Rings Generated by Their Units

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In 1953 K. Wolfson [14] and in 1954 D. Zelinsky [15] showed, independently, that every element of the ring of all linear transformations of a vector space over a division ring of characteristic not 2 is a sum of two nonsingular ones. Motivated by this result, in 1958, Skornyakov [12, p. 167] conjectured that all regular rings in which 2 is a unit are generated by their units. The conjecture was finally settled in the negative by G. Bergman [7] in 1974. At about the same time [6], we proved the conjecture in the affirmative for a large class of regular rings; that being, the class of regular rings which are strongly $\pi$-regular.

Included in this class are those regular rings with primitive factor rings Artinian. In fact, we prove in Theorem 1 that they are unit regular from which the generation by units follows immediately. As a corollary we obtain that regular rings which satisfy a polynomial identity are unit regular.

Our best theorem—Theorem 3—states that if $R$ is strongly $\pi$-regular and 2 is a unit in $R$, then each element of $R$ can be expressed as a sum of two units. The class of strongly $\pi$-regular rings includes algebraic algebras, locally Artinian rings, and rings with prime factor rings Artinian.

Throughout, $R$ will denote a ring with unity element. A ring $R$ is called (von Neumann) regular if for each $a$ in $R$ there exists an $x$ in $R$ such that $a = axa$. An element $a$ in $R$ is unit regular if there exists a unit $u$ in $R$ such that $a = uau$ and $R$ is called unit regular if each element of $R$ is unit regular.

In [3] G. Ehrlich proved that the class of unit regular rings includes semisimple Artinian rings, regular rings with no nonzero nilpotent elements, and regular group algebras over fields. The following theorem produces a sufficient condition for regular rings to be unit regular.
THEOREM 1. If $R$ is a regular ring with primitive factor rings Artinian, then $R$ is unit regular.

Proof. Suppose, to the contrary, that there exists $a \in R$ which is not unit regular in $R$. Then $\Omega = \{J \subset R: \bar{a} \text{ is not unit regular in } R/J\}$ is nonempty. We claim that Zorn's Lemma can be used to produce a maximal element $\bar{a} \in \bar{\Omega}$. Let $\{I,\}$ be a chain in $\bar{\Omega}$ and set $I = \bigcup I_a$. If $I \notin \bar{\Omega}$, then $\bar{a}$ is unit regular in $R/I$, i.e., there exists $u$ and $v$ in $R$ such that $\{a - au, 1 - uv, 1 - vu\} \subseteq I_a$. Hence, there is $\beta$ such that $\{a - au, 1 - uv, 1 - vu\} \subseteq I_\beta$, i.e., $\bar{a}$ is unit regular in $R/I_\beta$. This contradiction shows that $I \in \bar{\Omega}$.

If $R/A$ is simple, then $\bar{a}$ is not unit regular in $R/A$, which contradicts the choice of $A$.

If $R/A$ is not simple, then it contains a nontrivial central idempotent by [9, Theorem 3, p. 239]. Whence, $R/A = S \oplus T$ with $S \neq 0$, $T \neq 0$ and $a \in R/A$ has the form $\bar{a} = (s, t)$ with $s \in S$, $t \in T$. By the maximality of $A$, we have that $s$ is unit regular in $S$ and $t$ unit regular in $T$. Consequently, $\bar{a}$ is unit regular in $R/A$. This contradicts the choice of $A$ and therefore $R$ is unit regular.

This theorem produces a large class of unit regular rings as the following corollary shows. S. K. Jain informs us that he also has proven this corollary.

COROLLARY 1. If $R$ is a regular ring which satisfies a polynomial identity, then $R$ is unit regular.


LEMMA 1. (Ehrlich [3]). If $R$ is unit regular and 2 is a unit in $R$, then each element of $R$ can be expressed as a sum of two units.

Proof. If $a \in R$ where $u$ is a unit such that $a = au$, then $a = (2au - 1) \times 2^{-1}u^{-1} + 2^{-1}u^{-1}$ is a sum of two units.

COROLLARY 2. Let $R$ be a regular ring with primitive factor rings Artinian. If 2 is a unit in $R$, then each element of $R$ can be expressed as a sum of two units.

Proof. This follows immediately from Theorem 1 and the Lemma.

This Corollary can be improved but first we need a definition. A ring $R$ is $\pi$-regular if for each $a \in R$, there exists an $x \in R$ and a positive integer $n$ such that $a^n = a^nxa^n$. Regular rings, left Artinian rings, and rings with prime factor rings Artinian [4] are examples of $\pi$-regular rings.

THEOREM 2. Let $R$ be a $\pi$-regular ring with primitive factor rings Artinian. If $Z/R$ is not a homomorphic image of $R$, then each element of $R$ can be expressed as a sum of two units.
Proof. Suppose, to the contrary, that there exists \( a \in R \) which cannot be expressed as a sum of two units of \( R \). Then \( \mathcal{O} = \{ J < R : \bar{a} \text{ cannot be expressed as a sum of two units in } R/J \} \) is nonempty and again Zorn's Lemma can be used to produce a maximal element \( A \) in \( \mathcal{O} \). In order to simplify notation, we may assume that \( A \) is zero by passing to \( R/A \).

If \( R \) is primitive, then \( R \cong D_n \) where \( D \) is a division ring. If \( n > 1 \), then \( a \) can be expressed as a sum of two units of \( R \) by Henriksen [8, Theorem 12]. If \( n = 1 \), then the hypothesis guarantees that \( D \) is not \( \mathbb{Z}/2\mathbb{Z} \) and hence \( a \) can be expressed as a sum of two units of \( R \). In either case, we have contradicted the choice of \( A \).

If \( R \) is semiprimitive but not primitive, then as before \( R = S \oplus T \) with \( S \neq 0, T \neq 0 \) by [9, Theorem 3, p. 239]. If \( a = (s, t) \) with \( s \in S, t \in T \), then the maximality of \( A \) guarantees \( s = s_1 + s_2, t = t_1 + t_2, s_i \) units in \( S \), and \( t_i \) units in \( T \). Consequently, \( a = (s_1, t_1) + (s_2, t_2) \) with \( (s_i, t_i) \) units in \( R \). Again we have contradicted the choice of \( A \).

If \( R \) is not semiprimitive with Jacobson radical \( J(R) \neq 0 \), then \( \bar{a} = \bar{u}_1 + \bar{u}_2 \) where \( u_1, u_2 \) are units in \( R/J(R) \). Hence, \( a = u_1 + u_2 + j \) where \( u_1, u_2 \) are units in \( R \) and \( j \in J(R) \). Thence, \( a = u_1 + u_2(1 + u_2^{-1} j) \) is a sum of two units of \( R \). In this final case we have again contradicted the choice of \( A \) and wherefore the theorem is proved.

We have the following Corollary which was discovered independently by Burgess and Stephenson [2, Corollary 26] and the authors [6, p. A-84].

**Corollary 3.** A regular ring \( R \) of bounded index of nilpotency is unit regular. If, moreover, \( \mathbb{Z}/2\mathbb{Z} \) is not a homomorphic image of \( R \), then every element of \( R \) can be expressed as a sum of two units.

**Proof.** By Kaplansky [11, Theorem 2.3] the primitive factor rings of \( R \) are Artinian. Consequently, \( R \) is not only generated by its units (Theorem 2); but moreover, \( R \) is unit regular (Theorem 1).

In passing—it should perhaps be remarked—that regular rings with primitive factor rings Artinian which are not of bounded index abound, e.g., \( R = \bigoplus \sum_{i=2}^\infty F_n \) where \( F \) is a field.

Now we proceed to show that Theorem 2 can be improved by showing that it is true for certain \( \pi \)-regular rings whose primitive factor rings are not necessarily Artinian. In order to do this we have to use different techniques. Following Azumaya [1], an element \( a \in R \) is left (right) \( \pi \)-regular if there exists \( x \in R \) and \( n \in \mathbb{Z}^+ \) such that \( a^n = x a^{n+1} (a^n = a^{n+1} x) \). And \( a \in R \) is called strongly \( \pi \)-regular if \( a \) is both left and right \( \pi \)-regular. Finally, \( R \) is called left, right, or strongly \( \pi \)-regular if each element of \( R \) is left, right, or strongly \( \pi \)-regular, respectively. The class of strongly \( \pi \)-regular rings is the one that we are interested in and the following examples show that it is large.
Examples. (i) Any ring which has the property that each element is contained in a left Artinian subring [5, Lemma 2.1] is strongly \( \pi \)-regular. In particular, algebraic algebras [9, p. 210] and the locally left Artinian rings introduced in [5].

(ii) Rings with prime factor rings Artinian are strongly \( \pi \)-regular [4, Theorem 2.1].

(iii) Strongly \( \pi \)-regular rings need not have primitive factor rings Artinian. For example, let \( F \) be your favorite field and let \( S \) be the ring of \( \mathbb{K}_0 \times \mathbb{K}_0 \) row finite matrices with entries in \( F \). If \( T \) is the subring of \( S \) generated by \( F \cdot 1 \) and all matrices with at most finitely many nonzero entries, then \( T \) is locally left Artinian and hence strongly \( \pi \)-regular.

(iv) Finally, commutative \( \pi \)-regular rings are obviously strongly \( \pi \)-regular.

Lemma 2. If \( R \) is strongly \( \pi \)-regular, then a power of each element of \( R \) is unit regular.

Proof. Let \( a \in R \). By Azumaya [1, Theorem 3] there exists an \( x \in R \) and \( n \in \mathbb{Z}^+ \) such that \( ax = xa \) and \( a^n = a^{n+1}x \). Then \( a^n = a^{n+1}x = a^{n+2}x^2 = \cdots = a^{2n}x^n \) and so \( \epsilon = a^n x^n \) is an idempotent and \( \epsilon^2 = (ea^ne)(ex^n) \). If \( t = ea^ne + 1 - \epsilon \), then \( t^{-1} = ex^n + 1 - \epsilon \). Also \( et - a^n \) or \( \epsilon = a^{n-1}t \). Hence \( a^n t^{-1}a^n \) and \( a^n \) is unit regular.

Corollary 4. (Ehrlich [3]). If \( a = axa \) and \( ax = xa \), then \( a \) is unit regular.

Proof. Evident.

We have the following theorem, which was originally motivated by Henriksen’s [8, Theorem 17]. All aspects of its extension should be apparent.

Theorem 3. Let \( R \) be a strongly \( \pi \)-regular ring. If \( 2 \) is a unit in \( R \), then each element of \( R \) can be expressed as a sum of two units.

Proof. Let \( a \in R \). By Azumaya [1, Theorem 3] there exists an \( x \in R \) and \( n \in \mathbb{Z}^+ \) such that \( ax = xa \) and \( a^n = a^{n+1}x \). Then \( a = (a - a^2x) + (a - a^2x) ax + \cdots + (a - a^2x)a^{n-2}x^{n-2} + a^n x^{n-1} \). Moreover, \( a - a^2x \) is nilpotent [1, Lemma 4] and since everything in sight commutes, we have \( y = (a - a^2x) + (a - a^2x) ax + \cdots + (a - a^2x)a^{n-2}x^{n-2} \) is nilpotent.

By the above Corollary, we have that \( a^n x^{n-1} \) is unit regular because \( (a^n x^{n-1}) x(a^n x^{n-1}) = a^n-1(a^{n+1}x)x^{2n-2} = a^{2n-1}x^{2n-2} = \cdots = a^n x^{n-1} \) and \( a^n x^{n-1} \) commutes with \( x \). Therefore, \( a^n x^{n-1} = u_1 + u_2 \) with \( u_1 \) and \( u_2 \) units
in $R$ by Lemma 1. Careful scrutiny of the proofs of Corollary 4 and Lemma 1 shows that $u_1$ commutes with $y$. Clearly $u_1^{-1}$ commutes with $y$ since $u_1(u_1^{-1}y - yu_1^{-1}) = 0$. Hence $u_1^{-1}y$ is nilpotent and $1 + u_1^{-1}y$ is a unit. Whence $u_1 + y$ is a unit and $a = (u_1 + y) + u_2$ is a sum of two units of $R$.

**Corollary 5.** Let $R$ be a ring with prime factor rings Artinian. If $2$ is a unit in $R$, then each element of $R$ can be expressed as a sum of two units.

**Proof.** Since each prime factor ring is both left and right $\pi$-regular, $R$ is both left and right $\pi$-regular by Fisher-Snider [4, Theorem 2.1]. Hence $R$ is strongly $\pi$-regular.

This corollary is not valid with “prime factor rings Artinian” replaced by “primitive factor rings Artinian,” as is shown by $\mathbb{Q}[x]$ where $\mathbb{Q}$ is the rational numbers.

**Corollary 6.** Let $R$ be integral over a $\pi$-regular subring $S$ of the center of $R$. If $2$ is a unit in $R$, then each element of $R$ can be expressed as a sum of two units.

**Proof.** We claim that $R$ is strongly $\pi$-regular. Let $a \in R$. Commutative $\pi$-regular rings are characterized as those with prime ideals maximal [13]. Since $S[a]$ is a commutative integral extension of $S$, it has prime ideals maximal and hence is $\pi$-regular. Therefore, there exists $x \in S[a]$ and $n \in \mathbb{Z}^+$ such that $a^n = xa^{n+1} = a^{n+1}x$ and $R$ is strongly $\pi$-regular.

**Corollary 7.** Let $S$ be a commutative $\pi$-regular ring and $M$ a finitely generated $S$-module. If $2$ is a unit in $S$, then each element of $R = \text{End}_S(M)$ can be expressed as a sum of two units.

**Proof.** Suppose that $M$ has a generating set consisting of $n$ elements. We claim that $\text{End}_S(M) \cong I(A)/A$ where $I(A)$ is the idealizer in $S_n$ of an appropriate right ideal $A$ of $S_n$. This is proven for cyclic modules in [9, p. 25] and, of course, can be extended to finitely generated modules by appropriate tensoring. Let $a \in I(A)$. Then $a$ satisfies its characteristic polynomial by the Cayley–Hamilton theorem and hence is integral over $S$.

Hence, $\text{End}_S(M)$ is integral over $(S + A)/A$ and the result follows from the previous corollary.

**Remark.** Every ring in this paper which we have shown is generated by its units has been directly finite, i.e., $xy = 1$ implies $yx = 1$. However, regular plus directly finite is not sufficient to guarantee that $R$ be generated by its units as Bergman’s example [7] shows.
REFERENCES