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The q-Analogue of the Laguerre Polynomials

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1. INTRODUCTION

Let $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and $(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$, $0<q<1$. We can define $(a;q)_n$ for arbitrary complex n by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n,q)_\infty}, \qquad 0<q<1.$$

Jackson [6] defined a q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} 0<q<1. \tag{1.1}$$

Note that Γ_q satisfies the functional equation

$$\Gamma_q(x+1) = \frac{q^x-1}{q-1}\Gamma_q(x). \tag{1.2}$$

He also showed that $\lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x)$. Askey [2] proved the integral formula

$$\int_0^\infty \frac{x^\alpha\,dx}{(-(1-q)x;q)_\infty} = \frac{\Gamma(-\alpha)\,\Gamma(\alpha+1)}{\Gamma_q(-\alpha)}, \qquad 0<q<1, \quad \mathrm{Re}(\alpha)>0. \tag{1.3}$$

Using the q-binomial theorem [3, p. 66],

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \tag{1.4}$$

one can easily show that $1/(-(1-q)x;q)_\infty \to e^{-x}$ as $q\to 1^-$. We can estimate the integral in (1.3) by the discrete approximation

$$\sum_{k=-\infty}^{\infty}\frac{(cq^k)^\alpha}{(-(1-q)cq^k;q)_\infty}c(1-q)q^k, \qquad c>0, \quad 0<q<1. \tag{1.5}$$

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This can be written as

$$\int_0^\infty \frac{x^\alpha d(x, c; q)}{(-(1-q)x; q)_\infty}, \tag{1.6}$$

where the measure $d(x, c; q)$ has a point mass of size $c(1-q)q^k$ at cq^k. Note that as $q \to 1$, $\int_0^\infty f(x)\, d(x, c; q) \to \int_0^\infty f(x)\, dx$ for all continuous and integrable functions $f(x)$ on $(0, \infty)$.

It turns out that if we normalize $x^\alpha\, dx/(-(1-q)\,x;\, q)_\infty$ and $x^\alpha d(x, c; q)/(-(1-q)x; q)_\infty$ to have total mass one, then the moment sequences and orthogonal polynomials for these measures are the same. These orthogonal polynomials are q-extensions of the Laguerre polynomials and were discovered by Hahn [7] although he said little about them.

The above moment sequence is clearly an indeterminate Stieltjes moment sequence, and the set of all measures that generate this moment sequence is clearly a convex set. The extreme points of this set will be found; they are described in Section 7.

2. The Orthogonality Relation for the q-Laguerre Polynomials

The ordinary Laguerre polynomials are defined as

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k x^k}{(\alpha+1)_k k!}, \tag{2.1}$$

where $(a)_k = a(a+1)(a+2) \cdots (a+k-1)$. These polynomials satisfy the orthogonality relation

$$\begin{aligned} \int_0^\infty L_n^{(\alpha)}(x)\, L_m^{(\alpha)}(x)\, x^\alpha e^{-x}\, dx &= \Gamma(\alpha+n+1)/n!, \qquad m = n, \\ &= 0, \qquad m \neq n. \end{aligned} \tag{2.2}$$

There is a q-analogue of these polynomials which is defined as

$$L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k q^{\binom{k}{2}} (1-q)^k (q^{n+\alpha+1} x)^k}{(q^{\alpha+1}; q)_k (q; q)_k}. \tag{2.3}$$

Note that $L_n^{(\alpha)}(x; q) \to L_n^{(\alpha)}(x)$ as $q \to 1^-$.

One orthogonality relation is

THEOREM 1. *For* $\alpha > -1$,

$$\int_0^\infty L_m^{(\alpha)}(x;q)\,L_n^{(\alpha)}(x;q)\frac{x^\alpha\,dx}{(-(1-q)x;q)_\infty}$$
$$= \frac{\Gamma(\alpha+1)\,\Gamma(-\alpha)(q^{\alpha+1};q)_n}{\Gamma_q(-\alpha)(q;q)_n q^n}, \qquad m = n,$$
$$= 0, \qquad m \neq n. \tag{2.4}$$

Proof. We first show that

$$\int_0^\infty L_n^{(\alpha)}(x;q)x^m \frac{x^\alpha\,dx}{(-(1-q)x;q)_\infty} = 0, \qquad m < n. \tag{2.5}$$

In fact by (1.3),

$$\int_0^\infty L_n^{(\alpha)}(x;q)\frac{x^{\alpha+m}}{(-(1-q)x;q)_\infty}\,dx$$
$$= \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(q^{n+\alpha+1})^k(1-q)^k}{(q^{\alpha+1};q)_k(q,q)_k}$$
$$\times \int_0^\infty \frac{x^{k+\alpha+m}}{(-(1-q)x;q)_\infty}\,dx$$
$$= \frac{(q^{\alpha+1};q)_n}{(q;q)_n}$$
$$\times \sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(q^{n+\alpha+1})^k\Gamma(-k-\alpha-m)\Gamma(1+k+\alpha+m)}{(1-q)^{-k}(q^{\alpha+1};q)_k(q,q)_k\Gamma_q(-k-\alpha-m)}.$$

By the reflection formula for the gamma function and the functional equation for the q-gamma function, we obtain

$$\frac{\pi(q^{\alpha+1};q)_n\csc(-\alpha\pi-m\pi)}{(q;q)_n\Gamma_q(-\alpha-m)}$$
$$\times \sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(q^{n+\alpha+1})^k(-)^k(q^{-\alpha-k-m};q)_k}{(q^{\alpha+1};q)_k(q;q)_k}$$
$$= \frac{\pi(q^{\alpha+1};q)_n\csc(-\alpha\pi-m\pi)}{(q;q)_n\Gamma_q(-\alpha-m)}$$
$$\times \sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(q^{n+\alpha+1})^k q^{-\alpha k-mk-\binom{k+1}{2}}(q^{\alpha+m+1};q)_k}{(q^{\alpha+1};q)_k(q;q)_k}$$

$$= \frac{(q^{\alpha+1};q)_n \Gamma(-\alpha)\,\Gamma(\alpha+1)(q^{-\alpha-m};q)_m(-)^m}{(q;q)_n(1-q)^m\,\Gamma_q(-\alpha)}$$
$$\times \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^{\alpha+m+1};q)_k(q^{n-m})^k}{(q^{\alpha+1};q)_k\,(q;q)_k}.$$

There is a sum due to Heine [see 3, p. 68],

$$\sum_{k=0}^{\infty} \frac{(a;q)_k(b;q)_k}{(c;q)_k(q;q)_k}\left(\frac{c}{ab}\right)^k = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}. \tag{2.6}$$

In particular when $a = q^{-n}$,

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k(b;q)_k}{(c;q)_k(q;q)_k}\left(\frac{cq^n}{b}\right)^k = \frac{(c/b;q)_n}{(c;q)_n}. \tag{2.7}$$

Hence the integral (2.5) is

$$\int_0^\infty L_n^{(\alpha)}(x;q)\,\frac{x^{m+\alpha}\,dx}{(-(1-q)x;q)_\infty}$$
$$= \frac{(q^{\alpha+1};q)_n\Gamma(-\alpha)\,\Gamma(\alpha+1)(q^{\alpha+1};q)_m(q^{-m};q)_n}{(q;q)_n\Gamma_q(-\alpha)(1-q)^m q^{\alpha m+\binom{m+1}{2}}(q^{\alpha+1};q)_n}$$
$$= 0, \qquad m < n,$$
$$= \frac{\Gamma(-\alpha)\,\Gamma(\alpha+1)(q^{\alpha+1};q)_n(-)^n}{\Gamma_q(-\alpha)\,q^{\alpha n+n^2+n}(1-q)^n}, \qquad m = n.$$

Since $L_n^{(\alpha)}(x;q) = ((-)^n q^{n^2+n\alpha}(1-q)^n/(q;q)_n)\,x^n + \cdots$,

$$\int_0^\infty (L_n^{(\alpha)}(x;q))^2\,\frac{x^\alpha\,dx}{(-(1-q)x;q)_\infty} = \frac{\Gamma(-\alpha)\,\Gamma(\alpha+1)(q^{\alpha+1};q)_n}{\Gamma_q(-\alpha)(q;q)_n q^n},$$

which proves the theorem. ∎

We can normalize the measure so that the total mass is one, and we obtain

$$\int_0^\infty L_n^{(\alpha)}(x;q)\,L_m^{(\alpha)}(x;q)\,\frac{x^\alpha\Gamma_q(-\alpha)}{(-(1-q)x;q)_\infty\,\Gamma(-\alpha)\,\Gamma(\alpha+1)}\,dx$$
$$= \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}, \qquad m = n,$$
$$= 0, \qquad m \neq n. \tag{2.8}$$

There is another orthogonality relation using Ramanujan's sum

$$\sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(b;q)_k} x^k = \frac{(ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty (b/a;q)_\infty}{(x;q)_\infty (b/ax;q)_\infty (b;q)_\infty (q/a;q)_\infty}. \tag{2.9}$$

(See Askey [2].) Equation (2.9) can be rewritten as

$$\sum_{k=-\infty}^{\infty} \frac{(bq^k;q)_\infty}{(aq^k;q)_\infty} x^k = \frac{(ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty (b/a;q)_\infty}{(x;q)_\infty (b/ax;q)_\infty (a;q)_\infty (q/a;q)_\infty}. \tag{2.10}$$

Let $b=0$, and $x=q^\beta$ in (2.10). Then

$$\sum_{k=-\infty}^{\infty} \frac{q^{\beta k}}{(aq^k;q)_\infty} = \frac{(aq^\beta;q)_\infty (q^{1-\beta}/a;q)_\infty (q;q)_\infty}{(q^\beta;q)_\infty (a;q)_\infty (q/a;q)_\infty}. \tag{2.11}$$

This leads to the following result:

THEOREM 2.

$$\begin{aligned} \sum_{k=-\infty}^{\infty} L_n^{(\alpha)}(cq^k;q)\, L_m^{(\alpha)}(cq^k;q) \frac{q^{k\alpha+k}}{(-c(1-q)\,q^k;q)_\infty A} &= \frac{(q^{\alpha+1};q)_n}{q^n(q;q)_n}, \qquad m=n, \\ &= 0, \qquad m \neq n, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} A &= \sum_{k=-\infty}^{\infty} \frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty} \\ &= \frac{(-cq^{\alpha+1}(1-q);q)_\infty (-1/cq^\alpha(1-q);q)_\infty (q;q)_\infty}{(q^{\alpha+1};q)_\infty (-q/c(1-q);q)_\infty (-c(1-q);q)_\infty}. \end{aligned}$$

Remarks. The sum can be regarded as an integral with respect to a discrete measure with mass $q^{k\alpha+k}/(-c(1-q)q^k;q)_\infty A$ at cq^k. The normalization factor A is put in so that the measure has total mass one, and the sum was evaluated using (2.11).

Proof. It suffices to show that this discrete measure has the same moments as the measure

$$d\Psi(x) = \frac{x^\alpha \Gamma_q(-\alpha)\, dx}{(-(1-q)x;q)_\infty \Gamma(-\alpha)\, \Gamma(1+\alpha)} \qquad \text{on } [0,\infty).$$

Now by (1.2),

$$\int_0^\infty \frac{t^{n+\alpha}\Gamma_q(-\alpha)}{(-(1-q)t;q)_\infty \Gamma(-\alpha)\,\Gamma(\alpha+1)}\,dt$$
$$=\frac{\Gamma_q(-\alpha)\,\Gamma(-\alpha-n)\,\Gamma(\alpha+n+1)}{\Gamma_q(-\alpha-n)\,\Gamma(-\alpha)\,\Gamma(\alpha+1)}=\mu_n.$$

Using the functional equations and the reflection formula for $\Gamma(x)$ and the functional equation for the $\Gamma_q(x)$, we get

$$\mu_n=\frac{(1-q^{-\alpha-1})(1-q^{-\alpha-2})\cdots(1-q^{-\alpha-n})\csc(-\pi\alpha-\pi n)}{(1-q)^n\csc(-\pi\alpha)}$$

or

$$\mu_n=\frac{(q^{\alpha+1};q)_n q^{-\alpha n-\binom{n+1}{2}}}{(1-q)^n}. \tag{2.13}$$

The nth moment of the discrete measure is

$$\sum_{k=-\infty}^{\infty}\frac{(cq^k)^n q^{k\alpha+k}}{A(-c(1-q)\,q^k;q)_\infty}=c^n\sum_{k=-\infty}^{\infty}\frac{q^{nk+\alpha k+k}}{(-c(1-q)q^k;q)_\infty A}.$$

By (2.11) this is

$$\frac{\left(\begin{array}{l}c^n(q^{\alpha+1};q)_\infty(-c(1-q);q)_\infty(-q/c(1-q);q)_\infty\\ \qquad\times(-c(1-q)\,q^{\alpha+n+1};q)_\infty(-q^{-\alpha-n}/c(1-q);q)_\infty(q,q)_\infty\end{array}\right)}{\left(\begin{array}{l}(-c(1-q)q^{\alpha+1};q)_\infty(-q^{-\alpha}/c(1-q);q)_\infty(q;q)_\infty\\ \qquad\times(q^{\alpha+n+1};q)_\infty(-c(1-q);q)_\infty(-q/c(1-q);q)_\infty\end{array}\right)}.$$

Simplifying, we get

$$\frac{c^n(q^{\alpha+1};q)_n(-q^{-\alpha-n}/c(1-q);q)_n}{(-c(1-q)\,q^{\alpha+1};q)_n}$$
$$=\frac{\left(\begin{array}{l}c^n(1+q^{-\alpha-n}/c(1-q))(1+q^{1-\alpha-n}/c(1-q))\\ \qquad\cdots(1+q^{-\alpha-1}/c(1-q))(1-q^{\alpha+1})\cdots(1-q^{\alpha+n})\end{array}\right)}{(1+c(1-q)\,q^{\alpha+1})(1+c(1-q)\,q^{\alpha+2})\cdots(1+c(1-q)\,q^{\alpha+n})}$$
$$=\frac{q^{-\alpha n-\binom{n+1}{2}}(q^{\alpha+1};q)_n}{(1-q)^n},$$

which is the same nth moment as before.

Hence the orthogonality relation for the discrete measure is the same as that of the absolutely continuous one, and the theorem is proved. ∎

This theorem implies that the moment problem

$$\int_0^\infty x^n \, d\Psi(x) = \mu_n = \frac{q^{-\alpha n - \binom{n+1}{2}} (q^{\alpha+1}; q)_n}{(1-q)^n}, \qquad n = 0, 1, 2, ..., \tag{2.14}$$

has many solutions $d\Psi(x)$, in other words, the moment problem (2.14) is indeterminate. By a theorem of Riesz, the orthogonal polynomials $\{p_n(x)\}_{n=\infty}^{\infty}$ are complete in $L^2(d\Psi)$ if and only if $d\Psi(x)$ is an extreme point of the convex set of solutions of the moment problem. The external solutions are purely discrete measures which have mass points located at the zeros of an entire function. See Shohat and Tamarkin [10, pp. 60, 61] and Stone [11, pp. 577–614] for proofs of these facts. The discrete measures we found are not extremal since 0 is a limit point of the mass points and no entire function can have a finite limit point of its roots. We will give a description of all the extreme measures in Section 7 after we establish some preliminary results.

3. The Three-Term Recurrence Relation for $L_n^{(\alpha)}(x, q)$

Suppose we have a set of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ which are orthogonal with respect to a positive measure $d\Psi(x)$ with $p_n(x) = k_n x^n + k'_n x^{n-1} + \cdots$ and $\int (p_n(x))^2 \, d\Psi(x) = h_n > 0$, $n \geqslant 0$, $h_0 = p_0 = 1$. Here we assume that the measure $d\Psi(x)$ is not a measure with finitely many mass points. From the general theory of orthogonal polynomials, the polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfy a three-term recurrence relation

$$x p_n(x) = A_n \, p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x), \qquad n \geqslant 0, \tag{3.1}$$

where $A_n = k_n/k_{n+1}$ and $B_n = k'_n/k_n - k'_{n+1}/k_{n+1}$ for $n \geqslant 0$, while $C_n = k_{n-1} h_n / k_n h_{n-1}$, $n \geqslant 1$. Here $k'_0 = p_{-1} = 0$ and $p_0 = 1$. Since $p_{-1} = 0$, C_0 can be arbitrary. In Section 7 we will find second solutions of (3.1) with $C_0 = -1$.

For the q-Laguerre polynomials, the three-term recurrence relation is

$$\begin{aligned} x L_n^{(\alpha)}(x; q) = & -\frac{(1-q^{n+1})}{(1-q)\, q^{2n+\alpha+1}} L_{n+1}^{(\alpha)}(x; q) \\ & + \left[\frac{(1-q^n)}{(1-q) q^{2n+\alpha}} + \frac{(1-q^{n+\alpha+1})}{(1-q)\, q^{2n+\alpha+1}} \right] L_n^{(\alpha)}(x; q) \\ & - \frac{(1-q^{n+\alpha})}{(1-q)\, q^{2n+\alpha}} L_{n-1}^{(\alpha)}(x; q), \qquad n \geqslant 0. \end{aligned} \tag{3.2}$$

We have already seen that as $q \to 1^-$, $L_n^{(\alpha)}(x; q) \to L_n^{(\alpha)}(x)$. Also (3.2) becomes the three-term recurrence relation for the ordinary Laguerre polynomials which is

$$\begin{aligned} xL_n^{(\alpha)}(x) = & -(n+1)\,L_{n+1}^{(\alpha)}(x) \\ & + (2n+\alpha+1)\,L_n^{(\alpha)}(x) - (n+\alpha)\,L_{n-1}^{(\alpha)}(x). \end{aligned} \tag{3.3}$$

In some applications, a three-term recurrence relation of the form (3.1) occurs, and one uses it to generate a sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$. By a famous theorem of Favard [5], a set of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfying (3.1) with A_n, B_n, and C_n real, $n \geqslant 0$, and $A_{n-1}C_n > 0$, $n \geqslant 1$, is orthogonal with respect to at least one positive measure on the real line which is the solution of a certain moment problem. Consequently the orthogonality measure is not unique and is usually difficult to find. See Shohat and Tamarkin [10, Chap. 2].

The conditions $A_{n-1}C_n > 0$, $n \geqslant 1$, are necessary for the polynomials to be orthogonal with respect to a positive measure. For $0 < q < 1$ this means that $q^{\alpha} < q^{-1}$ so that α is either complex or $\alpha > -1$. The case $0 < q < 1$ and $\alpha > -1$ is the one considered in this paper. Other cases of orthogonality include $q > 1$ and $q^{\alpha} > q^{-1}$, or $q < -1$ and $q^{\alpha} < q^{-1}$. In the case $|q| > 1$, the solution of the associated moment problem is unique.

4. Relations between $L_n^{(\alpha)}(x;q)$ and $L_n^{(\alpha+1)}(x;q)$

We have shown that the $\{L_n^{(\alpha)}(x;q)\}_{n=0}^{\infty}$ are orthogonal with respect to $x^{\alpha}\,dx/(-(1-q)x;q)_{\infty}$ on $[0,\infty)$. Karlin and McGregor [9] stated a number of theorems about the relationships between the polynomials orthogonal with respect to $d\Psi(x)$ and those orthogonal with respect to $x\,d\Psi(x)$, where $d\Psi(x)$ is a positive measure. We state some of them here.

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a set of polynomials satisfying:

$$\begin{aligned} \int_0^{\infty} p_n(x)\,p_m(x)\,d\Psi(x) &= h_n, \qquad n = m, \\ &= 0, \qquad n \neq m, \end{aligned} \tag{4.1}$$

where $p_0(x) = 1$, $\int_0^{\infty} d\Psi(x) = 1$, $h_0 = 1$, and $p_n(0) = 1$, $n = 0, 1, 2,\ldots$, and $h_n > 0$, $n = 0, 1, 2,\ldots$.

It follows that the p_n's satisfy the three-term recurrence relation

$$xp_n(x) = A_n p_{n+1}(x) - (A_n + C_n)\,p_n(x) + C_n p_{n-1}(x), \qquad n \geqslant 0, \tag{4.2}$$

where $p_0 = 1$, $p_{-1} = 0$, and $A_{n-1}h_n = C_n h_{n-1}$, $n \geqslant 1$, $C_0 = 0$. Let $q_0(x) = ($
and

$$q_{n+1}(x) = -\frac{A_n}{h_n}[p_{n+1}(x) - p_n(x)], \qquad n \geqslant 0. \tag{4.3}$$

It turns out that the set $\{q_{n+1}(x)/x\}_{n=0}^{\infty}$ is orthogonal with respect t($x d\Psi(x)$. See Karlin and McGregor [9].

The following relations follow easily from (4.2) and (4.3):

$$\begin{aligned} xq_{n+1}(x) = A_n q_n(x) - (A_n + C_{n+1})\, q_{n+1}(x) \\ + C_{n+1} q_{n+2}(x), \qquad n \geqslant 0, \end{aligned} \tag{4.4}$$

$$p_n(x) = 1 - \sum_{k=0}^{n-1} \frac{h_k}{A_k} q_{k+1}(x), \tag{4.5}$$

$$q_{n+1}(x) = -x \sum_{k=0}^{n} \frac{p_k(x)}{h_k}, \tag{4.6}$$

$$\begin{aligned} (x-y) \sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{h_k} &= \frac{-A_n}{h_n}[p_{n+1}(y)p_n(x) - p_n(y)p_{n+1}(x)] \\ &= q_{n+1}(y)p_n(x) - q_{n+1}(x)p_n(y). \end{aligned} \tag{4.7}$$

Now we let $p_n(x) = ((q;\, q)_n/(q^{\alpha+1};\, q)_n)\, L_n^{(\alpha)}(x;\, q)$. Then $h_n = (q;q)_n/q^n(q^{\alpha+1};q)_n$ and $A_n = -(1-q^{n+\alpha+1})/(1-q)q^{2n+\alpha+1}$, $n \geqslant 0$. Thus

$$\begin{aligned} q_{n+1}(x) &= \frac{(q^{\alpha+1};q)_{n+1}}{(q;q)_n(1-q)\,q^{n+\alpha+1}} \\ &\quad \times \left[\frac{(q;q)_{n+1}}{(q^{\alpha+1};q)_{n+1}} L_{n+1}^{(\alpha)}(x;q) - \frac{(q;q)_n}{(q^{\alpha+1};q)_n} L_n^{(\alpha)}(x;q)\right] \\ &= \frac{(q^{\alpha+1};q)_{n+1}}{(q;q)_n(1-q)\,q^{n+\alpha+1}} \sum_{k=1}^{n+1} [(q^{-n-1};q)_k q^{nk+k} - (q^{-n};q)_k q^{nk}] \\ &\quad \times \frac{q^{\binom{k}{2}+\alpha k+k}(1-q)^k x^k}{(q^{\alpha+1};q)_k(q;q)_k}, \end{aligned}$$

$$q_{n+1}(x) = -\frac{(q^{\alpha+2};q)_n x}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^{\binom{k}{2}+(\alpha+1)k+nk+k}(1-q)^k x^k}{(q^{\alpha+2};q)_k(q;q)_k},$$

or

$$q_{n+1}(x) = -xL_n^{(\alpha+1)}(x;q). \tag{4.8}$$

Substituting all this into (4.4) through (4.7) we get

$$\frac{(q;q)_{n+1}}{(q^\alpha;q)_{n+1}} L_{n+1}^{(\alpha-1)}(x;q) = 1 - \frac{(1-q)q^\alpha x}{(1-q^\alpha)} \sum_{k=0}^{n} \frac{q^k (q;q)_k}{(q^{\alpha+1};q)_k} L_k^{(\alpha)}(x;q), \tag{4.9}$$

$$L_n^{(\alpha+1)}(x;q) = \sum_{k=0}^{n} q^k L_k^{(\alpha)}(x;q), \tag{4.10}$$

$$\begin{aligned}(x-y) \sum_{k=0}^{n} & \frac{q^k (q;q)_k}{(q^{\alpha+1};q)_k} L_k^{(\alpha)}(x;q)\, L_k^{(\alpha)}(y;q) \\ &= \frac{(q;q)_{n+1}}{(1-q)(q^{\alpha+1};q)_n q^{n+\alpha+1}} \\ &\times [L_{n+1}^{(\alpha)}(y;q)\, L_n^{(\alpha)}(x;q) - L_n^{(\alpha)}(y;q)\, L_{n+1}^{(\alpha)}(x;q)] \\ &= \frac{(q;q)_n}{(q^{\alpha+1};q)_n} [x L_n^{(\alpha+1)}(x;q)\, L_n^{(\alpha)}(y;q) - y L_n^{(\alpha)}(y;q)\, L_n^{(\alpha)}(x;q)].\end{aligned} \tag{4.11}$$

Other relations for q-Laguerre polynomials which follow easily from the above relations are

$$L_n^{(\alpha-1)}(x;q) = q^{-n}[L_n^{(\alpha)}(x;q) - L_{n-1}^{(\alpha)}(x;q)], \tag{4.12}$$

$$\begin{aligned}\frac{(1-q^{\alpha+n})}{(1-q)} L_n^{(\alpha-1)}(x;q) &= \frac{(1-q^{n+1})\, q^{-n-1}}{(1-q)} L_{n+1}^{(\alpha)}(x;q) \\ &\quad + \left[q^{\alpha+n} x - \frac{(1-q^{n+1})}{(1-q)} q^{-n-1}\right] L_n^{(\alpha)}(x;q),\end{aligned} \tag{4.13}$$

$$\begin{aligned}x L_n^{(\alpha+1)}(x;q) &= \frac{(1-q^{\alpha+n+1})}{(1-q)} q^{-\alpha-n-1} L_n^{(\alpha)}(x;q) \\ &\quad - \frac{(1-q^{n+1})}{(1-q)} q^{-\alpha-n-1} L_{n+1}^{(\alpha)}(x;q),\end{aligned} \tag{4.14}$$

$$\begin{aligned}x L_n^{(\alpha+1)}(x;q) &= \left[x q^n - \frac{(1-q^n)}{(1-q)} q^{-\alpha-n}\right] L_n^{(\alpha)}(x;q) \\ &\quad + \frac{(1-q^{\alpha+n})}{(1-q)} q^{-\alpha-n} L_{n-1}^{(\alpha)}(x;q),\end{aligned} \tag{4.15}$$

$$L_n^{(\alpha)}(x;q) - L_n^{(\alpha)}(xq;q) = -x(1-q)\, q^{\alpha+1} L_{n-1}^{(\alpha+1)}(xq;q), \tag{4.16}$$

$$\sum_{n=0}^{\infty} r^n L_n^{(\alpha)}(x;q) = \frac{(rq^{\alpha+1};q)_\infty}{(r;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+\alpha k}[-(1-q)xr]^k}{(q;q)_k (rq^{\alpha+1};q)_k}. \tag{4.17}$$

Equations (4.12)–(4.16) are straightforward to verify, and (4.17) can be proved using the q-binomial theorem (1.4).

5. Some Basic Properties of the Roots of $L_n^{(\alpha)}(x; q)$

Many properties of $L_n^{(\alpha)}(x; q)$ can be obtained from the orthogonality relation, others from the q-difference equation which is satisfied by $L_n^{(\alpha)}(x; q)$. Szegö [12, pp. 44–47] proves a number of theorems about the zeros of orthogonal polynomials in general. We will not give any of those proofs here, but we will see how these theorems apply to q-Laguerre polynomials. We will then establish some additional properties of the roots of $L_n^{(\alpha)}(x; q)$. The behavior of these roots as $n \to \infty$ is important because the roots converge to the mass points of an extreme solution of the moment problem (2.14).

DEFINITION. Let f and g be two functions defined on $(-\infty, \infty)$. We say that the roots of f and g interlace if a root of f lies strictly between two consecutive and distinct roots of g and vice versa.

THEOREM A. *Let $\{p_n(x)\}_{n=0}^{\infty}$ be a set of polynomials orthogonal with respect to a positive measure $d\Psi(x)$ on the interval (a, b). Then the roots of each of the polynomials are real and simple and lie in the orthogonality interval. Moreover, the roots of p_{n+1} and p_n interlace for $n = 1, 2, 3,\ldots$.*

COROLLARY. *Let $0 < x_{1,n}^{(\alpha)}(q) < x_{2,n}^{(\alpha)}(q) < \cdots < x_{n,n}^{(\alpha)}(q)$ be the roots of $L_n^{(\alpha)}(x; q)$. Then*

$$\begin{aligned} 0 &< x_{1,n+1}^{(\alpha)}(q) < x_{1,n}^{(\alpha)}(q) < x_{2,n+1}^{(\alpha)}(q) < {}_{2,n}^{(\alpha)}(q) \\ &< \cdots < x_{n,n+1}^{(\alpha)} < x_{n,n}^{(\alpha)}(q) < x_{n+1,n+1}^{(\alpha)}(q). \end{aligned} \tag{5.1}$$

THEOREM B. *Let $\{p_n(x)\}_{n=0}^{\infty}$ be a set of polynomials orthogonal with respect to a positive measure $d\Psi(x)$. Then for any n, $p_n(x)$ has at most one root on any interval where $\Psi(x)$ is constant.*

COROLLARY. *Let a and b be two consecurtive roots of $L_n^{(\alpha)}(x; q)$, $0 < a < b$. Then $b/a > q^{-1}$.*

Proof. By Theorem 2, there is a discrete measure with point masses at the points $\{aq^n\}_{n=-\infty}^{\infty}$. Since the interval $[a, aq^{-1}]$ contains at most one root of $L_n^{(\alpha)}(x; q)$, b lies outside the interval $[a, aq^{-1}]$ and $b > aq^{-1}$. ∎

In Section 6 we will show that the ratio between consecutive roots is in fact $>q^{-2}$. But to do that, we will have to use q-difference equations which will be discussed in Section 6. Now we will study the behavior of $L_n^{(\alpha)}(x; q)$ as α or q varies. Szegö [12, pp. 115, 116] proves a general theorem about the behavior of the zeros as a parameter is varied.

THEOREM C. *Let $w(x, \tau)$ be a weight function on the interval $[a, b]$ depending on a parameter τ such that $w(x, \tau)$ is positive and continuous for $a < x < b$, $\tau_1 < \tau < \tau_2$. Also, assume the existence and continuity of the partial derivative $w_\tau(x, \tau)$ for $a < x < b$, $\tau_1 < \tau < \tau_2$, and the convergence of the integrals*

$$\int_a^b x^\nu w_\tau(x, \tau)\, dx, \qquad \nu = 0, 1, 2, \dots, 2n-1,$$

uniformly in very closed interval $\tau' \leqslant \tau \leqslant \tau''$ of the open interval (τ_1, τ_2). If the zeros of $p_n(x) = p_n(x, \tau)$ are denoted by $x_1(\tau) < x_2(\tau) < \cdots < x_n(\tau)$, the νth zero $x_\nu(\tau)$ (for a fixed value of ν) is an increasing function of τ provied w_τ/w is an increasing function of x, $a < x < b$.

COROLLARY. *Let $\alpha > -1$, $0 < q < 1$, and*

$$0 < x_{1,n}^{(\alpha)}(q) < x_{2,n}^{(\alpha)}(q) < \cdots < x_{n,n}^{(\alpha)}(q) \tag{5.2}$$

be the roots of $L_n^{(\alpha)}(x; q)$. Then for any j, $1 \leqslant j \leqslant n$, $x_{j,n}^{(\alpha)}(q)$ is an increasing function of α and a decreasing function of q.

Proof. The proof of Theorem C given in Szegö [12] is still valid if "increasing" is replaced everywhere in the theorem by "decreasing." Also, the proof is valid if $b = \infty$.

Now consider the orthogonality relation (2.4). Let $w(x, \alpha, q) = x^\alpha/(-(1-q)x; q)_\infty$. Then $w_\alpha/w = \ln x$, which is an increasing function of x on $(0, \infty)$. So $x_{j,n}^{(\alpha)}(q)$ is an increasing function of α for each j. To prove the second part of the corollary, we need to show that $(\partial/\partial x)(w_q/w) < 0$ for $x > 0$.

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{w_q}{w} &= \frac{\partial}{\partial x}\sum_{n=0}^{\infty}\frac{[(n+1)\,q^n - nq^{n-1}]x}{(1+(1-q)\,q^n x)} \\
&= \sum_{n=0}^{\infty}\,[(n+1)q^n - nq^{n-1}]\,\frac{(1+(1-q)\,q^n x) - (1-q)\,q^n x}{(1+(1-q)\,q^n x)^2} \\
&= \sum_{n=0}^{\infty}\frac{(n+1)q^n - nq^{n-1}}{(1+(1-q)q^n x)^2} \\
&= \sum_{n=0}^{\infty}\frac{(n+1)q^n}{(1+(1-q)q^n x)^2} - \sum_{n=1}^{\infty}\frac{nq^{n-1}}{(1+(1-q)q^n x)^2} \\
&= \sum_{n=0}^{\infty}\frac{(n+1)q^n}{(1+(1-q)q^n x)^2} - \frac{(n+1)q^n}{(1+(1-q)q^{n+1}x)^2}.
\end{aligned} \tag{5.3}$$

For $x > 0$, the denominator of the second fraction is less than that of the first, so each term of the sum is negative. Hence $(\partial/\partial x)(w_q/w) < 0$, and $x_{j,n}^{(\alpha)}(q)$ is a decreasing function of q for $0 < q < 1$ and we are done. ∎

It turns out that not only are the roots of $L_n^{(\alpha+1)}(x; q)$ larger than those of $L_n^{(\alpha)}(x; q)$, but the roots of the two polynomials interlace. The proof, however, is entirely different from Szëgo's proof of Theorem B.

THEOREM 3. *Let* $\{x_{j,n}^{(\alpha)}(q)\}_{j=1}^{n}$ *be the roots of* $L_n^{(\alpha)}(x; q)$. *Then*

$$\begin{aligned} 0 &< x_{1,n}^{(\alpha)}(q) < x_{1,n}^{(\alpha+1)}(q) < x_{2,n}^{(\alpha)}(q) < x_{2,n}^{(\alpha+1)}(q) < \cdots \\ &< x_{n-1,n}^{(\alpha)}(q) < x_{n-1,n}^{(\alpha+1)}(q) < x_{n,n}^{(\alpha)}(q) < x_{n,n}^{(\alpha+1)}(q). \end{aligned} \tag{5.4}$$

Proof. Divide (4.11) by $(x-y)$ and let $y \to x$. Then we get

$$\begin{aligned} &\sum_{k=0}^{n} \frac{q^k (q;q)_k}{(q^{\alpha+1};q)_k} (L_k^{(\alpha)}(x;q))^2 \\ &= \frac{(q;q)_n}{(q^{\alpha+1};q)_n} \\ &\times \left[\frac{d}{dx} (x L_n^{(\alpha+1)}(x;q))\, L_n^{(\alpha)}(x;q) - x L_n^{(\alpha+1)}(x;q) \frac{d}{dx} L_\alpha^{(\alpha)}(x;q) \right]. \end{aligned} \tag{5.5}$$

Note that the left side is strictly positive. Now $(d/dx)(xL_n^{(+1)}(x; q))$ has opposite signs at $x = 0$ and $x = x_{1,n}^{(\alpha+1)}(q)$, which are two consecutive roots of $xL_n^{(\alpha+1)}(x; q)$. Hence $L_n^{(\alpha)}(0, q)$ and $L_n^{(\alpha)}(x_{1,n}^{(\alpha+1)}(q); q)$ have opposite signs, so $L_n^{(\alpha)}(x; q)$ has a root in between, i.e., $0 < x_{1,n}^{(\alpha)}(q) < x_{1,n}^{(\alpha+1)}(q)$. Using this same argument, we can show that a root of $L_n^{(\alpha)}(x; q)$ must lie between two consecutive roots of $L_n^{(\alpha+1)}(x; q)$ and vice versa. Relation (5.4) clearly follows. ∎

6. THE q-DIFFERENCE EQUATIONS SATISFIED BY $L_n^{(\alpha)}(x; q)$

For the classical orthogonal polynomials such as the Laguerre polynomials, one often derives results about the zeros from the appropriate differential equations using a comparison theorem. There is no convenient differential equation for $L_n^{(\alpha)}(x; q)$. The q-difference operator $\nabla_q f(x) = [f(qx) - f(x)]/[x(q-1)]$ turns out to be a good analogue of the derivative operator for q-Laguerre polynomials. Note also that $\lim_{q \to 1} \nabla_q f(x) = f'(x)$ assuming that $f'(x)$ exists. Now we state a number of q-difference rules.

$$\nabla_q(f(x)+g(x)) = \nabla_q f(x) + \nabla_q g(x), \tag{6.1}$$

$$\nabla_q(kf(x)) = k(\nabla_q f)(x), \qquad k \text{ a constant}, \tag{6.2}$$

$$\nabla_q(f(x)\,g(x)) = f(qx)(\nabla_q g)(x) \qquad (q\text{-product rule}), \tag{6.3}$$

$$\nabla_q\left(\frac{f(x)}{g(x)}\right) = \frac{((\nabla_q f)(x))\cdot g(x) - ((\nabla_q g)(x))\cdot f(x)}{g(x)\,g(qx)},$$

$$g(x)\cdot g(qx) \neq 0 \ (q\text{-quotient rule}), \tag{6.4}$$

$$\nabla_q(f(ax)) = a(\nabla_q f)(ax), \tag{6.5}$$

$$(\nabla_q^n)(f(x)\cdot g(x)) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (\nabla_q^{n-k} f)(q^k x)(\nabla_q^k g)(x), \tag{6.6}$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = (q;q)_n/(q;q)_k(q;q)_{n-k}$ is the Gaussian binomial coefficient (q-Leibnitz rule).

$$f(aqx) = f(ax) + ax(q-1)(\nabla_q f)(ax), \tag{6.7}$$

$$(\nabla_q^n f)(x) = \frac{(q;q)_n}{(1-q)^n}\,[x, xq, xq^2, ..., xq^n; f]$$

$$= \frac{(q;q)_n}{(1-q)^n n!}\, f^{(n)}(\xi). \tag{6.8}$$

Here the square bracket quantity denotes the nth order divided difference of f on the points $x, xq, ..., xq^n$; ξ lies between x and xq^n.

$$(\nabla_q^n f)(x) = \frac{1}{x^n(q-1)^n q^{\binom{n}{2}}} \sum_{j=0}^{n} (-)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q f(q^{n-j}x), \tag{6.9}$$

$$(\nabla_p^n f)(x) = p^{\binom{n}{2}}(\nabla_q^n f)(p^n x), \qquad p = q^{-1}, \tag{6.10}$$

$$(\nabla_{(q^n)} f)(x) = \frac{\left(\begin{array}{l} q^{n-1}(\nabla_q f)(q^{n-1}x) + q^{n-2}(\nabla_q f)(q^{n-2}x) \\ \qquad + \cdots + q(\nabla_q f)(qx) + (\nabla_q f)(x) \end{array}\right)}{q^{n-1} + q^{n-2} + \cdots + q + 1}. \tag{6.11}$$

It is straightforward to verify these identities directly from the definition. Exton [4] derived a q-difference equation for q-Laguerre polynomials which in our notation is,

$$x\nabla_q^2 L_n^{(\alpha)}(x;q) + \left[\frac{(1-q^{\alpha+1})}{(1-q)} - q^{\alpha+2}x\right] \nabla_q L_n^{(\alpha)}(qx;q)$$

$$+ \frac{(1-q^n)}{(1-q)}\, q^{\alpha+1} L_n^{(\alpha)}(qx;q) = 0 \tag{6.12}$$

Note that as $q \to 1^-$, (6.12) becomes

$$xL_n^{(\alpha)\prime\prime}(x) + (\alpha + 1 - x)L_n^{(\alpha)\prime}(x) + nL_n^{(\alpha)}(x) = 0, \tag{6.13}$$

which is the differential equation for the ordinary Laguerre polynomials.

We can rewrite (6.12) as a q-difference equation without a ∇_q term. The technique is similar but a bit more complicated than that of ordinary differential equations. Let $k(x)$ be a continuous function on $(0, \infty)$. For brevity, we let $L_n(x) = L_n^{(\alpha)}(x; q)$, then for $x > 0$,

$$\begin{aligned}\nabla_q^2(k(x)\,L_n(x)) = (\nabla_q^2 L_n(x))\,k(x) + (q+1)(\nabla_q L_n)(qx)(\nabla_q k)(x)\\ + L_n(q^2x)(\nabla_q^2 k)(x)\end{aligned}$$

using (6.6). By (6.7)

$$\begin{aligned}\nabla_q^2(k(x)L_n(x)) = (\nabla_q^2 L_n)(x)\,k(x) + [(\nabla_q k)(x) + q(\nabla_q k)(qs)](\nabla_q L_n)(qx)\\ + L_n(qx)(\nabla_q^2 k)(x).\end{aligned}$$

Using (6.12) and doing some algebra, we get

$$\begin{aligned}&\nabla_q^2(k(x)\,L_n(x))\\ &\quad= \left[\frac{k(q^2x) - k(x)}{x(q-1)} + \left(q^{\alpha+2} - \frac{(1-q^{\alpha+1})}{(1-q)x}\right)k(x)\right](\nabla_q L_n)(qx)\\ &\qquad+ \left[(\nabla_q^2 k)(x) - \frac{(1-q^n)}{(1-q)x}\,q^{\alpha+1}k(x)\right]L_n(qx).\end{aligned} \tag{6.14}$$

To make the $\nabla_q L_n(qx)$ coefficient vanish, we set

$$k(q^2x) - k(x) = (x(q-1)q^{\alpha+2} + q^{\alpha+1} - 1)\,k(x)$$

or

$$k(x) = \frac{1}{q^{\alpha+1}(1 + x(q-1)q)}\,k(q^2x).$$

One solution of the functional equation is

$$k(x) = \frac{x^{(\alpha+1)/2}}{(-xq(1-q); q^2)_\infty}.$$

Now let

$$k(x)=\frac{x^{(\alpha+1)/2}}{(-xq(1-q);q^2)_\infty},$$

and

$$u(x)=\frac{x^{(\alpha+1)/2}}{(-xq(1-q);q^2)_\infty}L_n^{(\alpha)}(x;q).$$

Substituting for $k(x)$ and simplifying in (6.14), we get

$$(\nabla_q^2 u)(x) + \left[\frac{(1+q)}{x^2(1-q)^2q} - \frac{(1+q^\alpha+xq^{n+\alpha+1}(1-q))(-xq^2(1-q);q^2)_\infty}{x^2(1-q)^2q^{(\alpha+1)/2}(-xq(1-q);q^2)_\infty}\right]u(qx)=0. \qquad (6.15)$$

We will use (6.15) to derive some results about the roots of $L_n^{(\alpha)}(x;q)$.

DEFINITION. Let $u(x)\in C(0,\infty)$. Then we say that the roots of $u(x)$ are *well separated* if $u(c)=u(d)=0$ and $0<c<d$ implies that $d/c>q^{-1}$. Moreover, if $u(c)=u(d)=0$ and $0<c<d$ implies that $d/c>q^{-2}$, then we say that the roots are *very well separated.*

We have already shown in the corollary to Theorem 4 that the roots of $L_n^{(\alpha)}(x;q)$ are well separated. But now we have an even stronger result which is:

THEOREM 4. *The roots of $L_n^{(\alpha)}(x;q)$ are very well separated.*

Proof. Equation (6.15) can be rewritten as

$$u(q^2x)+u(x)=\left[\frac{(xq^{n+\alpha+1}(1-q)+q^\alpha+1)(-xq^2(1-q);q^2)_\infty}{q^{(\alpha-1)/2}(-xq(1-q);q^2)_\infty}\right]u(qx). \quad (6.16)$$

Let c and d be two consecutive roots of $u(x)$, $0<c<d$, and let $p=q^{-1}$. Since the roots of $L_n^{(\alpha)}(x;q)$ are well separated, the roots of $u(x)=x^{(\alpha+1)/2}(1/(-x(1-q);q^2)_\infty)L_n^{(\alpha)}(x;q)$ are also well separated, so $pc<d$. Now $u(x)$ has one sign in the interval (c,d), and for simplicity in the argument, we assume that $u(x)>0$ on (c,d), the argument for $u(x)<0$ being analogous. See Fig. 1. Let $x=pc$ in (6.16) and assume that $c<pc<d\leqslant p^2c$. Since the roots of u are well separated, no root of $u(x)$ lies in the interval $(d,p^2c]$. $u'(d)<0$, so $u(x)<0$ in $(d,p^2c]$. But then $u(c)=0$,

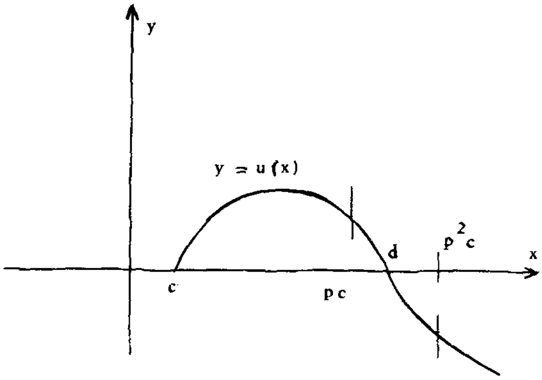

FIGURE 1

$u(pc) > 0$, and $u(p^2c) < 0$. So with $x = p^2c$ the square bracket quantity in (6.16) is positive, and the right side of (6.15) is positive while the left side is negative. This is a contradiction. Hence $d > p^2c$, and since c and d were arbitrary positive roots of $u(x)$, the roots of u are very well separated and the proof is complete. ∎

7. The Limiting Functions and the Extreme Measures

When a Stieltjes moment problem is indeterminate the associated orthogonal polynomials then converge to an entire function, the roots of which are the mass points of an extreme measure. See Shohat and Tamarkin [10, pp. 44–60]. We will show directly that the q-Laguerre polynomials converge to $[x(1-q)]^{-\alpha/2} J_\alpha(2\sqrt{x/(1-q)}; q)$, where $J_\alpha(x;q)$ is a q analogue of a Bessel function. The roots of the polynomials also converge as $n \to \infty$, which is clear from (5.1).

THEOREM 5. *Let*

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^{\binom{k}{2}} (q^{n+\alpha+1}x)^k (1-q)^k}{(q^{\alpha+1};q)_k (q;q)_k}. \tag{7.1}$$

Then $L_n^{(\alpha)}(x;q)$ converges uniformly in any bounded domain to the entire function

$$L_\infty^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+\alpha k}(1-q)^k(-x)^k}{(q^{\alpha+1};q)_k (q;q)_k}. \tag{7.2}$$

Proof. It suffices to show that

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^{n+\alpha+1}x)^k q^{\binom{k}{2}}(1-q)^k}{(q^{\alpha+1};q)_k(q;q)_k} \tag{7.3}$$

converges on bounded domains to

$$\sum_{k=0}^{\infty} \frac{q^{k^2+\alpha k}(1-q)^k(-x)^k}{(q^{\alpha+1};q)_k(q;q)_k}. \tag{7.4}$$

In fact

$$\begin{aligned}
&\left| \sum_{k=0}^{\infty} \frac{q^{k^2+\alpha k}(1-q)^k(-x)^k}{(q^{\alpha+1};q)_k(q;q)_k} \right. \\
&\qquad \left. - \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^{n+\alpha+1})^k q^{\binom{k}{2}}(1-q)^k x^k}{(q^{\alpha+1};q)_k(q;q)_k} \right| \\
&= \left| \sum_{k=0}^{n} [q^{k^2} - (q^{-n}-1)(q^{1-n}-1)\cdots(q^{k-n-1}_{-1})\, q^{nk+k+\binom{k}{2}}] \right. \\
&\qquad \left. \times \frac{q^{\alpha k}(1-q)^k(-x)^k}{(q^{\alpha+1};q)_k(q;q)_k} \right| \\
&\leqslant \sum_{k=0}^{\infty} |q^{k^2} - (1-q^n)(1-q^{n-1})\cdots(1-q^{1+n-k})\, q^{k^2}| \\
&\qquad \times \frac{q^{\alpha k}(1-q)^k\, |x|^k}{(q^{\alpha+1};q)_k(q;q)_k}.
\end{aligned} \tag{7.5}$$

The inequality $\prod_{k=1}^{n}(1-x_k) \geqslant 1-\sum_{k=1}^{n} x_k$, $0 \leqslant x_k \leqslant 1$, $k=1,\ldots,n$, can easily be proved by induction on n. Hence

$$\begin{aligned}
&|q^{k^2} - (1-q^n)(1-q^{n-1})\cdots(1-q^{1+n-k})q^{k^2}| \\
&\quad \leqslant q^{k^2} \left| 1 - \left(1 - \sum_{j=1}^{k} q^{1+n-j}\right) \right| \\
&\quad = q^{k^2+n}\frac{(1-q^{-k})}{(1-q^{-1})} = q^{k^2-k+n+1}\frac{(1-q^k)}{(1-q)} \leqslant \frac{q^{k^2-k+n+1}}{(1-q)}, \qquad k \leqslant n.
\end{aligned}$$

Therefore

$$|q^{k^2} - (1-q^n)(1-q^{n-1})\cdots(1-q^{1+n-k})q^{k^2}| \leqslant \frac{q^{k^2-k+n+1}}{1-q}. \tag{7.6}$$

Note that (7.6) is valid for all nonnegative values of k. Substituting this estimate back into (7.5), we get

$$\left| \sum_{k=0}^{\infty} \frac{q^{k^2+\alpha k}(1-q)^k(-x)^k}{(q^{\alpha+1};q)_k(q;q)_k} - \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^{n+\alpha+1})^k q^{\binom{k}{2}}(1-q)^k x^k}{(q^{\alpha+1};q)_k(q;q)_k} \right| \leqslant \frac{q^{n+1}}{(1-q)} \sum_{k=0}^{\infty} \frac{q^{k^2+\alpha k-k}(1-q)^k|x|^k}{(q^{\alpha+1};q)_k(q;q)_k}.$$

The last series has an infinite radius of convergence and is therefore bounded in every bounded domain. It follows that the first expression in (7.5) $\to 0$ as $n \to \infty$ uniformly in bounded domains. Hence the series (7.3) converges to (7.4) uniformly on bounded domains and the theorem clearly follows. ∎

$L_\infty^{(\alpha)}(x;q)$ is closely related to the q-Bessel function

$$J_\alpha(x;q) = \sum_{k=0}^{\infty} \frac{(-)^k q^{k^2+\alpha k}(x/2)^{2k+\alpha}}{\Gamma_q(k+\alpha+1)\Gamma_q(k+1)}. \tag{7.7}$$

In fact

$$L_\infty^{(\alpha)}(x;q) = [x(1-q)]^{-\alpha/2} J_\alpha\left(2\sqrt{\frac{x}{1-q}};q\right). \tag{7.8}$$

Ismail [8], has derived a number of results about $J_\alpha(x;q)$ and its roots. Some of these results can be obtained from the corresponding results for $L_n^{(\alpha)}(x;q)$, by letting $n \to \infty$. It is not hard to show that for each j, $\lim_{n\to\infty} x_{j,n}^{(\alpha)}(q) = x_{j,\infty}^{(\alpha)}(q)$ exists and is a root of $L_\infty^{(\alpha)}(x;q)$. So

$$0 < x_{1,\infty}^{(\alpha)}(q) < x_{1,\infty}^{(\alpha+1)}(q) < x_{2,\infty}^{(\alpha)}(q) < x_{2,\infty}^{(\alpha+1)}(q) < \cdots. \tag{7.9}$$

Letting $n \to \infty$ in (6.12), we see that

$$x\nabla_q^2 L_\infty^{(\alpha)}(x;q) + \left[\frac{(1-q^{\alpha+1})}{(1-q)} - q^{\alpha+2}x\right] \nabla_q L_\infty^{(\alpha)}(xq;q) + \frac{q^{\alpha+1}}{(1-q)} L_\infty^{(\alpha)}(xq;q) = 0, \tag{7.10}$$

and $x^{(\alpha+1)/2}\,(1/(-xq(1-q);q^2)_\infty)\,L_\infty^{(\alpha)}(x;q)$ satisfies

$$(\nabla_q^2 u)(x)+\left[\frac{(1+q)}{x^2(1-q)^2q}-\frac{(1+q^\alpha)(-xq^2(1-q);q^2)_\infty}{x^2(1-q)^2q^{(\alpha+1)/2}(-xq(1-q);q^2)_\infty}\right]u(qx)=0, \quad (7.11)$$

by (6.15), hence the roots of $L_\infty^{(\alpha)}(x;q)$ are very well separated.

Now we will begin to construct the extreme measures for the moment problem (2.14). We start by first obtaining the Stieltjes transform of the measure

$$I(z,\Psi)=\int\frac{d\Psi(t)}{z-t}. \quad (7.12)$$

The role played by $I(z,\Psi)$ in solving the moment problem is suggested by the following theorem.

THEOREM D. *If $\Psi(t)$ is any solution of the moment problem*

$$\int t^n\,d\Psi(t)=\mu_n, \qquad \mu_n>0, \quad n=0,1,2,..., \quad (7.13)$$

then $I(z;\Psi)$ is analytic in $\mathrm{Im}(z)>0$. $\mathrm{Im}(I(z;\Psi))\leqslant 0$, *and is asymptotically represented by the series $\sum_{n=0}^\infty \mu_n z^{-n-1}$ in any sector $\varepsilon\leqslant\arg z\leqslant\pi-\varepsilon$, $0<\varepsilon<\pi/2$.*

Conversely, if $f(z)$ is analytic in $\mathrm{Im}(z)>0$, $\mathrm{Im}(f(z))\leqslant 0$ in $\mathrm{Im}(z)>0$, and if $f(z)$ is asymptotically represented by the series $\sum_{n=0}^\infty \mu_n z^{-n-1}$ in any sector $\varepsilon\leqslant\arg z\leqslant\pi-\varepsilon$, $0<\varepsilon<\pi/2$, then there exists a unique solution $\Psi(t)$ of the moment problem, such that $f(z)=I(z;\Psi)$.

For a proof, see Shohat and Tamarkin [10, pp. 27–29].

Now we will try to construct a function $f(z)$ satisfying the above properties. First we consider the three-term recurrence relation.

$$xp_n(x)=A_np_{n+1}(x)+B_np_n(x)+C_np_{n-1}(x), \qquad n\geqslant 0, \quad C_0=-\mu_0, \quad (7.14)$$

$$p_0=1, \qquad p_{-1}=0, \quad (7.15)$$

for the polynomials $\{p_n(x)\}_{n=0}^\infty$ orthogonal with respect to a solution $\Psi(x)$ of the moment problem. Note that the $\{p_n(x)\}_{n=0}^\infty$ and the equation (7.14) depend only on the moments $\{\mu_n\}_{n=0}^\infty$ of the measure $d\Psi(x)$. Next we let the sequence of polynomials $\{q_n(x)\}_{n=0}^{\infty}$ satisfy (7.14) with the initial conditions $q_{-1}=1$, $q_0=0$. It follows easily by induction on n that

$$q_n(z)=\int_{-\infty}^{\infty}\frac{p_n(z)-p_n(t)}{z-t}\,d\Psi(t), \qquad n=0,1,2. \tag{7.16}$$

Consequently, if $p_n(z)=\sum_{k=0}^{n} a_k z^k$, then

$$\begin{aligned} q_n(z) &= (a_n\mu_0)z^{n-1}+(a_n\mu_1+a_{n-1}\mu_0)z^{n-2} \\ &\quad +\cdots+(a_n\mu_{n-1}+a_{n-1}\mu_{n-2}+\cdots+a_1\mu_0). \end{aligned} \tag{7.17}$$

$q_n(z)$ is called the numerator polynomial associated with $p_n(z)$.

LEMMA. *The continued fraction*

$$f_n(z,\tau)=\frac{-\left(\frac{C_0}{A_0}\right)\big|}{\big|\left(\frac{z-B_0}{A_0}\right)}-\frac{\left(\frac{C_1}{A_1}\right)\big|}{\big|\left(\frac{z-B_1}{A_1}\right)}-\frac{\left(\frac{C_2}{A_2}\right)\big|}{\big|\left(\frac{z-B_2}{A_2}\right)}-\cdots-\frac{\left(\frac{C_n}{A_n}\right)\big|}{\big|\left(\frac{z-B_n}{A_n}\right)-\sigma}$$

$$=\frac{q_{n+1}(z)-\sigma q_n(z)}{p_{n+1}(z)-\sigma p_n(z)}$$

converges as $n\to\infty$ for each real value of σ to $I(z,\Psi_\sigma)$, where Ψ_σ is a solution of the moment problem.

For a proof, see Shohat and Tamarkin [10, pp. 46–51].

Now for the moment problem (2.14), $p_n(z)=L_n^{(\alpha)}(z;q)$. We denote the associated numerator polynomial as $V_n^{(\alpha)}(z;q)$. Define

$$\begin{aligned}
A_{n+1}(z)&=\frac{-(q^2;q)_n}{(q^{\alpha+1};q)_n q^{n+\alpha+1}}\begin{vmatrix} V_{n+1}^{(\alpha)}(z;q) & V_{n+1}^{(\alpha)}(0;q)\\ V_n^{(\alpha)}(z;q) & V_n^{(\alpha)}(0;q)\end{vmatrix},\\
B_{n+1}(z)&=\frac{-(q^2;q)_n}{(q^{\alpha+1};q)_n q^{n+\alpha+1}}\begin{vmatrix} L_{n+1}^{(\alpha)}(z;q) & V_{n+1}^{(\alpha)}(0;q)\\ L_n^{(\alpha)}(z;q) & V_n^{(\alpha)}(0;q)\end{vmatrix},\\
C_{n+1}(z)&=\frac{-(q^2;q)_n}{(q^{\alpha+1};q)_n q^{n+\alpha+1}}\begin{vmatrix} V_{n+1}^{(\alpha)}(z;q) & L_{n+1}^{(\alpha)}(0;q)\\ V_n^{(\alpha)}(z;q) & L_n^{(\alpha)}(0;q)\end{vmatrix},\\
D_{n+1}(z)&=\frac{-(q^2;q)_n}{(q^{\alpha+1};q)_n q^{n+\alpha+1}}\begin{vmatrix} L_{n+1}^{(\alpha)}(z;q) & L_{n+1}^{(\alpha)}(0;q)\\ L_n^{(\alpha)}(z;q) & L_n^{(\alpha)}(0;q)\end{vmatrix},\\
\rho_n(z)&=\left(\sum_{k=0}^{n}\frac{q^n(q;q)_n}{(q^{\alpha+1};q)_n}\,|L_n^{(\alpha)}(z;q)|^2\right)^{-1}.
\end{aligned} \tag{7.18}$$

Using the three-term recurrence relation (3.2) but with $C_0 = -\mu_0 = -1$, we get

$$\begin{aligned} A_{n+1}(z) &= z \sum_{j=0}^{n} \frac{q^j (q;q)_j}{(q^{\alpha+1};q)_j} V_j^{(\alpha)}(z;q)\, V_j^{(\alpha)}(0;q), \\ B_{n+1}(z) &= -1 + z \sum_{j=0}^{n} \frac{q^j (q;q)_j}{(q^{\alpha+1};q)_j} L_j^{(\alpha)}(z;q)\, V_j^{(\alpha)}(0;q), \\ C_{n+1}(z) &= 1 + z \sum_{j=0}^{n} \frac{q^j (q;q)_j}{(q^{\alpha+1};q)_j} V_j^{(\alpha)}(z;q)\, L_j^{(\alpha)}(0;q), \\ D_{n+1}(z) &= z \sum_{j=0}^{n} \frac{q^j (q;q)_j}{(q^{\alpha+1};q)_j} L_j^{(\alpha)}(z;q)\, L_j^{(\alpha)}(0;q). \end{aligned} \tag{7.19}$$

DEFINITION. Let $d\Psi_1(x)$ and $d\Psi_2(x)$ be two measures on the real line. Then $d\Psi_1$ and $d\Psi_2$ are substantially equal if $\int f(x)\, d\Psi_1(x) = \int f(x)\, d\Psi_2(x)$ for all continuous functions f with compact support.

Let

$$I(z, \Psi) = \int_{-\infty}^{\infty} \frac{d\Psi(t)}{z-t}. \tag{7.20}$$

If $\Psi(t)$ is of bounded variation, then we have the Stieltjes inversion formula

$$\begin{aligned} &\tfrac{1}{2}[\Psi(t_1+0) + \Psi(t_1-0)] - \tfrac{1}{2}[\Psi(t_0+0) - \Psi(t_0-0)] \\ &\quad = \lim_{\epsilon\to 0} \frac{-1}{2\pi i} \int_{t_0}^{t_1} [I(t+i\varepsilon) - I(t-i\varepsilon)]\, dt. \end{aligned}$$

See Stone [11]. Thus $\Psi(t)$ is substantially uniequely determined by $I(z, \Psi)$.

With these preliminary definitions established, we are now ready to state the main theorems.

THEOREM E. *The polynomials $A_{n+1}(z)$, $B_{n+1}(z)$, $C_{n+1}(z)$, and $D_{n+1}(z)$ converge uniformly in bounded domains to entire functions $A(z)$, $B(z)$, $C(z)$, and $D(z)$. Moreover the series*

$$\frac{1}{\rho(z)} = \sum_{k=0}^{\infty} \frac{q^k (q;q)_k}{(q^{\alpha+1};q)_k} |L_k^{(\alpha)}(z;q)|^2 \tag{7.22}$$

converges uniformly in every bounded domain.

THEOREM F. *If $\Psi(t)$ is a measure which is a solution of the moment problem* (2.14) *then*

$$\int_{-\infty}^{\infty} \frac{d\Psi(t)}{z-t} = I(z;\,\Psi) = \frac{A(z)-\sigma(z)\,C(z)}{B(z)-\sigma(z)\,D(z)}, \tag{7.23}$$

where $\sigma(z)$ is analytic in the half plane, $\operatorname{Im}(z) > 0$, *satisfying* $\operatorname{Im}(\sigma(z)) \leqslant 0$ *for* $\operatorname{Im}(z) > 0$. *Moreover, $\sigma(z)$ is uniquely determined by the solution $\Psi(t)$. Conversely, for each function $\sigma(z)$ satisfying the above conditions there is substantially one solution $\Psi(t)$ of the moment problem which also satisfies* (7.23). *For each non-real value of z, the function*

$$\frac{A(z)-\sigma C(z)}{B(z)-\sigma D(z)}$$

describes the circumference of a circle $C(z)$ when the real parameter σ describes $[-\infty, \infty]$. The value of $I(z;\,\Psi)$ is always either on the boundary or in the interior of $C(z)$. To each point ζ_0 of the circumference of a circle $C(z_0)$ there corresponds a substantially unique solution of the moment problem $\Psi(t)$ such that $I(z_0;\,\Psi) = \zeta_0$. It is given by (7.23) *with $\sigma(z)$ replaced by the constant σ_0 determined from*

$$\zeta_0 = \frac{A(z_0)-\sigma_0\,C(z_0)}{B(z_0)-\sigma_0\,D(z_0)}.$$

The set of solutions $\Psi(t)$ satisfying $I(z_0;\,\Psi) = \zeta_0$, where ζ_0 is on the circumference of $C(z_0)$ does not depend on z_0. To each ζ_0 interior to the circle $C(z_0)$ there correspond continuously many solutions $\Psi(t)$ such that $I(z_0;\,\Psi) = \zeta_0$.

THEOREM G. *Let $\sigma \in [-\infty, \infty]$ and Ψ_σ be a solution of the moment problem* (2.14) *such that*

$$\frac{A(z)-\sigma C(z)}{B(z)-\sigma D(z)} = I(z;\,\Psi_\sigma).$$

Then Ψ_σ is a step function; the location of the mass points coincides with the set of zeros of the denominator $B(z)-\sigma D(z)$, and the mass concentrated at each mass point $x_j(\sigma)$ is $\rho(x_j(\sigma))$. The spectrum $\{x_j(\sigma)\}_{j=0}^{\infty}$ interlaces with the spectrum $\{x_j(\tau)\}_{j=0}^{\infty}$, $\tau \neq \sigma$. For any real value of x_0 there is a unique value σ_0 of σ, $-\infty \leqslant \sigma < \infty$, such that $x_0 \in \{x_j(\sigma)\}_{j=0}^{\infty}$. Moreover, for each j, $x_j(\sigma)$ is a continuous monotonic function of σ.

These theorems and their proofs are found in Shohat and Tamarkin [10, pp. 52–60], and Stone [11, 00. 577–614]. It follows that the measures $\{\Psi_\sigma\colon -\infty \leqslant \sigma < \infty\}$ are the extremal measures. These extreme measures have an important property stated in the following theorem.

THEOREM H. *Let $\Psi(t)$ be a solution of the moment problem* (2.14). *Then the q-Laguerre polynomials $L_n^{(\alpha)}(x; q)$ are complete in $L^2(d\Psi)$ if and only if $\Psi(t)$ is an extremal solution of the moment problem* (2.14).

For a proof see Shohat and Tamarkin [10, pp. 61, 62].

Now we will try to calculate $B(z) - \sigma D(z)$ as explicitly as we can, using the following lemma.

LEMMA. *The following relations hold.*

$$V_n^{(\alpha)}(0; q) = \frac{(1-q)q^\alpha}{(1-q^\alpha)} \left[1 - \frac{(q^{\alpha+1}; q)_n}{(q; q)_n}\right], \qquad \alpha \neq 0, \quad n \geqslant 0, \tag{7.24}$$

$$D_{n+1}(z) = zL_n^{(\alpha+1)}(z; q), \tag{7.25}$$

$$B_{n+1}(z) = -\frac{(q; q)_{n+1}}{(q^\alpha; q)_{n+1}} L_{n+1}^{(\alpha-1)}(z; q) - \frac{(1-q)q^\alpha}{(1-q^\alpha)} zL_n^{(\alpha+1)}(z; q), \qquad \alpha \neq 0. \tag{7.26}$$

Proof. We start with (7.24). The case $\alpha = 0$ has to be treated differently, because $V_n^{(\alpha)}(0; q)$ and $B_{n+1}(z)$ have singularities there. We will derive all these results for $\alpha \neq 0$ and later consider the limiting case as $\alpha \to 0$.

$$\begin{aligned} V_n^{(\alpha)}&(0; q) \\ &= \sum_{k=1}^n a_k \mu_{k-1} = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \\ &\quad \times \sum_{k=1}^n \frac{(q^{-n}; q)_k q^{\binom{k}{2}} (q^{n+\alpha+1})^k (q^{\alpha+1}; q)_{k-1} q^{-\alpha(k-1) - \binom{k}{2}}}{(q^{\alpha+1}; q)_k (q; q)_k (1-q)^{-1}} \\ &= \frac{(q^{\alpha+1}; q)_n q^\alpha (1-q)}{(q; q)_n (1-q^\alpha)} \sum_{k=1}^n \frac{(q^{-n}; q)_k (q^\alpha; q)_k}{(q^{\alpha+1}; q)_k (q; q)_k} (q^{n+1})^k. \end{aligned} \tag{7.27}$$

Now by Heine's identity (2.6),

$$\begin{aligned} V_n^{(\alpha)}(0; q) &= \frac{(q^{\alpha+1}; q)_n q^\alpha (1-q)}{(q; q)_n (1-q^\alpha)} \left(\frac{(q; q)_n}{(q^{\alpha+1}; q)_n} - 1\right) \\ &= \frac{q^\alpha (1-q)}{(1-q^\alpha)} \left(1 - \frac{(q^{\alpha+1}; q)_n}{(q; q)_n}\right). \end{aligned}$$

proving (7.24).

To prove (7.25), we use (7.19) and (4.20) to write $D_{n+1}(z)$ as

$$D_{n+1}(z) = z \sum_{j=0}^{n} q^j L_j^{(\alpha)}(z; q)$$

$$= z L_n^{(\alpha+1)}(z; q).$$

Similarly,

$$B_{n+1}(z) = -1 + \frac{(1-q)q^\alpha z}{(1-q^\alpha)} \left[\sum_{j=0}^{n} \frac{q^j (q; q)_j}{(q^{\alpha+1}; q)_j} L_j^{(\alpha)}(z, q) - q^j L_j^{(\alpha)}(z; q) \right]$$

using (7.24) and (7.19). By (4.9) and (4.10),

$$B_{n+1}(z) = -\frac{(q; q)_{n+1}}{(q^\alpha; q)_{n+1}} L_{n+1}^{(\alpha-1)}(z; q) - \frac{(1-q)q^\alpha z}{(1-q^\alpha)} L_n^{(\alpha+1)}(z; q),$$

which is (7.26). ∎

THEOREM 14. *Let $d\Psi(x)$ be a measure which is a solution of the moment problem*

$$\int_{-\infty}^{\infty} x^n \, d\Psi(x) = \mu_n = \frac{(q^{\alpha+1}; q)_n q^{-\alpha n - \binom{n+1}{2}}}{(1-q)^n},$$

where $\alpha > -1$, and $0 < q < 1$.

Then $\Psi(x)$ is an extremal solution of the above moment problem if and only if for some value of σ, $-\infty \leqslant \sigma < \infty$, $\Psi(x)$ is substantially equal to the measure $\Psi_\sigma(x)$, which has jumps of size

$$\rho(\tau_k^{(\alpha)}(\sigma; q)) = \left(\sum_{k=0}^{\infty} \frac{q^k (q; q)_k}{(q^{\alpha+1}; q)_k} (L_k^{(\alpha)}(\tau_k^{(\alpha)}(\sigma; q)))^2 \right)^{-1} \tag{7.28}$$

at the point $\tau_k^{(\alpha)}(\sigma; q)$, which is the kth root (written in increasing order) of the entire function

$$G^{(\alpha)}(z, \sigma; q) = L_\infty^{(\alpha)}(z; q) - \sigma z L_\infty^{(\alpha+1)}(z; q). \tag{7.29}$$

In particular

$$G^{(\alpha)}\left(z, \frac{(1-q)q^\alpha}{(1-q^\alpha)}\right) = \frac{1}{(1-q^\alpha)} L_\infty^{(\alpha-1)}(z; q), \qquad \alpha > -1, \quad \alpha \neq 0. \tag{7.30}$$

For each real number τ there is a unique real number σ, $-\infty \leqslant \sigma < \infty$, such that Ψ_σ has a mass point at τ, and σ is given by

$$\begin{aligned} \sigma &= \frac{L_\infty^{(\alpha)}(\tau; q)}{\tau L_\infty^{(\alpha+1)}(\tau; q)} \qquad \text{if} \quad \tau L_\infty^{(\alpha+1)}(\tau; q) \neq 0, \\ &= -\infty \qquad\qquad\qquad \text{if} \quad \tau L_\infty^{(\alpha+1)}(\tau; q) = 0. \end{aligned} \tag{7.31}$$

Moreover, as σ increases from $-\infty$ to 0, $\tau_1^{(\alpha)}(\sigma; q)$ decreases from 0 to $-\infty$, and as σ increases from $-\infty$ to ∞, $\tau_2^{(\alpha)}(\sigma; q)$ decreases from $x_1^{(\alpha+1)}(q)$ to 0, and for $k > 2$, $\tau_k^{(\alpha)}(\sigma; q)$ decreases from $x_{k-1}^{(\alpha+1)}(q)$ to $x_{k-2}^{(\alpha+1)}(q)$.

Proof. Our first task is to show that $B(z) - \sigma D(z) = C \cdot G^{(\alpha)}(x, f(\sigma); q)$, where C is a nonzero constant and $f(\sigma)$ is a continuous monotonic function of σ.

Let us assume for the moment that $\alpha \neq 0$. Then by (7.25) and (7.26),

$$B(z) - cD(z) = \frac{-(q; q)_\infty}{(q^\alpha; q)_\infty} L_\infty^{(\alpha-1)}(z; q) - \left[\frac{(1-q)q^\alpha}{(1-q^\alpha)} + \sigma\right] zL_\infty^{(\alpha+1)}(z; q). \tag{7.32}$$

Now replace n by $n+1$ in (4.12), and add that equation to $(1-q)q^\alpha$ times Eq. (4.14). We then get (replacing x by z),

$$L_{n+1}^{(\alpha-1)}(z; q) + (1-q)\, q^\alpha z L_n^{(\alpha+1)}(z; q) = L_{n+1}^{(\alpha)}(z; q) - q^\alpha L_n^{(\alpha)}(z; q). \tag{7.33}$$

By letting $n \to \infty$ we obtain

$$L_\infty^{(\alpha-1)}(z; q) = (1-q^a)\, L_\infty^{(\alpha)}(z; q) - (1-q)\, q^\alpha z L_\infty^{(\alpha+1)}(z; q). \tag{7.34}$$

Then (7.34) becomes

$$\begin{aligned} B(z) - \sigma D(z) &= \frac{-(q; q)_\infty}{(q^{\alpha+1}; q)_\infty} L_\infty^{(\alpha)}(z; q) \\ &\quad + \left[\frac{(1-q)q^\alpha}{(1-q^\alpha)}\left(\frac{(q; q)_\infty}{(q^{\alpha+1}; q)_\infty} - 1\right) - \sigma\right] zL_\infty^{(\alpha+1)}(z; q). \end{aligned} \tag{7.35}$$

Note that $B(z) - \sigma D(z)$ has a removable singularity at $\alpha = 0$ so that (7.35) remains valid for all $\alpha > -1$. We can now factor out $-(q; q)_\infty/(q^{\alpha+1}; q)_\infty$ and then reparameterize σ to obtain $G^{(\alpha)}(z, \sigma; q)$. The first part of the theorem now follows easily from Theorem 12. Using the definition of $G^{(\alpha)}(z, \sigma; q)$ and (7.34), we quickly obtain (7.30). (7.31) follows by setting $G^{(\alpha)}(\tau, \sigma; q) = 0$ and solving for σ.

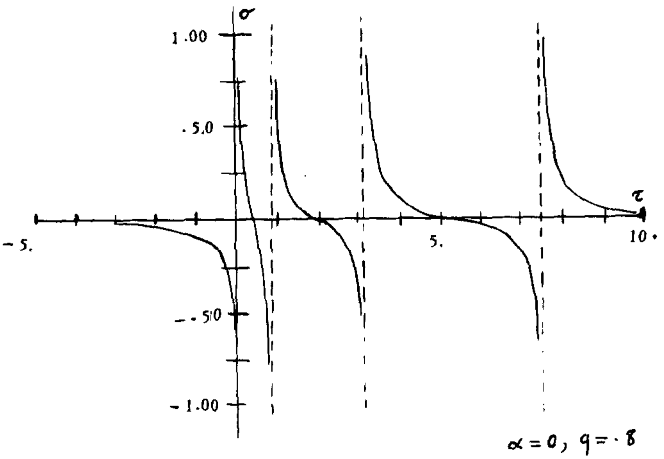

FIGURE 2

To study the behavior of the roots as σ changes, we differentiate (7.31) to get

$$\frac{\partial\sigma}{\partial\tau}=\frac{\frac{\partial}{\partial\tau}(L_{\infty}^{(\alpha)}(\tau;q))\,\tau L_{\infty}^{(\alpha+1)}(\tau;q)-L_{\infty}^{(\alpha)}(\tau;q)\,\frac{\partial}{\partial\tau}(\tau L_{\infty}^{(\alpha+1)}(\tau;q))}{(\tau L_{\infty}^{(\alpha+1)}(a;q))^2}. \qquad (7.36)$$

Letting $n\to\infty$ and $x=\tau$ in (5.5) we obtain

$$\frac{\partial\sigma}{\partial\tau}=\frac{-(q^{\alpha+1};q)_{\infty}\left[\sum_{k=0}^{\infty}\frac{q^k(q;q)_k}{(q^{\alpha+1};q)_k}(L_k^{(\alpha)}(\tau;q))^2\right]}{(q;q)_{\infty}(\tau L_{\infty}^{(\alpha+1)}(\tau;q))^2}<0. \qquad (7.37)$$

This implies that each root $\tau_k^{(\alpha)}(\sigma;q)$ of $G^{(\alpha)}(x,\tau;q)$ is a decreasing function of σ. Moreover, when we take into account Theorem D and the fact that $G^{(\alpha)}(\tau,0;q)=L_{\infty}^{(\alpha)}(\tau;q)$ has no negative roots, we see that the graph of σ vs τ must look like Fig. 2. The rest of Theorem 11 follows easily. ∎

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