NORTH-HOLLAND

Stable Matchings and Linear Programming

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ABSTRACT

This paper continues the work of Abeledo and Rothblum, who study nonbipartite stable matching problems from a polyhedral perspective. We establish here additional properties of fractional stable matchings and use linear programming to obtain an alternative polynomial algorithm for solving stable matching problems.

1. INTRODUCTION

In a *stable matching problem* agents have to be matched in pairs while having strict preferences over their potential mates. The goal is to find a matching where no two agents prefer being matched to each other over their outcomes in the matching, where singlehood is considered worse than being paired with any admissible mate. Gale and Shapley [5] call such a matching *stable*, since it will remain unchanged if the agents act rationally.

A special case of the stable matching problem is the *stable marriage problem*, where each agent is labeled as either "man" or "woman" and all matchable pairs consist of a man and a woman. Gale and Shapley [5]

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described a polynomial algorithm that computes a stable matching for any given stable marriage problem. A nonbipartite stable matching problem is also known as a *stable roommates problem*. In this case, Gale and Shapley [5] showed that stable matchings do not always exist. Settling a question raised by Knuth [9], Irving [7] obtained the first polynomial algorithm that either finds a stable matching, or determines that none exists, for problems where all pairs of agents are matchable. Gusfield and Irving [6] showed how to modify this algorithm to handle arbitrary stable matching problems.

Stable matching problems have a wealth of structural properties that have been explored over the past three decades using combinatorial techniques. A new approach for studying stable matching problems was initiated by Vande Vate [15] and Rothblum [13], who showed, for the stable marriage problem, that a simple system of linear inequalities describes the *stable matching polytope*, i.e., the convex hull of the incidence vectors of the stable matchings. This result allowed Roth, Rothblum, and Vande Vate [11] to use linear programming theory to derive known properties of stable marriage problems and to extend these to fractional stable matchings. Abeledo and Rothblum [2] expanded this approach to the general case, where the underlying graph is not necessarily bipartite, by studying properties of the *fractional stable matching polytope*, which is a relaxation of the stable matching polytope and whose integral vectors are also the incidence vectors of the stable matchings.

In this paper we continue the analysis of stable matching problems from a polyhedral perspective. One of our main results here shows that we can associate with each fractional stable matching a polytope which is a subset of the fractional stable matching polytope and which contains an integral vector if (and only if) the original problem has a stable matching. We then show how to generate, using linear programming, a sequence of fractional stable matchings whose associated polytopes are monotonically decreasing. This procedure stops when it identifies a stable matching or reaches a conclusion that no such matching exists. As linear programming problems can be solved in polynomial time, this procedure yields an alternative polynomial algorithm for solving stable matching problems. We remark, however, that it is not competitive with Irving's from a complexity standpoint.

For further results, applications, and history of stable matching problems, see the books by Gusfield and Irving [6] and Roth and Sotomayor [12].

2. PRELIMINARIES

2.1. *Graphs and Matchings*

We shall only be concerned here with finite undirected graphs without loops or multiple edges. For graph theory definitions see, e.g., [3]. Let

$G = (V, E)$ be a graph with vertex set V and edge set E. For each $v \in V$, $N(v) = \{u: \{v, u\} \in E\}$ is the set of neighbors of v.

We will consider vectors $x \in R^E$, indexed by the edges of G, such that $0 \leqslant x_e \leqslant 1$ for every $e \in E$. We call such vectors *half-integral* if they belong to $\{0, \frac{1}{2}, 1\}^E$. Let $x \in R^E$ be a nonnegative vector; then the set of edges $E_+(x) \equiv \{\{u, v\} \in E: x_{u,v} > 0\}$ is the *support* of x. Also, for each nonnegative scalar δ, we define the set of edges $E_\delta(x) \equiv \{\{u, v\} \in E: x_{u,v} = \delta\}$.

A *matching* for $G = (V, E)$ is a subset $\mu \subseteq E$ such that no two edges in μ have a common vertex. A matching μ defines a one-to-one mapping $\mu(\cdot)$ from the set V onto itself where $\mu(v) = u$ if $\{u, v\} \in \mu$ and $\mu(v) = v$ if no edge in μ contains v. We call $\mu(v)$ the *outcome of* v *under* the matching μ. If $\mu(v) = v$, then v is *single* or *unmatched* in μ. Otherwise, if $\mu(v) \neq v$, we say that v is *matched to* $\mu(v)$. The matching μ can be represented by an *incidence vector* $x \in \{0, 1\}^E$ such that $x_{u,v} = 1$, if $\{u, v\} \in \mu$, and $x_{u,v} = 0$ otherwise.

2.2. *The Stable Matching Problem*

We formally define a *stable matching problem* as a pair $(G; P)$ where $G = (V, E)$ is a graph and P is a mapping on V such that, for each vertex $v \in V$, $P(v)$ is a strict linear order on $N(v) \cup \{v\}$ that has v as its last element. We call G the *acceptability graph*, P the *preference profile*, and $P(v)$ the *preference* of vertex v.

We will usually represent the preference of a vertex v by $<_v$. In particular, for u, w in $N(v) \cup \{v\}$, we write $u <_v w$ if $P(v)$ orders w before u; and we say that v prefers w to u. Note that, as $P(v)$ ranks v last, we have that $u >_v v$ for each $u \in N(v)$. We express by $u \leqslant_v w$ that either $u <_v w$ or $u = w$. Finally, for a vertex v and a nonempty set of vertices $S \subseteq N(v)$, let $\min_v S$ and $\max_v S$ denote, respectively, the least preferred and the most preferred element in S with respect to the preference of v. Further, we define $\min_v \varnothing = \max_v \varnothing = \{v\}$.

A pair $\{u, v\} \in E$ is a *blocking pair* for a matching μ if

$$\mu(u) <_u v \quad \text{and} \quad (v) <_v u,$$

i.e., $\{u, v\}$ is a blocking pair for μ if both vertices prefer being matched to each other over their outcome under μ. A matching μ is *stable* if it has no blocking pair. Equivalently, μ is a stable matching for $(G; P)$ if the following *stability condition* holds for each $\{u, v\} \in E$:

$$\mu(u) \geqslant_u v \quad \text{or} \quad \mu(v) \geqslant_v u. \tag{1}$$

To *solve* a stable matching problem means to either find a stable matching for the problem or to determine that no such matching exists.

The algorithm of Gale and Shapley [5], combined with an appropriate labeling of the vertices, shows that stable matching problems with bipartite graphs always have a stable matching. Abeledo and Isaak [1] showed that, given a nonbipartite graph, it is always possible to define a preference profile so that the resulting problem does not have a stable matching.

3. FRACTIONAL STABLE MATCHINGS

3.1. *Basic Definitions and Results*

Henceforth, $(G; P)$ denotes a given stable matching problem with graph $G = (V, E)$. We define the *stable matching polytope* $\mathrm{SM}(G; P)$ of the problem $(G; P)$ as the convex hull of the incidence vectors of its stable matchings. We describe below a system of linear inequalities that must be satisfied by all vectors in $\mathrm{SM}(G; P)$.

LEMMA 3.1 [2]. *The incidence vectors of the stable matchings of* $(G; P)$ *are the integer solutions of the following system of linear inequalities*:

$$\sum_{u \in N(v)} x_{u,v} \leqslant 1 \qquad \textit{for each} \quad v \in V, \tag{2}$$

$$x_{u,v} \geqslant 0 \qquad \textit{for each} \quad \{u, v\} \in E, \tag{3}$$

$$\sum_{i >_u v} x_{u,i} + \sum_{j >_v u} x_{v,j} + x_{u,v} \geqslant 1 \qquad \textit{for each} \quad \{u, v\} \in E, \tag{4}$$

where $i >_u v$ *denotes* $\{i \in N(u): i >_u v\}$ *and* $j >_v u$ *denotes* $\{j \in N(v): j >_v u\}$.

We call constraints (4) the *stability constraints*. Solutions of (2), (3), and (4) are called *fractional stable matchings* of $(G; P)$. The *fractional stable matching polytope* is the set of all fractional stable matchings and will be denoted $\mathrm{FSM}(G; P)$. The following assertion follows trivially.

LEMMA 3.2. $\mathrm{FSM}(G; P) \supseteq \mathrm{SM}(G; P)$. *Further*, $\mathrm{FSM}(G; P) = \mathrm{SM}(G; P)$ *if and only if the extreme points of* $\mathrm{FSM}(G; P)$ *are integer.*

Vande Vate [15] and Rothblum [13] showed that the stable matching polytope of a problem with a bipartite graph is described by the linear system of Lemma 3.1.

THEOREM 3.3 [13]. *If G is a bipartite graph, then* $\mathrm{SM}(G; P) = \mathrm{FSM}(G; P)$.

The next theorem, proved in [2], asserts that the fractional stable matching polytope is always nonempty and establishes the half-integrality of its extreme points.

THEOREM 3.4 [2]. $\mathrm{FSM}(G; P)$ *is nonempty, and its extreme points are half-integral.*

In particular, as $\mathrm{SM}(G; P)$ is empty when $(G; P)$ has no stable matching, the above result shows that the inequalities (2), (3), and (4) do not always describe $\mathrm{SM}(G; P)$. Further, the possibility of obtaining an NP-description (see [10]) of a class of linear inequalities that describes the stable matching polytope of every problem seems unlikely. This is a consequence of a result by Feder [4], who proved that finding a stable matching that maximizes a linear objective function is NP-hard. Thus, any NP-description of the stable matching polytope would imply that NP = co-NP (see Karp and Papadimitriou [8]). In spite of the above difficulties, it is shown in [2] that many properties of stable matchings are shared by all fractional stable matchings and thus are captured by the simple linear system that defines the fractional stable matching polytope.

The next theorem, applied to the incidence vectors of stable matchings, shows that the set of matched vertices is invariant for every stable matching.

THEOREM 3.5 [2]. *V is partitioned into sets V^0 and V^1 such that for each $x \in \mathrm{FSM}(G; P)$:*

$$\sum_{j \in N(v)} x_{j,v} = 0 \qquad \text{if} \quad v \in V^0,$$

$$\sum_{j \in N(v)} x_{j,v} = 1 \qquad \text{if} \quad v \in V^1.$$

Henceforth, we shall refer by V^0 and V^1 to the sets defined in Theorem 3.5.

COROLLARY 3.6. *If* $|V^1|$ *is odd, then* $(G; P)$ *has no stable matching.*

A pair $\{u, v\} \in E$ is called a *fractional stable pair* if there exists a fractional stable matching $x \in \mathrm{FSM}(G; P)$ with $x_{u,v} > 0$. This definition extends the notion of *stable* [6] or *achievable* [12] pairs, which means the pairs that are matched together in some stable matching. Of course, stable pairs are fractional stable pairs, but the converse is not always valid.

THEOREM 3.7 [2]. *Let* $\{u, v\}$ *be a fractional stable pair. For every* $x \in \mathrm{FSM}(G; P)$

$$\sum_{i >_u v} x_{i,u} + \sum_{j >_v u} x_{j,v} + x_{u,v} = 1. \tag{5}$$

We remark that the above result, specialized to stable pairs and incidence vectors of stable matchings, implies that if two vertices are matched together in a particular stable matching, then there exists no other stable matching that is preferred by both vertices.

3.2. *Extended Supports of Fractional Stable Matchings*

Let $x \in \mathrm{FSM}(G; P)$. We define the mappings $\overline{\sigma}_x(\cdot)$ and $\underline{\sigma}_x(\cdot)$ from V into itself by

$$\underline{\sigma}_x(v) \equiv \min_v\{u \in N(v) : x_{u,v} > 0\}$$

and

$$\overline{\sigma}_x(v) \equiv \max_v\{u \in N(v) : x_{u,v} > 0\},$$

where we remind the reader that $\min_v \varnothing = \max_v \varnothing = v$. Clearly, if x is the incidence vector of a stable matching μ, then for all $v \in V$ we have $\underline{\sigma}_x(v) = \overline{\sigma}_x(v) = \mu(v)$. The next result asserts that the mappings $\overline{\sigma}_x(\cdot)$ and $\underline{\sigma}_x(\cdot)$ are the inverse of each other.

LEMMA 3.8 [2]. *Let* $x \in \mathrm{FSM}(G; P)$. *Then, for* $u, v \in V$,

$$v = \underline{\sigma}_x(u) \quad \textit{if and only if} \quad u = \overline{\sigma}_x(v).$$

LEMMA 3.9. *Let* $\{u, v\}$ *be a fractional stable pair, and let* x *be a fractional stable matching. Then* $u <_v \underline{\sigma}_x(v)$ *if and only if* $v >_u \overline{\sigma}_x(u)$.

Proof. As $\{u, v\}$ is a fractional stable pair, both u and v are in V^1 and therefore

$$\sum_{i \in N(u)} x_{i,u} = \sum_{j \in N(v)} x_{j,v} = 1. \tag{6}$$

Further, Theorem 3.7 implies

$$\sum_{i >_u v} x_{i,u} + \sum_{j >_v u} x_{j,v} + x_{u,v} = 1. \tag{7}$$

Thus, $\sum_{j >_v u} x_{j,v} = 1$ if and only if $\sum_{i \geqslant_u v} x_{i,u} = 0$. This combines with (6) to show that $u <_v \underline{\sigma}_x(v)$ if and only if $v >_u \overline{\sigma}_x(u)$. ■

A main concept that we introduce now is the *extended support* of a fractional stable matching. Let $x \in \mathrm{FSM}(G; P)$ then its extended support $T(x)$ is the set of edges

$$T(x) \equiv \{\{u, v\} \in E: \underline{\sigma}_x(v) \leqslant_v u \leqslant_v \overline{\sigma}_x(v) \text{ and } \underline{\sigma}_x(u) \leqslant_u v \leqslant_u \overline{\sigma}_x(u)\}. \tag{8}$$

It immediately follows that the extended support contains the support

$$E_+(x) \subseteq T(x). \tag{9}$$

Further, if x is the incidence vector of a stable matching μ, then $\underline{\sigma}_x(v) = \overline{\sigma}_x(v) = \mu(v)$ for every $v \in V$, and $T(x) = \mu$.

The next lemma characterizes the set of edges that are not in the extended support of a given fractional stable matching.

LEMMA 3.10. *Let $x \in \mathrm{FSM}(G; P)$. Then $\{u, v\} \in E \setminus T(x)$ if and only if $u <_v \underline{\sigma}_x(v)$ or $v <_u \underline{\sigma}_x(u)$.*

Proof. The "if" part of the lemma follows from the definition of $T(x)$ in (8). To prove the "only if" part, let $\{u, v\} \in E \setminus T(x)$ and assume to the contrary that $u \geqslant_v \underline{\sigma}_x(v)$ and $v \geqslant_u \underline{\sigma}_x(u)$. As $\{u, v\} \notin T(x)$, the definition of $T(x)$ in (8) implies that at least one of the two inequalities $u >_v \overline{\sigma}_x(v)$ and $v >_u \overline{\sigma}_x(u)$ is met. Without loss of generality, assume that $u >_v \overline{\sigma}_x(v)$. Then $\sum_{j \geqslant_v u} x_{v,j} = 0$. Now, if $u \in V^0$ then $\sum_{i >_u v} x_{u,i} = \sum_{i \in N(u)} x_{u,i} = 0$. If,

alternatively, $u \in V^1$, then the assumption that $v \geqslant_u \underline{\sigma}_x(u)$ implies that $\sum_{i >_u v} x_{u,i} \leqslant 1 - x_{u,\underline{\sigma}_x(u)} < 1$. In either case, $\sum_{i >_u v} x_{u,i} + \sum_{j >_v u} x_{v,j} + x_{u,v} < 1$. So x does not satisfy (4), a contradiction to our assumption. ■

Given $x \in \mathrm{FSM}(G; P)$, Lemma 3.10 implies the following representation of $T(x)$

$$T(x) = \{\{u, v\} \in E : \underline{\sigma}_x(v) \leqslant_v u \quad \text{and} \quad \underline{\sigma}_x(u) \leqslant_u v\}. \tag{10}$$

We note that in [2], the extended support of a fractional stable matching is defined by (10). This equality shows that the two definitions are equivalent.

3.3. *Problem Reduction*

In this subsection we define for each fractional stable matching x a polytope, denoted $\mathrm{FSM}_{T(x)}(G; P)$, that consists of the fractional stable matchings whose support is included in $T(x)$. Our main result here shows how to use a given stable matching μ and a fractional stable matching x to compute a stable matching μ' for $(G; P)$ whose incidence vector is in $\mathrm{FSM}_{T(x)}(G; P)$. As an immediate consequence of this result we conclude that $\mathrm{FSM}(G; P)$ has an integral point if and only if $\mathrm{FSM}_{T(x)}(G; P)$ has one. Thus, given any fractional stable matching x, solving $(G; P)$ is equivalent to determining whether the polytope $\mathrm{FSM}_{T(x)}(G; P)$ has an integral point. This property is essential for the algorithm we develop in the next section, where it will be combined with a procedure for generating a decreasing sequence of such polytopes, each strictly contained in the previous one, and each of them containing an integer point if and only if the original $\mathrm{FSM}(G; P)$ has one.

We begin by introducing some additional notation. Given a set of edges $T \subseteq E$, we denote by $\mathrm{FSM}_T(G; P)$ the polytope defined by (2), (3), (4), and

$$x_{u,v} = 0 \qquad \text{for each} \quad \{u, v\} \in E \setminus T. \tag{11}$$

It is clear that for any set of edges $T \subseteq E$, $\mathrm{FSM}_T(G; P) \subseteq \mathrm{FSM}(G; P)$ *and, in particular, every integral solution of* $\mathrm{FSM}_T(G; P)$ *is the incidence vector of a stable matching for* $(G; P)$. *Further, any such integral solution is the incidence vector of a matching whose edges are all in* T. *We now prove the main result of this section.*

THEOREM 3.11. *Let $x \in \mathrm{FSM}(G; P)$, and assume μ is a stable matching for $(G; P)$. Define the mapping μ' from V into itself by*

$$\mu'(v) \equiv \begin{cases} \underline{\sigma}_x(v), & \mu(v) <_v \underline{\sigma}_x(v), \\ \mu(v), & \underline{\sigma}_x(v) \leqslant_v \mu(v) \leqslant_v \overline{\sigma}_x(v), \\ \overline{\sigma}_x(v), & \overline{\Sigma}_x(v) <_v \mu(v). \end{cases} \tag{12}$$

Then μ' is a stable matching for $(G; P)$, and its incidence vector is in $\mathrm{FSM}_{T(x)}(G; P)$.

Proof. Let y denote the incidence vector of μ. We first prove that the mapping μ' represents a matching. It has to be shown that $\mu'(v) = u$ if and only if $\mu'(u) = v$. Let $v \in V$ and set $u \equiv \mu'(v)$. Note that for all three cases of (12), $\{u, v\}$ is a fractional stable pair. We consider each case separately:

(i) $\mu(v) <_v \underline{\sigma}_x(v)$: Then $u = \mu'(v) = \underline{\sigma}_x(v)$, and by Lemma 3.8, $v = \overline{\sigma}_x(u)$. Further, as $\mu(v) <_v u = \underline{\sigma}_x(v)$, Lemma 3.9, applied to y implies that $\overline{\sigma}_x(u) = v <_u \mu(u)$. By (12) it follows that $\mu'(u) = \overline{\sigma}_x(u) = v$.

(ii) $\underline{\sigma}_x(v) \leqslant_v \mu(v) \leqslant \overline{\sigma}_x(v)$: Then $u = \mu'(v) = \mu(v)$, and by Lemma 3.9, $\underline{\sigma}_x(u) \leqslant_u v = \mu(u) \leqslant_u \overline{\sigma}_x(u)$. The definition in (12) now implies that $\mu'(u) = \mu(u) = v$.

(iii) $\overline{\sigma}_x(v) <_v \mu(v)$: Then $u = \mu'(v) = \overline{\sigma}_x(v)$, and symmetric arguments to case (i) prove that in this case $\mu'(u) = \underline{\sigma}_x(u) = v$.

Let z be the incidence vector of μ'. As μ' is a matching, z satisfies (2) and (3). Further, by the construction of μ', $\underline{\sigma}_x(v) \leqslant_v \mu'(v) \leqslant_v \overline{\sigma}_x(v)$ for every $v \in V$. Hence, by the definition of $T(x)$ we have that $\mu' = E_+(z) \subseteq T(x)$. It follows that $z_{u',v'} = 0$ whenever $\{u', v'\} \in E \setminus T(x)$, implying that z satisfies (11) [for $T \equiv T(x)$].

It remains to show that z satisfies the stability constraints of $(G; P)$. That is, we have to show that for each $\{u, v\} \in E$

$$\sum_{i >_u v} z_{u,i} + \sum_{j >_v u} z_{v,j} + z_{u,v} \geqslant 1.$$

Let $\{u, v\} \in E$. If $\{u, v\} \notin T(x)$, then by Lemma 3.10 we can assume without loss of generality that $\underline{\sigma}_x(v) >_v u$. Therefore $\underline{\sigma}_x(v) \neq v$ and so $v \in V^1$. Further, by the construction of μ', $\mu'(v) \geqslant_v \underline{\sigma}_x(v) >_v u$. Hence $\sum_{i \geqslant_v u} z_{v,i} = \sum_{i \in N(v)} z_{v,i} = 1$, and the stability constraint of $\{u, v\}$ holds. Assume next $\{u, v\} \in T(x)$. Note that $\mu'(v) \in \{\underline{\sigma}_x(v), \mu(v), \overline{\sigma}_x(v)\}$ and $\mu'(u) \in \{\underline{\sigma}_x(u), \mu(u), \overline{\sigma}_x(u)\}$, and consider two cases:

(i) $\mu'(v) = \overline{\sigma}_x(v)$ or $\mu'(u) = \overline{\sigma}_x(u)$: As $\{u, v\} \in T(x)$, it follows that $u \leqslant_v \overline{\sigma}_x(v) = \mu'(v)$. Hence, $\sum_{j \geqslant_v u} z_{v,j} = 1$, and the stability constraint of $\{u, v\}$ holds. (The second case follows from symmetric arguments.)

(ii) $\mu'(v) \in \{\underline{\sigma}_x(v), \mu(v)\}$ and $\mu'(u) \in \{\underline{\sigma}_x(u), \mu(u)\}$: In this case (12) implies that $\mu'(v) \geqslant_v \mu(v)$ and $\mu'(u) \geqslant_u \mu(u)$. Hence,

$$\sum_{i >_u v} z_{u,i} \geqslant \sum_{i >_u v} y_{u,i} \quad \text{and} \quad \sum_{j \geqslant_v u} z_{v,j} \geqslant \sum_{j \geqslant_v u} y_{v,j}.$$

In particular,

$$\sum_{i >_u v} z_{u,i} + \sum_{j \geqslant_v u} z_{v,j} \geqslant \sum_{i >_u v} y_{u,i} + \sum_{j \geqslant_v u} y_{v,j} \geqslant 1,$$

where the last inequality holds because $y \in \mathrm{FSM}(G; P)$.

We have seen that z satisfies (2), (3), (4), and (11). Hence $z \in \mathrm{FSM}_{T(x)}(G; P)$ and, as z is integral, it is a stable matching for $(G; P)$. ■

The next corollary follows immediately from Theorem 3.11 and is the key to our algorithm.

COROLLARY 3.12. *Let $x \in \mathrm{FSM}(G; P)$. Then $(G; P)$ has a stable matching if and only if it has one whose incidence vector is in $\mathrm{FSM}_{T(x)}(G; P)$.*

4. THE ALGORITHM

In this section we describe a polynomial procedure for solving stable matching problems. Our procedure uses linear programming to generate a finite sequence of fractional stable matchings that converges to the incidence vector of a stable matching, when such a matching exists. As linear programming is solvable in polynomial time (see, e.g., Schrijver [14]), we are able to obtain an algorithm for the stable matching problem with polynomial complexity.

For given $T \subseteq E$ and $\{u, v\} \in T$, we consider the linear programs

$$\max\{y_{u,v} : y \in \mathrm{FSM}_T(G; P)\}$$

and

$$\min\{y_{u,v} : y \in \mathrm{FSM}_T(G; P)\}.$$

We represent optimal solutions to the above linear programs by y^+ and y^-, respectively. As T and $\{u, v\}$ will always be identified by the context, no confusion will arise with this notation. We remark that these linear programs do not necessarily have unique optimal solutions.

Our algorithm solves the stable matching problem $(G; P)$ by iteratively solving a sequence of linear programs of the above type. The following results are used to establish the correctness of this procedure.

LEMMA 4.1. *Let* $x \in \mathrm{FSM}(G; P)$ *and* $y \in \mathrm{FSM}_{T(x)}(G; P)$. *Then* $T(y) \subseteq T(x)$.

Proof. As y satisfies (11) with respect to $T = T(x)$, $y_{u,v} > 0$ implies $\underline{\sigma}_x(v) \leqslant_v u \leqslant_v \overline{\sigma}_x(v)$ and $\underline{\sigma}_x(u) \leqslant_u v \leqslant_u \overline{\sigma}_x(u)$. In particular, for every $v \in V$, $\underline{\sigma}_x(v) \leqslant_v \underline{\sigma}_y(v) \leqslant_v \overline{\sigma}_y(v) \leqslant_v \overline{\sigma}_x(v)$. By the definition of $T(x)$ and $T(y)$ in (8), it follows that $T(y) \subseteq T(x)$. ∎

LEMMA 4.2. *Assume* $x \in \mathrm{FSM}(G; P)$ *and* $E_+(x) \neq E_1(x)$. *Let* $\{u, v\}$ *be a pair in* $E_+(x) \setminus E_1(x)$ *such that* $u = \underline{\sigma}_x(v)$, *and let* $T \equiv T(x)$.

(a) *If* $y^+_{u,v} = 1$, *then* $T(y^+) \subset T(x)$.
(b) *If* $y^-_{u,v} = 0$, *then* $T(y^-) \subset T(x)$.
(c) *If* $y^+_{u,v} < 1$ *and* $y^-_{u,v} > 0$, *then* $(G; P)$ *has no stable matching.*

Proof. Note first that if $E_+(x) \neq E_1(x)$, then trivially there is a pair $\{u, v\} \in E_+(x) \setminus E_1(x)$ such that $u = \underline{\sigma}_x(v)$. Now, if $y^+_{u,v} = 1$, then $\underline{\sigma}_x(v) = \underline{\sigma}_{y^+}(v) = \overline{\sigma}_{y^+}(v) <_v \overline{\sigma}_x(v)$ and, by Lemma 4.1, $T(y^+) \subseteq T(x)$. Further, $\{v, \overline{\sigma}_x(v)\} \in T(x) \setminus T(y^+)$, implying that $T(y^+) \subset T(x)$. We now consider when $y^-_{u,v} = 0$. We recall that by Lemma 3.8, $v = \overline{\sigma}_x(u)$. Since, by Lemma 4.1, $T(y^-) \subseteq T(x)$, we conclude that $\overline{\sigma}_{y^-}(u) <_u \overline{\sigma}_x(u) = v$. Thus, as $\{u, v\} \notin T(y^-)$, we conclude that $T(y^-) \subset T(x)$. Finally, if $y^+_{u,v} < 1$ and $y^-_{u,v} > 0$ then $0 < y_{u,v} < 1$, for all $y \in \mathrm{FSM}_{T(x)}(G; P)$. Hence there is no stable matching for $(G; P)$ whose incidence vector is in $\mathrm{FSM}_T(G; P)$, and by Corollary 3.12, $(G; P)$ has no stable matching. ∎

THE ALGORITHM.

begin
  Let x be a fractional stable matching for $(G; P)$
  while $E_+(x) \neq E_1(x)$ **do**
    choose $\{u, v\} \in E_+(x) \setminus E_1(x)$ such that $u = \underline{\sigma}_x(v)$.
    set $T := T(x)$
    if $y^+_{u,v} = 1$ **then** $x := y^+$
    if $y^+_{u,v} < 1$ and $y^-_e = 0$ **then** $x := y^-$
    if $y^+_{u,v} < 1$ and $y^-_{u,v} > 0$ **then**
      output "no stable matching" **stop**
  output the stable matching $\mu := T(x)$
end

THEOREM 4.3. *The algorithm terminates after at most $|E|$ iterations, either with a stable matching μ for $(G; P)$ or with the conclusion that none exists.*

Proof. Termination of this algorithm is guaranteed, since as long as neither of the two termination criteria is met, parts (a) and (b) of Lemma 4.2 imply that the algorithm generates a sequence of fractional matchings whose extended supports have monotonically decreasing cardinality. As $|T(x)| \leqslant |E|$ for each $x \in \mathrm{FSM}(G; P)$ and, in particular, for the initial fractional stable matching, it follows that the algorithm must stop after at most $|E|$ iterations.

Assume now that at termination the first criterion $[E_+(x) = E_1(x)]$ is met, and let x be the final fractional stable matching. Then by Lemma 3.1, x is the incidence vector of a stable matching for $(G; P)$. Further, by the definition of $T(x)$ it is clear that $\mu = T(x)$ is the matching whose incidence vector is x.

Finally, if the alternative termination criterion is met ($y^+_{u,v} < 1$ and $y^-_{u,v} > 0$), part (c) of Lemma 4.2 asserts that no stable matching exists. ■

Since the initial fractional stable matching x can be found by solving a single linear program and each iteration requires solving at most two linear programs, this algorithm yields a method that solves a stable matching problem $(G; P)$ by solving at most $2|E| + 1$ linear programs.

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