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Lifting chains of prime ideals

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Abstract

We give an elementary proof that for a ring homomorphism $A \to B$ satisfying the property that every ideal in A is contracted from B the following property holds: for every chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$ in A there exists a chain of prime ideals $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_r$ in B such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$.

MSC: 13B24

Let A and B be commutative rings and let $\varphi : A \to B$ be a ring homomorphism. This induces a continuous mapping $\varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ by sending a prime ideal $\mathfrak{q} \subset B$ to $\varphi^{-1}(\mathfrak{q})$. Properties of the ring homomorphism are then often reflected by topological properties of φ^*. For example, if $A \to B$ is integral, then "going up" holds, and if $A \to B$ is flat, then "going down" holds (see [4, Proposition 4.15 and Lemma 10.11]). If moreover $\varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective and going up or going down holds, then also the following property holds: for every given chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$ in A there exists a chain of prime ideals $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_r$ in B lying over it.

In this note we give a direct and elementary proof showing that this chain lifting property holds also under the condition that every ideal in A is contracted from B, i.e. $I = \varphi^{-1}(IB)$ holds for every ideal $I \subseteq A$. This result can be found for pure homomorphisms in Picavet's paper (see [10, Proposition 60 and Theorem 37]) and is proved using valuation theory. Our direct method allows to find explicitly chains of prime ideals and characterizes which prime ideals $\mathfrak{q}_0$ over $\mathfrak{p}_0$ may be extended to a chain. We start with the following lemma.

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Lemma 1. *Let B be a commutative ring, let $\mathfrak{a}_0,\dots,\mathfrak{a}_r$ be ideals and $F_0,\dots,F_r$ multiplicatively closed systems. Define inductively (set $S_{r+1}=\{1\}$) for $i=r,\dots,0$ the following multiplicatively closed sets*

$$S_i=\{s\in B:(s,\mathfrak{a}_i)\cap F_i\cdot S_{i+1}\neq\emptyset\}.$$

Then the following are equivalent.

(i) $0\notin S_0$.
(ii) $\mathfrak{a}_i\cap F_i\cdot S_{i+1}=\emptyset$ *for* $i=0,\dots,r$.
(iii) *There exists a chain of prime ideals* $\mathfrak{q}_0\subseteq\cdots\subseteq\mathfrak{q}_r$ *such that* $\mathfrak{a}_i\subseteq\mathfrak{q}_i$ *and* $\mathfrak{q}_i\cap F_i\cdot S_{i+1}=\emptyset$.
(iv) *There exists a chain of prime ideals* $\mathfrak{q}_0\subseteq\cdots\subseteq\mathfrak{q}_r$ *such that* $\mathfrak{a}_i\subseteq\mathfrak{q}_i$ *and* $\mathfrak{q}_i\cap F_i=\emptyset$.

Proof. It is clear that the S_i are multiplicatively closed and that $S_{i+1}\subseteq S_i$. (i) $\Leftrightarrow$ (ii). If $0\in S_0$, then $\mathfrak{a}_0\cap F_0\cdot S_{i+1}\neq\emptyset$, and if $\mathfrak{a}_i\cap F_i\cdot S_{i+1}\neq\emptyset$ for some i, then $0\in S_i$ and thus also $0\in S_0$.

We show (ii) $\Rightarrow$ (iii) by induction. Since $\mathfrak{a}_0\cap F_0S_1=\emptyset$, there exists [2, Chapter 2, Section 5, Corollary 2] a prime ideal $\mathfrak{q}_0$ such that $\mathfrak{a}_0\subseteq\mathfrak{q}_0$ and $\mathfrak{q}_0\cap F_0S_1=\emptyset$.

Thus suppose that the chain $\mathfrak{q}_0\subset\cdots\subset\mathfrak{q}_i$ is already constructed. We have to look for a prime ideal $\mathfrak{q}_{i+1}$ which includes both $\mathfrak{q}_i$ and $\mathfrak{a}_{i+1}$ and which is disjoint to $F_{i+1}\cdot S_{i+2}$. If such a prime ideal would not exist, then $(\mathfrak{q}_i+\mathfrak{a}_{i+1})\cap F_{i+1}\cdot S_{i+2}\neq\emptyset$, say $q+a=f\cdot s$, where $q\in\mathfrak{q}_i$, $a\in\mathfrak{a}_{i+1}$, $f\in F_{i+1}$ and $s\in S_{i+2}$. Then by definition $q\in S_{i+1}$ contradicting the induction assumption.

(iii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (ii) are clear, so we have to show (iv) $\Rightarrow$ (iii). We show this by descending induction, the beginning for $i=r$ is clear. Suppose that $\mathfrak{q}_{i-1}\cap F_{i-1}S_i\neq\emptyset$, and let $q=fs$ be an element in the intersection, $q\in\mathfrak{q}_{i-1}$, $f\in F_{i-1}$, $s\in S_i$. Since F_{i-1} is disjoined to the prime ideal $\mathfrak{q}_{i-1}$, it follows that $s\in\mathfrak{q}_{i-1}$. On the other hand, since $s\in S_i$ we have an equation $bs+q'=f's'$, where $b\in B$, $q'\in\mathfrak{q}_i$, $f'\in F_i$, $s'\in S_{i+1}$, and this contradicts the induction hypothesis. □

Remark 2. The referee (whom I thank for his careful reading) pointed out that there exists a similar and more general result in a preprint of Bergman (see [1]). Bergman studies for a partially ordered set I and ideals $\mathfrak{a}_i$ and multiplicatively closed subsets S_i in a commutative ring the existence of prime ideals $\mathfrak{p}_i$, $\mathfrak{a}_i\subseteq\mathfrak{p}_i$, $\mathfrak{p}_i\cap S_i=\emptyset$ such that $\mathfrak{p}_i\subset\mathfrak{p}_j$ holds for $i\leqslant j$. Bergman [1, Proposition 9] gives a characterization for the existence of such prime ideals for a tree order I in terms of an inductively defined system of equations which is related to our characterization in Lemma 1(ii). It is possible that using Bergman's result one may obtain a stronger version of the following theorem.

Theorem 3. *Let A and B be commutative rings and let $\varphi:A\to B$ be a ring homomorphism such that $I=\varphi^{-1}(IB)$ holds for every ideal $I\subseteq A$. Then for every chain of prime ideals $\mathfrak{p}_0\subset\cdots\subset\mathfrak{p}_r$ in* Spec A *there exists a chain of prime ideals $\mathfrak{q}_0\subset\cdots\subset\mathfrak{q}_r$ in B such that $\mathfrak{p}_i=\mathfrak{q}_i\cap A$ for $i=0,\dots,r$.*

Proof. Let a chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$ in A be given. We shall apply the preceding lemma to the ideals $\mathfrak{a}_i = \mathfrak{p}_i B$ and the multiplicatively closed sets $F_i = A - \mathfrak{p}_i \subset B$. Note that the fiber over $\mathfrak{p}$ consists of the prime ideals $\mathfrak{q}$ for which $\mathfrak{p}B \subset \mathfrak{q}$ and $\mathfrak{q} \cap \varphi(A - \mathfrak{p}) = \emptyset$ hold. Define $S_i \subseteq B$ as before and suppose that $0 \in S_0$. This means that there exists an element $a_0 \in \mathfrak{a}_0$ such that $a_0 = f_0 \cdot s_1$, where $f_0 \in F_0$, $s_1 \in S_1$. This means by definition that we have an equation

$$b_1 s_1 + a_1 = f_1 s_2, \quad \text{where } b_1 \in B,\ a_1 \in \mathfrak{a}_1,\ f_1 \in F_1 \text{ and } s_2 \in S_2.$$

Going on recursively we find equations

$$b_j s_j + a_j = f_j s_{j+1}, \quad \text{where } b_j \in B,\ a_j \in \mathfrak{a}_j,\ f_j \in F_j \text{ and } s_{j+1} \in S_{j+1}$$

and eventually

$$b_r s_r + a_r = f_r, \quad \text{where } b_r, s_r \in S_r,\ a_r \in \mathfrak{a}_r,\ f_r \in F_r.$$

We multiply the last equation by $f_{r-1} \cdots f_0$ and get

$$b_r (s_r f_{r-1}) f_{r-2} \cdots f_0 + a_r f_{r-1} \cdots f_0 = f_r f_{r-1} \cdots f_0.$$

We may replace $b_r (s_r f_{r-1}) f_{r-2} \cdots f_0$ by

$$b_r (b_{r-1} s_{r-1} + a_{r-1}) f_{r-2} \cdots f_0 = b_r b_{r-1} (s_{r-1} f_{r-2}) \cdots f_0 + b_r a_{r-1} f_{r-2} \cdots f_0$$

and so going on we find that

$$f_r \cdots f_0 = b_r \cdots b_1 a_0 + b_r \cdots b_2 a_1 f_0 + b_r \cdots b_3 a_2 f_1 f_0 + \cdots + b_r a_{r-1} f_{r-2} \cdots f_0 + a_r f_{r-1} \cdots f_0.$$

This equation shows that

$$f_r \cdots f_0 \in (\mathfrak{p}_0 + \mathfrak{p}_1 f_0 + \mathfrak{p}_2 f_1 f_0 + \cdots + \mathfrak{p}_{r-1} f_{r-2} \cdots f_0 + \mathfrak{p}_r f_{r-1} \cdots f_0)B$$

and this yields an equation in A (here we apply the condition that every ideal is contracted),

$$p_0 + p_1 f_0 + p_2 f_1 f_0 + \cdots + p_{r-1} f_{r-2} \cdots f_0 + p_r f_{r-1} \cdots f_0 - f_r \cdots f_0 = 0,$$

where $p_i \in \mathfrak{p}_i$. We may write this as

$$p_0 = -f_0 (p_1 + p_2 f_1 + \cdots + p_{r-1} f_{r-2} \cdots f_1 + p_r f_{r-1} \cdots f_1 - f_r \cdots f_1)$$

and therefore $p_1 + p_2 f_1 + \cdots + p_{r-1} f_{r-2} \cdots f_1 + p_r f_{r-1} \cdots f_1 - f_r \cdots f_1 \in \mathfrak{p}_0 \subset \mathfrak{p}_1$. Then again we may multiply out f_1 and so on until we find $p_{r-1} + p_r f_{r-1} - f_r f_{r-1} \in \mathfrak{p}_{r-2} \subset \mathfrak{p}_{r-1}$ and then $p_r f_{r-1} - f_r f_{r-1} \in \mathfrak{p}_{r-1}$, hence $p_r - f_r \in \mathfrak{p}_{r-1}$ and $f_r \in \mathfrak{p}_r$, which is a contradiction. □

Remark 4. The condition that every ideal is contracted is fulfilled for example if $\varphi : A \to B$ is a pure homomorphism. This means that for every A-module M the natural mapping $M \to M \otimes_A B$ is injective. If B contains A as a direct summand, then $A \subseteq B$ is pure. Direct summands arise often in invariant theory: if a linearly reductive group acts on a ring B, then the ring of invariants $A = B^G$ is a direct summand in B

(see [5, Chapter 1, Section 1]). Example 7 shows that for a direct summand neither going up nor going down hold in general.

Picavet studies in [10] the property of a ring homomorphism that over every chain of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there lies a chain of prime ideals above. He calls a ring homomorphism with this property subtrusif and shows that a homomorphism $\varphi : A \to B$ is universally subtrusif if and only if for every valuation domain $A \to V$ the corresponding homomorphism $V \to B \otimes_A V$ is pure.

Picavet proved the theorem for universally subtrusive morphisms [10, Proposition 60 in connection with Theorem 37] using several facts from valuation theory: that for a chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$ in a domain A there exists a valuation ring $A \subseteq V \subseteq Q(A)$ and a chain of prime ideals $\mathfrak{r}_0 \subset \cdots \subset \mathfrak{r}_r$ in V with $\mathfrak{r}_i \cap A = \mathfrak{p}_i$, see [6, Corollary 19.7] (see also [8,3] for recent developments in the lifting of chains to valuation rings), and that a valuation domain is a Bezout domain and hence a torsion free module over it is flat (see [9, Theorem 63] and [2, Chapter 1, Section 4, Proposition 3]).

Corollary 5. *Let A and B be commutative rings and let $\varphi : A \to B$ be a ring homomorphism such that $I = \varphi^{-1}(IB)$ holds for every ideal $I \subseteq A$. Then* $\dim B \geqslant \dim A$.

Proof. This is clear from the theorem. □

Corollary 6. *Let A be a commutative Noetherian ring and let B be an A-Algebra of finite type such that every ideal of A is contracted from B. Then g:* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is submersive, i.e.* $\operatorname{Spec} A$ *carries the quotient topology.*

Proof. We have to show that a subset $W \subseteq \operatorname{Spec} A$ is open if its preimage is open. Since g is surjective, we know that $W = g(g^{-1}(W))$, hence W is constructible by [7, Théorème 7.1.4]. For the openness it is therefore enough to show that it is closed under generalization, and this follows directly from our property: let $\mathfrak{p}' \in W$ and let $\mathfrak{p} \subset \mathfrak{p}'$ be a generalization. Let $\mathfrak{q} \subset \mathfrak{q}'$ be prime ideals lying over them. Then $\mathfrak{q}' \in g^{-1}(W)$ and since $g^{-1}(W)$ is open it is closed under generalization, hence $\mathfrak{q} \in g^{-1}(W)$, and this means $\mathfrak{p} \in W$. □

It is easy to give an example of a direct summand such that $\operatorname{Spec} B \to \operatorname{Spec} A$ fulfills neither the going down nor the going up property.

Example 7. Let K be a field and let the polynomial ring $B = K[X, Y, Z]$ be $\mathbb{Z}$-graded by $\deg X = \deg Y = 1$, $\deg Z = -1$. Then the ring of degree zero is

$$A = B_0 = K[XZ, YZ] \cong K[U, V].$$

A is a direct summand in B, hence the chain lifting property holds.

We consider the chain $(XZ) \subset (XZ, YZ)$ in A. The principal prime ideal ZB maps to (XZ, YZ), but no prime ideal $\subset ZB$ maps to (XZ), hence going down does not hold.

The prime ideal $(X, Y^2Z - 1)B$ maps to (XY). But a prime ideal lying over (XZ, YZ) must contain either ZB or $(X, Y)B$, hence also going up fails to hold.

Remark 8. A surjective (even bijective) mapping between affine varieties may not fulfill the chain lifting property, since there exist bijective mappings which are not homeomorphisms.

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