Asymptotic properties of solutions of nonlinear difference equations

Rigoberto Medina

Departamento de Ciencias Exactas, Universidad de Los Lagos, Casilla 933, Osorno, Chile

Received 25 September 1994

Abstract

Using discrete inequalities and Schauder's fixed point theorem we study the problem of asymptotic equilibrium for difference equations.

Keywords: Difference inequalities; Asymptotic equilibrium; Asymptotic equivalence; Asymptotic formulae

AMS classification: primary 39A10

1. Introduction

We consider the difference equation

\[ \Delta x(n) = f(n, x(n)), \]

where \( x \) and \( f \) are \( s \)-dimensional vectors, \( N_0 = \{n_0, n_0 + 1, \ldots, n_0 + k, \ldots \} \) with \( n_0 \in \{0, 1, 2, \ldots \} \); \( f(n, x) \) is defined on \( N_0 \times \mathbb{R}^s \), and \( \Delta \) will denote the forward difference operator, that is, \( \Delta x(n) = x(n + 1) - x(n) \).

Eq. (1) has asymptotic equilibrium if:

(i) there exists \( r > 0 \) such that any solution \( x = x(n, n_0, x(n_0)) \) of Eq. (1) with \( |x(n_0)| < r \) satisfies \( x(n) = \xi + o(1) \) as \( n \to \infty \); (2)

(ii) corresponding to each \( \xi \in \mathbb{R}^s \), there is a solutions of Eq. (1) satisfying (2).

We will prove the asymptotic equilibrium of Eq. (1) for \( f = f(n, x) \) summable in \( n \) for \( x \) fixed, which are not necessarily Lipschitz or bounded with respect to \( x \).

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1 Research supported by Dirección de Investigación Universidad de Los Lagos and Fondecyt under Grant No. 1940779.
Furthermore, for quasilinear systems,
\[ \Delta x(n) = A(n)x(n) + g(n, x(n)), \]  
we find that its solutions \( x \) satisfy the asymptotic formula
\[ x(n) = \Phi(n) [\zeta + o(1)] \quad \text{as } n \to \infty, \]
where \( \Phi \) is a fundamental matrix solution of the unperturbed linear system
\[ \Delta y(n) = A(n)y(n). \]

As an important application of our main results we establish the asymptotic equivalence between the solutions of Eqs. (3) and (5).

To prove our results we will use a general nonlinear discrete inequality combined with the well-known Schauder fixed point theorem. Any nonlinear situation requires a compact operator and hence a compactness criterium in \( \ell_{\infty} = \ell_{\infty} (N_0, \mathbb{R}^q) \), the Banach space of the uniformly bounded sequences. In this paper we use the following simple criterium: "A set \( E \subset \ell_{\infty} \) is relatively compact if \( E \) is uniformly bounded and equiconvergent to some \( \xi \in \mathbb{R}^q \)."

We recall that \( E \subset \ell_{\infty} \) is said equiconvergent to \( \xi \in \mathbb{R}^q \) if for every \( \varepsilon > 0 \), there exists \( T \geq n_0 \) such that \( |x(n) - \xi| < \varepsilon \) for any \( x \in E \) and \( n \geq T \).

Some papers dealing with the subject of this work, for difference equations, can be found in \([2-5, 13, 11]\). Other important references about qualitative theory of difference equations are \([8, 1, 6]\). The asymptotic equilibrium of differential equations is a classical problem, see \([7, 12]\).

2. Preliminaries

In this section we establish a useful comparison theorem, which gives an explicit solution of a discrete inequality of Bihari type whose nonlinear nature is very general.

Consider the inequality
\[ u(n) \leq c + \sum_{i=1}^{p} \left\{ \sum_{k=n_0}^{n-1} \lambda_i(k) \omega_i(u(k)) \right\}, \quad p \in \mathbb{N} \]  
under the following conditions (C):
\[ (C.1) \quad c \geq 0; \quad u(n) \geq 0 \quad \text{for any } n \in N_0. \]
\[ (C.2) \quad 0 \leq \lambda_i \in \ell_1 (N_0), \quad (1 \leq i \leq p) \quad \text{and} \]
\[ (C.3) \quad \text{The functions } \omega_i (1 \leq i \leq p) \text{ are continuous and nondecreasing on } [0, \infty) \text{ and positive on } (0, \infty) \text{ such that } \omega_{i+1}/\omega_i (1 \leq i \leq p - 1) \text{ are nondecreasing on } (0, \infty). \]

To solve inequality (6), we define
(i) The functions
\[ W_k(u) = \int_{u_k}^{u} \frac{ds}{\omega_k(s)}, \quad u > 0, \quad u_k > 0 \quad (1 \leq k \leq p) \]  
and \( W_k^{-1} \) their inverse function.
(ii) The functions \( \varphi_0(u) = u \) and
\[ \varphi_k = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1, \quad \psi_k(u) = W_k^{-1} [W_k(u) + \alpha_k], \]
where 
\[ \alpha_k = \sum_{i=n_0}^{\infty} \lambda_k(i). \]

It is easy to see that the functions \( \varphi_k \) (and \( \psi_k \)) do not depend on the choice of \( u_k \) in (7), and any \( \varphi_k \) \((1 \leq k \leq p)\) is a continuous, positive and nondecreasing function on its domain (see [10, Remark 4]).

Thus, we have the following theorem.

**Theorem A** (Medina and Pinto [9]). Under conditions (C), if \( u \) satisfies the inequality (6) and \( c < \varphi_p^{-1} (\infty) \), then for any \( n \in N_0 \) we have
\[ u(n) \leq \varphi_p(c). \tag{9} \]

Using the discrete “integral” \( \sum_{k=0}^{n} f(k) \) and the discrete “derivative” \( \Delta \), the proof of Theorem A is quite similar to the proof of the continuous case, which can be seen in [10, Theorem 6, pp. 126-129].

We remark that if
\[ \int_{1}^{\infty} \frac{ds}{\omega_i(s)} = \infty, \quad (1 \leq i \leq p), \tag{10} \]
then any \( \varphi_k \) (and \( \psi_k \)) is defined for all \( u \). Thus, (9) is valid for all \( c > 0 \). The dual condition to (10), namely
\[ \int_{0}^{1} \frac{ds}{\omega_i(s)} = \infty, \quad (1 \leq i \leq p) \tag{11} \]
implies that any \( \varphi_k \) (and \( \psi_k \)) is defined for all \( u \) small enough. Then (9) is valid if \( c \) is small enough. Moreover, (11) implies
\[ \varphi_k(0^+) = 0, \quad (1 \leq k \leq p) \tag{12} \]
which is actually a stability condition.

**Example 2.1.** Calculus of \( \psi_i \) and \( \varphi = \psi_2 \circ \psi_1 \) for \( \omega_i(u) = u^n_i, \ i = 1, 2 \). We have that \( \psi_i(u) = W_i^{-1} [W_i(u) + \alpha_i] \) is given by
\[ \psi_i(u) = \begin{cases} [u^{1-n_i} + \alpha_i(1 - n_i)]^{-1/n_i-1} & \text{if } n_i \neq 1, \\ u \exp \alpha_i & \text{if } n_i = 1. \end{cases} \]

Thus, for \( n_i \neq 1 \), \( \varphi = \psi_2 \circ \psi_1 \) takes the form
\[ \varphi(u) = \{[u^{1-n_1} + \alpha_1(1 - n_1)]^{n_2-1/n_1-1} + \alpha_2(1 - n_2)\}^{-1/n_2-1}. \tag{13} \]

Moreover, since \( \psi_i^{-1}(u) = W_i^{-1} [W_i(u) - \alpha_i] \) then \( \varphi^{-1}(u) \) is obtained from (13) replacing \( \alpha_k \) by \(-\alpha_k\). Thus for \( n_i > 1 \), \( \varphi(u) \) is defined for all \( u < \varphi^{-1}(\infty) \), where
\[ \varphi^{-1}(\infty) = \{[\alpha_1(n_1 - 1)]^{n_2-1/n_1-1} + \alpha_2(n_2 - 1)\}^{-1/n_2-1}. \]
For the case \( n_1 = 1 < n_2 \) we get
\[
\varphi(u) = \left[(ue^{\alpha_1})^{1-n_2} + \alpha_2(1 - n_2)\right]^{-1/n_2 - 1} \\
= ue^{\alpha_1} \left[1 + \alpha_2(1 - n_2)(ue^{\alpha_1})^{n_2 - 1}\right]^{-1/n_2 - 1},
\]
defined for all \( u < \varphi^{-1}(\infty) = [\alpha_2(n_2 - 1)]^{-1/n_2 - 1} \) and if \( n_1 < 1 = n_2 \), then \( \varphi(u) = e^{\alpha_2\left[u^{1-n_1} + \alpha_1(1 - n_1)\right]^{-1/n_1 - 1}} \) defined for all \( u < \varphi^{-1}(\infty) = \infty \).

3. The main results

**Theorem 3.1.** Assume that for \( n \in N_0 \) and \( x \in \mathbb{R}^n \) the function \( f \) satisfies
\[
|f(n, x)| \leq \sum_{i=1}^{p} \lambda_i(n) \omega_i(|x|), \quad p \in \mathbb{N},
\]
where \( \omega_i, 1 \leq i \leq p, \) satisfies conditions (C.3), and \( \lambda_i \) are nonnegative sequences such that \( \lambda_i \in \ell_1(N_0) \) for \( 1 \leq i \leq p \). In addition, suppose that there exists a constant \( c > 0 \) such that
\[
\sum_{k=n_0}^{\infty} \lambda_p(k) < \int_{\varphi_{p-1}(c)}^{\infty} \frac{ds}{\varphi_p(s)}, \quad (14)
\]
where \( \varphi = \varphi_{p-1} \) is given by (8).

Then any solution \( x \) of Eq. (1) such that \( |x(n_0)| \leq c \) is defined on \( N_0 \) and satisfies (2).

**Proof.** Any solution \( x = x(n, n_0, x(n_0)) \) of Eq. (1) satisfies the equation
\[
x(n) = x(n_0) + \sum_{k=n_0}^{n-1} f(k, x(k)) \quad (15)
\]
for \( n_0 \leq n \leq T \). So, if \( |x(n_0)| \leq c \), we have
\[
|x(n)| \leq |x(n_0)| + \sum_{k=n_0}^{n-1} |f(k, x(k))| \\
\leq c + \sum_{i=1}^{p} \left\{ \sum_{k=n_0}^{n-1} \lambda_i(k) \omega_i(|x(k)|) \right\},
\]
where \( c < \varphi_{p-1}^{-1}(\infty) \). Theorem A implies that
\[
|x(n)| \leq W_p^{-1} \left[ W_p(\varphi_{p-1}(c)) + \sum_{k=n_0}^{n-1} \lambda_p(k) \right],
\]
which, by (14), is valid for any \( n \in N_0 \). Moreover, for \( n \in N_0 \),
\[
|x(n)| \leq W_p^{-1} \left[ W_p(\varphi_{p-1}(c)) + \sum_{k=n_0}^{n-1} \lambda_p(k) \right] < \varphi_p(c), \quad (16)
\]
where \( \varphi_p \) is given by (8).
Furthermore, for any of these solutions \( x \) we get \( f(n, x(n)) \in \ell_1(N_0) \) since

\[
|f(n, x(n))| \leq \sum_{i=1}^{p} \lambda_i(n) \omega_i(|x(n)|)
\]

\[
\leq \sum_{i=1}^{p} \lambda_i(n) \omega_i(\phi_p(c)) \in \ell_1(N_0).
\]

Then every solution \( x \) of Eq. (1) such that \(|x(n_0)| \leq c\) is defined on \( N_0 \), and by (15) \( x \) is convergent and satisfies (2). □

**Remark 3.2.**
(a) If (10) holds, then conditions (14) of Theorem 3.1 is satisfied for all \( c > 0 \).
(b) If (11) holds, then (see [10]) there exists always \( c \) small enough satisfying condition (14).
(c) Finally, if \( 1/\omega_i \in L_1((0, \infty)) \), \( 1 \leq i \leq p \), then the inequality

\[
\sum_{k=n_0}^{\infty} \lambda_i(k) \geq \int_{0}^{\infty} \frac{ds}{\omega_i(s)}
\]

for some \( i \) implies that there is no \( c > 0 \) satisfying condition (14) of Theorem 3.1. Otherwise, there exists always \( c \) small enough satisfying condition (14). In every case, the biggest \( c \) satisfying condition (14) is

\[
c = \phi^{-1}_p(\infty),
\]

so, we obtain:

**Corollary 3.3.**
(A) If (10) holds, then the result of Theorem 3.1 is true for all solutions.
(B) If (11) holds, then the result of Theorem 3.1 is true only for a solution \( x \) such that \(|x(n_0)|\) is small enough, namely

\[
|x(n_0)| < \phi^{-1}_p(\infty).
\]

**Example 3.4.** The ordinary difference equation

\[
\Delta x(n) = \frac{e-1}{e^n} x^2(n), \quad x(1) = x_0, \quad n \geq 1
\]

and the solution \( x(n) = e^n \) shows that the result of Theorem 3.1 is not true for arbitrary solutions. In fact, here \( \lambda_1(n) = (e-1)/e^n \), \( \alpha_1 = \sum_{k=1}^{\infty} \lambda_1(k) = e \) and \( \omega_1(u) = u^2 \), \( W_1(u) = u^{-1} \).

Then \( \phi^{-1}_1(\infty) = W^{-1}_1(-\alpha_1) = e^{-1} \). Corollary 3.3(B) ensures that any solution \( x \) of Eq. (18) converges if

\[
|x(1)| < \phi^{-1}_1(\infty) = e^{-1}.
\]

The last condition is not satisfied by the solution \( x(n) = e^n \). Moreover, this example shows that the condition \(|x(n_0)| < \phi^{-1}_p(\infty)\) must be satisfied in Corollary 3.3 and that, in general, the ball \( B(0, \rho) \) cannot be extended to a closed ball.

The converse of Theorem 3.1 solves problem (ii).
Theorem 3.5. Assume the hypotheses of Theorem 3.1. In addition, assume that $f(n, x)$ is continuous in $x$ for any $n$ fixed. Then for each $\rho > 0$ there is $n_0$ large enough such that for every $\xi \in \mathbb{R}^p$ with $|\xi| \leq \rho$, there exists a solution $x(n)$ of Eq. (1) tending to $\xi$ when $n \to \infty$.

Proof. Let $\xi$ be an arbitrary vector of $\mathbb{R}^p$, and let $\rho > 2|\xi|$. Consider the set $B = B(0, \rho) = \{x = (x_n) \in \ell_\infty/\|x\| \leq \rho\}$, where

$$\|x\| = \sup_{n \in N_0} |x(n)|,$$

and consider the operator $S$ on $B$ defined by the relation

$$Sx(n) = \xi - \sum_{k=n}^{\infty} f(k, x(k)), \quad n \in N_0.$$  \hfill (19)

It satisfies the following:

(i) There exists $n_0$ such that $S$ maps $B$ into itself. In fact, taking $n_0$ so large that

$$\sum_{i=1}^{p} \left\{ \sum_{k=n_0}^{\infty} \lambda_i(k) \varphi_i(\|x\|) \right\} \leq \frac{1}{2} \rho,$$

which is possible since $\lambda_i \in \ell_1(N_0)$, $1 \leq i \leq p$. Then $x \in B$ implies that for $n \geq n_0$ we have

$$|Sx(n)| \leq |\xi| + \sum_{i=1}^{p} \left\{ \sum_{k=n}^{\infty} \lambda_i(k) \varphi_i(\|x\|) \right\} \leq \rho.$$

(m) The mapping $S$ is continuous.

Let $x \in B$ and $\{x_j\}$ be an arbitrary sequence of elements of $B$ such that $\|x_j - x\| \to 0$ as $j \to \infty$. Since $g_j(k) = |f(k, x_j(k)) - f(k, x(k))| \leq g(k)$ with $g \in \ell_1(N_0)$, and $g_j(k) \to 0$ as $j \to \infty$ by the continuity of $f(n, \cdot)$, then

$$|Sx_j(n) - Sx(n)| = \left| \sum_{k=n}^{\infty} f(k, x_j(k)) - \sum_{k=n}^{\infty} f(k, x(k)) \right|$$

$$\leq \sum_{k=n}^{\infty} |f(k, x_j(k)) - f(k, x(k))| \to 0$$

as $j \to \infty$. Hence $S$ is continuous.

(n) The set $SB$ is relatively compact. It suffices to prove that $SB$ is bounded and equiconvergent to $\xi$.

For any $x \in B$ we have

$$|Sx(n)| \leq |\xi| + \sum_{i=1}^{p} \left\{ \sum_{k=n_0}^{\infty} \lambda_i(k) \varphi_i(\|x\|) \right\} < \infty.$$
Therefore, the set $SB$ is an uniformly bounded subset of the space $\ell_\infty$. Moreover, it is equiconvergent to $\xi$, since for every $\varepsilon > 0$, there exists $T = T(\varepsilon)$ such that
\[
|Sx(n) - \xi| \leq \sum_{i=1}^{p} \left\{ \sum_{k=n}^{\infty} \lambda_i(k) \omega_i(\|x\|) \right\} < \varepsilon
\]
for every $n \geq T$ and all $x \in B$. Then the Compactness Criterion of Section 1 proves that $SB$ is relatively compact. Therefore, by Schauder’s fixed point theorem, there exists $x \in B$ such that $x = Sx$, that is, there exists a solution $x(n)$ of equation
\[
x(n) = \xi - \sum_{k=n}^{\infty} f(k, x(k)), \quad n \in N_0.
\]
Obviously, $x$ is a solution of the terminal value problem (ii) and the proof of Theorem 3.5 is complete. □

Remark 3.6. It is clear from the proofs that Theorems 3.1 and 3.5 remain in force when Eq. (1) is a difference equation in a Banach space.

Now we study Eq. (3).

Theorem 3.7. Let $\Phi$ be a fundamental matrix of the linear system (5). Assume that for $(n, x) \in N_0 \times \mathbb{R}^p$ we have
\[
|\Phi^{-1}(n + 1)g(n, \Phi(n)x)| \leq \sum_{i=1}^{p} \lambda_i(n)\omega_i(|x|),
\]
where $\lambda_i, \omega_i (1 \leq i \leq p)$ satisfy the hypotheses of Theorem 3.1. Then any solution $x$ of Eq. (3) with $|x(n_0)| \leq c$ is defined on $N_0$ and satisfies (4).

Proof. Let $\Phi$ be such that $\Phi(n_0) = I$ (the identity matrix); then $u(n) = \Phi^{-1}(n)x(n)$ satisfies the equation
\[
\Delta u(n) = \Phi^{-1}(n + 1)g(n, \Phi(n)u(n)), \quad u(n_0) = x(n_0).
\]
This system satisfies the hypotheses of Theorem 3.1, then $\lim_{n \to \infty} u(n)$ exists, and (2) follows. □

Note. If $x(n_0) \neq 0$, then there exists $n_0$ sufficiently large for which $\xi \neq 0$. In order to prove this assertion, it suffices to suppose that there is a solution $x(n)$ of Eq. (1), starting from $x(n_0) \neq 0$, such that $\lim_{n \to \infty} x(n) = 0$. Then our hypotheses allow us to obtain a contradiction.

In the same way, we can prove.

Theorem 3.8. Assume the hypotheses of Theorem 3.5. Then for each $\rho > 0$ there is $n_0$ large enough such that for every $\xi \in \mathbb{R}^p$ with $|\xi| \leq \rho$, there exists a solution of Eq. (3) defined on $N_0$ and satisfying (4).
4. Applications

We next apply our results to the question of asymptotic equivalence. Two systems of difference equations are said to be asymptotically equivalent if, corresponding to each solution of one system, there exists a solution of the other system such that the difference between these two solutions tends to zero. If we know that two systems are asymptotically equivalent, and if we also know the asymptotic behavior of the solutions of one of them, then it is clear that we obtain information about the asymptotic behavior of the solutions of the other system.

**Theorem 4.1.** Assume the hypotheses of Theorems 3.5 and 3.7. Suppose that all solutions of Eq. (5) are bounded on \( N_0 \), and

\[
|\det \Phi(n)| = \left| \prod_{k=n_0}^{n-1} \det(I + A(k)) \right| \det \Phi(n_0) > \alpha > 0,
\]

where \( I \) is the identity matrix, and \( \alpha \) some positive constant.

Then, corresponding to each bounded solution \( x(n) \) of Eq. (3) leaving an interior of some sphere \( B(0, \rho) \) at time \( n_0 = 0 \), there is a bounded solution \( y(n) \) of Eq. (5) such that

\[
\lim_{n \to \infty} |x(n) - y(n)| = 0. \tag{21}
\]

Conversely, corresponding to each bounded solution \( y(n) \) of Eq. (5), there is a bounded solution \( x(n) \) of Eq. (3) with the property (21).

**Proof.** Let \( \Phi(n) = \prod_{k=n_0}^{n-1} (I + A(k)) \) be the fundamental solution matrix of Eq. (5) such that \( \Phi(0) = I \). The substitution \( x(n) = \Phi(n)z(n) \) transforms Eq. (3) into the system

\[
\Delta z(n) = h(n, z), \tag{22}
\]

where \( h(n, z) = \Phi^{-1}(n + 1)g(n, \Phi(n)z) \), and it is easily seen that, under our conditions, \( h(n, z) \) satisfies all the conditions of Theorems 3.5 and 3.7. There is therefore \( \rho > 0 \) such that, for \( x(0) = z(0) \in B(0, \rho) \), the limit

\[
y_0 = \lim_{n \to \infty} z(n, 0, x(0))
\]

exists. The solution \( y(n, 0, y_0) \) corresponding to \( x(n, 0, x(0)) \) \((x(0) \in B(0, \rho))\) has the property (21), because

\[
0 \leq \lim_{n \to \infty} |x(n, 0, x(0)) - y(n, 0, y_0)|
\]

\[
\leq \lim_{n \to \infty} |\Phi(n)||z(n, 0, x(0)) - y_0| \leq 0.
\]
Now, let \( y(n) \) be a solution of Eq. (5). Corresponding to \( y_0 = \Phi^{-1}(n_0)y(n_0) \), there is a solution \( z(n, n_0, z_0) \) of Eq. (22) tending to \( y_0 \) when \( n \to \infty \). The required solution is thus
\[
x(n) = \Phi(n)z(n, n_0, z_0).
\]

Now, we will illustrate Theorems 3.1 and 3.5 showing explicitly the radius of attraction, that is, we make precise the initial conditions and the estimates for the solutions of the equilibrium problem.

**Example 4.2.** Consider the ordinary difference equation
\[
\Delta x(n) = \lambda_1(n)x^{k_1}(n) + \lambda_2(n)x^{k_2}(n) + \lambda_3(n)x^{k_3}(n) \quad \text{for } n \in N_0,
\]
where \( \lambda_i \in \ell_1(N_0) \) (1 \( \leq i \leq 3 \)) and \( 1 < k_1 \leq k_2 \leq k_3 \).

(i) The function \( f(n, x) = \sum_{i=1}^{3} \lambda_i(n)x^{k_i} \) satisfies the hypotheses of Theorems 3.1 and 3.5 with \( \omega_i(u) = u^{k_i}, 1 \leq i \leq 3 \).

(ii) To see that condition (14) of Theorem 3.1 is satisfied we observe that
\[
\varphi_0(c) = c, \quad \varphi_1(c) = \left[ c^{1-k_1} + \alpha_1(1-k_1) \right]^{1/(1-k_1)}
\]
for \( 0 < c < c_1 \), where
\[
c_1 = \left[ \alpha_1(k_1 - 1) \right]^{1/(1-k_1)},
\]
\[
\varphi_2(c) = \left[ (c^{1-k_1} + \alpha_1(1-k_1))^{(k_3-1)/(k_1-1)} + \alpha_2(1-k_2) \right]^{1/(1-k_1)},
\]
for \( 0 < c < c_2 \), where
\[
c_2 = \left[ \alpha_1(k_1 - 1) + (\alpha_2(k_2 - 1))^{(k_3-1)/(k_2-1)} \right]^{1/(1-k_1)}.
\]

We have
\[
\int_{\varphi_{i-1}(c)}^{\infty} \frac{ds}{\omega_i(s)} = \frac{1}{k_i - 1} \left( \varphi_{i-1}(c) \right)^{1-k_i}, \quad 1 \leq i \leq 3 \text{ for } c \in \text{Dom } \varphi_{i-1}.
\]

Then condition (14) of Theorem 3.1 is equivalent to
\[
\alpha_3 < \frac{1}{k_3 - 1} (\varphi_2(c))^{1-k_3} \quad \text{or} \quad \varphi_2(c) < \left( \frac{1}{\alpha_3(k_3 - 1)} \right)^{1/(k_3-1)}.
\]

Since, \( \varphi_i (i = 1, 2, 3) \) are monotone functions and by (12) \( \lim_{c \to 0^+} \varphi_i(c) = 0 \), choosing \( c \) small enough we get that (24) is satisfied. Solving the equation
\[
\varphi_2(c^*) = \left( \frac{1}{\alpha_3(k_3 - 1)} \right)^{1/(k_3-1)}
\]
we obtain
\[
c^* = \left\{ [\alpha_3(k_3 - 1)]^{(k_3-1)/(k_3-1)} + \alpha_2(k_2 - 1) \right\}^{(k_3-1)/(k_3-1)} + \alpha_1(k_1 - 1) \right\}^{1/(1-k_1)}
\]
(actually $c^* = \varphi_3^{-1}(\infty)$). Then taking $c \leq c^*$, condition (14) in Theorem 3.1 is satisfied. Thus, we conclude that any solution $x = x(n, n_0, x(n_0))$ of Eq. (23) with $|x(n_0)| < c^*$ is defined on $N_0$ and satisfies (2). Moreover, for every $\xi \in \mathbb{R}$ with $|\xi| < c^*$ there exists a solution $x(n)$ of Eq. (23) tending to $\xi$ when $n \to \infty$. Hence, Eq. (23) has asymptotic equilibrium. Furthermore, by (16), all solution $x = x(n, n_0, x(n_0))$ such that $|x(n_0)| < c^*$ is bounded, and the following estimate is true:

$$|x(n, n_0, x(n_0)| \leq \varphi_3(c)$$

$$= \left\{ \left( c_1^{1-k_1} + \alpha_1(1 - k_1) \right)^{(k_2-1)/(k_1-1)} + \alpha_2(1 - k_2)^{(k_3-1)/(k_2-1)} + \alpha_3(1 - k_3) \right\}^{1/(k_3-1)}.$$

We note that the radius of attraction and the estimate of the solutions depend directly on the series $\lambda_i(n)$ ($i = 1, 2, 3$).

References