml-1} R.$$ We can simplify the expression in brackets in (2.8) by using the form of $\Phi$ and the condition that $u$ and $v$ satisfy (2.3). We are left with the result

$$k(k - 1) \sum_{i,j=1}^n f_{ij} A_i A_j + k(m - 1) ab(uv)^{m-2} W^2 = \Phi^{2ml-1} R,$$

where $W^2$ is given by (2.5). From (2.7) we can obtain an alternative expression for $A_i$, which, when used in (2.9), leads to the equation

$$(1 - l) \Phi^{2ml-2} \sum_{i,j=1}^n f_{ij} \Phi_{x_i} \Phi_{x_j} + k(m - 1) ab(uv)^{m-2} W^2 = \Phi^{2ml-1} R,$$

from which we can identify $R$ as the nonlinear terms given in (2.4).

The nonlinear terms of (2.4) are clearly analogous to the nonlinear terms of [4, Eq. (18)], obtained with (2.2) as a function of a single independent variable. The nonlinear partial differential equation corresponding to (2.1) and

$$\Phi = (au^m + mbv^n + cv^m)^{j/m}, \quad j + n = m,$$

can be written down immediately from [4, Eqs. (24)–(25)].

We note that if $u = C v$, where $C$ is any constant, then $W^2$ given by (2.5) vanishes. In this sense, $W$ is a generalization of the Wronskian of ordinary
second-order differential equations. It is also possible that $W$ may vanish when $u \neq Cv$: this possibility is demonstrated in the next section. For the case $u = Cv$ we have the important equation

$$L_k \Phi = (1 - l) \Phi \sum_{i,j=1}^{n} f_{ij}(x) \Phi_{x_i} \Phi_{x_j},$$  \hspace{1cm} (2.12)$$

which is satisfied by

$$\Phi = u^k,$$  \hspace{1cm} (2.13)$$

where we have $C = [(1 - a)/b]^{1/m}$. We would also get (2.12) and (2.13) if $v = 0, u \neq 0,$ and $a = 1$. Equation (2.12) generalizes a class of equations with quadratic nonlinearities, such as those considered by Montroll [21].

The form of (2.12) also follows from (2.4) with $m = 1$ and $u \neq Cv$, but the solution becomes

$$\Phi = (au + bv)^k.$$  \hspace{1cm} (2.14)$$

With $f_{ij}(x) = 0, i \neq j$, $f_{ii}(x) = \text{const}$, $k = 2$, and with $g_i(x)$ and $h(x)$ appropriately defined, (2.12) and (2.14) extend the one-dimensional quantum probability equation considered in [22] to two and three dimensions.

3. AN EXAMPLE AND AN ALGORITHM

Although the linear base equation (2.3) follows from (2.4) with $k = m = 1$, it is worthwhile to point out that (2.3) and its solution exist independently of the nonlinear equation (2.4). Thus any two particular solutions $u$ and $v$ may be selected to form the function $\Phi$, and, since the explicit forms of $U(x)$ and $V(x)$ are known, the coefficients of the nonlinear terms of (2.4) are explicit functions of $x$. Therefore, the solutions to the base equation can be used, through the combination $(uv)^m - W^2$, to generate a large set of nonlinear equations satisfied by the form $\Phi$.

We now consider an explicit example in two variables $x_1$ and $x_2$. The solution of the equation [23, p. 176]

$$w_2 h_{z,zz} - x_2^2 e^{2x_1} - 2x_1 e^{x_2} - w = 0$$  \hspace{1cm} (3.1)$$

is given by

$$\phi = x_2^2 F_1(x_1) + x_2 F_2(x_1 x_2),$$  \hspace{1cm} (3.2)$$

where $F_1$ and $F_2$ are arbitrary functions of their arguments. For the particular choices

$$F_1 = \cos x_1, \hspace{1cm} u = x_2^2 \cos x_1,$$  \hspace{1cm} (3.3)$$

$$F_2 = x_1 x_2, \hspace{1cm} v = x_1 x_2^2,$$  \hspace{1cm} (3.4)$$
we find that \( W^2 \) is identically zero. Thus the nonlinear equation

\[
x_1 x_2 \Phi_{x_1 x_2} - x_2^2 \Phi_{x_2^2} - 2 x_1 \Phi_{x_1} + 2 x_2 \Phi_{x_2} - 2k\Phi
\]

\[
= (1 - l) \Phi^{-1}(x_1 x_2 \Phi_{x_1} \Phi_{x_2} - x_2^2 \Phi_{x_2})
\]

has the solution

\[
\Phi = x_2^{2k}(a \cos^m x_1 + bx_1^{-m})^{k/m}.
\]

A similar result holds when \( \cos x_1 \) is replaced by \( \sin x_1 \). Other choices of \( u \) and \( v \) from (3.2) yield nonvanishing \( W \). For example, if \( F_1 \) and \( F_2 \) are arbitrary nonzero constants we have

\[
W^2 = -(F_1 F_2)^2 x_2^2.
\]

For \( m \neq 1 \), the corresponding nonlinear equation is

\[
L\Phi = (1 - l) \Phi^{-1}(x_1 x_2 \Phi_{x_1} \Phi_{x_2} - x_2^2 \Phi_{x_2}) + abx_2^{2m} \Phi^{1-2m},
\]

with the solution

\[
\Phi = (F_1 x_2)^k [a - b x_2^{m/F_2^{2m}k(m - 1)}]^{k/m},
\]

where \( L\Phi \) in (3.8) denotes the left side of (3.5).

These examples suggest the following algorithm by which the solution of a given nonlinear equation may be found by our technique: The parameters \( k \) and \( m \) are adjusted and particular solutions \( u \) and \( v \) are found to match, through \((uv)^{m-2} W^2 \), the given nonlinear terms. The arbitrary constants \( a \) and \( b \) are chosen to give the required product. If these steps are possible for a given nonlinear equation then the question of initial conditions can be considered.

4. Initial Conditions

Initial value problems for equations of the form (2.12) have been considered in [15]. The solution (2.13), involving only one solution of the base equation, is particularly simple to adapt to initial value problems, provided \( u \) and \( \partial u/\partial x_i \) meet certain requirement. We will illustrate this statement. Following [15], let \( x_0 \) denote initial values for any \( m \) of the \( n \) independent variables and let \( x \) denote the remaining \( n - m \) variables. Thus, we have

\[
\Phi(x, x_0) = \Phi_0(x)
\]
as a specified function of \( z \), while the function

\[ u(z, x_0) = u_0(z) \]  

(4.2)

is to be determined. From (2.13) it follows that

\[ u_0(z) = [\Phi_0(z)]', \]  

(4.3)

and we note that an initial value problem is possible if \( \Phi_0(z) \neq 0 \) when \( l < 0 \).

A similar statement can be made for the first derivative

\[ u_{x_i} |_{x_i = x_{i0}} = l[\Phi_i(z, x_{i0})]^{-1} g(z), \]  

(4.4)

where

\[ g(z) = \Phi_{x_i} |_{x_i = x_{i0}} \]  

(4.5)

is some specified function.

The circumstances are more complicated when both base solutions \( u \) and \( v \) are involved, as in (2.4). Initial conditions (4.3) and (4.4) still apply if it is possible to find a solution \( v(x) \) such that the coefficient \( (uv)^{m-2} W^2 \) can be matched with the given equation and such that

\[ v(z, x_0) = v_0(z) = 0, \quad m \neq 1. \]  

(4.6)

These initial conditions are straightforward generalizations of the one-dimensional problem solved by Pinney [2]. For convenience we have set \( a = 1 \). Clearly, when \( m = 2 \) the product \( uv \) is of no consequence, but the function \( W \) is generally not a constant.

5. Nonlinear Term with Constant Coefficient

We now raise the question: Under what conditions is the factor \( (uv)^{m-2} W^2 \) constant? This is an important question because the coefficients of the nonlinear terms are constant in many problems of physical interest. The supposition that \( v = u^{-1} \) is a solution of the base equation leads to a condition which is sufficient to insure that the factor is constant. Under this supposition, \( uv = 1 \), and only the function \( W^2 \) remains. Using \( u^{-1} \) in (2.3), we find that \( u \) must also satisfy the equation

\[ u^{-2} \sum_{i,j=1}^{n} f_{ij}(x) u_{x_i} u_{x_j} + h(x) = 0. \]  

(5.1)
With $v = u^{-1}$ in (2.5) the function $W^2$ becomes

$$W^2 = 4u^{-2} \sum_{i,j=1}^{n} f_{ij}(x) u_{x_i} u_{x_j}.$$  

(5.2)

Comparison of (5.1) and (5.2) shows that

$$W^2 = -4h(x).$$  

(5.3)

Therefore, a sufficient condition for the factor $(uv)^{-m-2} W^2$ to be constant, with $v = u^{-1}$, is that $h(x)$ be constant.

When $f_{ij}$ is constant and $h$ is identically zero in (2.3), the solutions, with constant $a_i$,

$$u = \sum_{i,j=1}^{n} a_i x_i, \quad v = \text{const},$$  

(5.4)

yield a constant for $W$. For this case, however, $(uv)^{-m-2}$ is constant only for $m = 2$.

An equation equivalent to (5.1) can be established by use of the identity

$$u_{x_i}^{n} u_{x_j} = nu^{n-1} u_{x_ix_j} + n(n-1) u^{n-2} u_{x_i} u_{x_j},$$  

(5.5)

with $n = 2$. We then find that $u^n$ satisfies the differential equation

$$\sum_{i,j=1}^{n} f_{ij} u_{x_i}^{n} + \sum_{i=1}^{n} g_i u_{x_i}^{n} + n^2 h u^n = 0.$$  

(5.6)

This equation is linear in $u^n$ and has the form of (2.3). This is particularly important if the coefficients $f_{ij}$, $g_i$, and $h$ are constant, since $u^n$ then can be represented by a Fourier integral.

6. APPLICATIONS TO A PROBLEM IN FIELD THEORY

We now consider a special case of (2.4) which is of particular physical interest. The physical application is the field equation arising in the study of spin zero mesons, including a self interaction. This equation is

$$\partial^\mu \partial_\mu \Phi + M^2 \Phi + \lambda \Phi^8 = 0,$$  

(6.1)

with the Klein–Gordon as base equation, i.e.,

$$\partial_\mu \partial^\mu u + M^2 u = 0.$$  

(6.2)
In (6.1) and (6.2) $M$ is the meson mass, $\lambda$ is the self-interaction coupling constant, and $\partial_u = (\partial^0, -\nabla)$. Indices are lowered by the metric tensor $g^{uv} = \text{diag}(1, -1, -1, -1)$. Solutions of the Cauchy problem for this differential equation have been discussed by several authors [24–27]. Explicit solutions have been given for one value of the momentum by Chew [25] and Raczka [26]. The solutions which we give here differ from the previous in that they describe self-interacting systems for all times [28], rather than reducing to free particle solutions at $t = \pm \infty$.

In order to exploit the general method of solution described above, we must have

$$l = 1, \quad (m - 1) ab(uv)^{m-2} W^2 \Phi^{1-2m} = \lambda \Phi^2. \quad (6.3)$$

Consequently, we find $m = -1$. If we further require

$$uv = 1, \quad W^2 = 4M^2, \quad (6.4)$$

then we have

$$-8abM^2 = \lambda. \quad (6.5)$$

Now consider the restriction imposed by (6.4). This condition requires that the second solution of the base equation must be the inverse of the first. Thus, we find that $u$ must satisfy the additional equation

$$\partial^u u \partial_u u + M^2 u^2 = 0. \quad (6.6)$$

Therefore, if $u$ satisfies (6.2) and (6.6) then a solution of (6.1) is

$$\Phi = u(1 - \lambda u^2/(8M^2))^{-1}. \quad (6.7)$$

As explicit examples, we may choose

$$u^{(\pm)} = A^{(\pm)} e^{\mp \bar{\xi} \cdot x (2\omega)^{-1/2}}, \quad (6.8)$$

where $A^{(\pm)}$ are constants and

$$\bar{k} \cdot \bar{x} = k_0 x_0 - k \cdot r, \quad \omega_k = (k^2 + M^2)^{1/2}. \quad (6.9)$$

The general solutions, for one value of momentum, are elliptic functions [25–27]. Solutions (6.7) and (6.8) have been employed in the study of the quantization of self-interacting fields [28–29] which have the property that they describe self-interacting systems for all times.

With the help of (6.2) and (6.6) we can obtain a differential equation for $u^n$ for arbitrary $n$. This is

$$\partial_u \partial^u u^n + n^2 M^2 u^n = 0. \quad (6.10)$$
As mentioned in the previous section, since this is a linear second-order differential equation with constant coefficients, the function $u^n$ is guaranteed to have a Fourier decomposition.

As a final comment on these solutions, we remark that the nonlinear term in (6.1) may be replaced by $\Phi^{2q+1}$, $q \neq 0, -1$, with no change in the details of the solutions. The particular solution for this case is

$$\Phi_q - u[1 - (\lambda/4(q + 1) M^q) u^{2q}]^{-1/\eta}, \quad (6.11)$$

where $u$ must still satisfy (6.2) and (6.6).

7. NONLINEAR BASE EQUATION

In this section we present a nonlinear partial differential equation which is equivalent to a nonlinear base equation. The nonlinear equation

$$L \Phi = G(x) \Phi^{1-m_1} + (1-l) \Phi^{-1} \sum_{i,j=1}^{n} f_{ij} \Phi_{x_i} \Phi_{x_j} + k(m-1) ab(\nu c)^{m_2} W^2 \Phi^{1-2m_1}, \quad kl = 1, \quad (7.1)$$

has a solution of the form of (2.2), provided now that $u$ and $v$ satisfy the nonlinear equations

$$L \beta = G(x) \Phi^{1-m}/2\beta, \quad (7.2)$$

where $\beta = a$ for $\Phi = u$ and $\beta = b$ for $\Phi = v$. When $a = b$, the functions $u$ and $v$ satisfy the same base equation.

Equation (7.1) may be obtained by adding the term $G\Phi^{1-m_1}$ to the left side of (2.4). It follows that the reduction of the equation analogous to (1.1) is unaffected provided that $u$ and $v$ satisfy (7.2).

A particular case of (7.1) in one independent variable is encountered in the classical motion of charged particles in crossed electric and magnetic fields with cylindrical symmetry [30].

REFERENCES

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