Finite Normalizing Extensions

AHMAD SHAMSUDDIN

Department of Mathematics, American University of Beirut,
Beirut, Lebanon

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1. INTRODUCTION

Let $S$ be a ring and let $R$ be a subring of $S$ (with the same 1). We say that $S$ is a finite normalizing extension of $R$ if there exist elements $a_1, \ldots, a_n \in S$ such that $S = \sum_{i=1}^{n} Ra_i$, where $Ra_i = a_i R$ for $i = 1, \ldots, n$. The reader is referred to [2, Chap. 10] for a detailed study of such extensions. The purpose of this note is first, to supply a simple proof of properness, namely, that if $N$ is a proper left ideal of $R$ then $SN$ is a proper left ideal of $S$—see [2, 10.2.14], and second, to bound the global dimension of $R$ by that of $S$ in case $R$ has finite global dimension.

2. DESCENT OF FLATNESS

Let $f: R \to S$ be a ring homomorphism such that $f(1) = 1$. Raynaud and Gruson in [3] consider the following conditions:

1. If $E \otimes_R S$ is a flat right $S$-module then $E_R$ is a flat right $R$-module.
2. If $S \otimes_R E$ is a flat left $S$-module then $E_R$ is a flat left $R$-module.
3. If $S \otimes_R E = 0$ then $E_R = 0$.
4. If $S \otimes_R E = 0$ then $E_R = 0$.
5. If $\text{Hom}_R(S, E) = 0$ then $E_R = 0$.
6. If $\text{Hom}_R(S, E) = 0$ then $E_R = 0$.

Conditions (P$_f$) and (P$_r$) are called “Descent of flatness.” We shall show that all these conditions are always verified when $S$ is a finite normalizing extension of $R$.

Proposition 2.1. Let $S$ be a finite normalizing extension of $R$. Then all the conditions (P$_f$)–(O'$_r$) are satisfied.
Proof. We shall prove conditions (P), (O), and (O') only, since the proofs of the remaining conditions are entirely similar.

(P) Suppose that \( E \otimes_R S \) is a flat \( S \)-module. Then the left \( S \)-module \( \text{Hom}_Z(E \otimes_R S, \mathbb{Q}/\mathbb{Z}) \) is injective (see, e.g., [1, Lemma 19.14]). However, there is an isomorphism

\[
\text{Hom}_Z(E \otimes_R S, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}_R(S, \text{Hom}_Z(E, \mathbb{Q}/\mathbb{Z}))
\]

of left \( S \)-modules. It follows from [4, Corollary 2] that the left \( R \)-module \( \text{Hom}_Z(E, \mathbb{Q}/\mathbb{Z}) \) is injective. This proves that \( E_R \) is flat, as desired.

(O') This is just Corollary 4(ii) of [4].

(O) We have

\[
0 = \text{Hom}_Z(E \otimes_R S, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}_R(S, \text{Hom}_Z(E, \mathbb{Q}/\mathbb{Z}))
\]

and so, by (O'), \( \text{Hom}_Z(E, \mathbb{Q}/\mathbb{Z}) = 0 \). It now follows that \( E = 0 \).

An immediate corollary to (O,) is the following simple proof of "properness":

PROPOSITION 2.2. If \( N \) is a proper left ideal of \( R \) then \( SN \) is a proper left ideal of \( S \).

Proof. Suppose that \( S = SN \). Then \( S \otimes_R R/N \cong S/SN = 0 \). It follows from (O,) that \( R/N = 0 \), i.e., \( N = R \).

3. DESCENT OF PROJECTIVITY

A module \( _RM \) is said to be a Mittag-Leffler (ML) module if for every family \( \{ N_i : i \in I \} \) of right \( R \)-modules, the canonical map

\[
\left( \prod_{i \in I} N_i \right) \otimes_R M \rightarrow \prod_{i \in I} (N_i \otimes_R M)
\]

is a monomorphism. Raynaud and Gruson [3] proved the following results concerning descent of ML and projectivity. Note that the results to be quoted are proved under the blanket assumption that the rings considered in the sections in which these results appear are all commutative. However, a close inspection of the proofs reveals that no commutativity assumption is actually needed to prove these particular results.

PROPOSITION 3.1. Let \( f : R \rightarrow S \) be a ring homomorphism for which condition (O,) is satisfied. If \( _RM \) is a flat \( R \)-module for which \( S \otimes_R M \) is a Mittag-Leffler left \( S \)-module then \( _RM \) is a Mittag-Leffler left \( R \)-module.

Proof. This is Proposition 2.5.2 of [3].
Theorem 3.2. Let \( f: R \to S \) be a ring homomorphism for which the property that whenever \( _R \mathcal{M} \) is a flat \( R \)-module for which \( S \otimes_R \mathcal{M} \) is a Mittag-Leffler left \( S \)-module then \( _R \mathcal{M} \) is a Mittag-Leffler left \( R \)-module holds. Then whenever \( _R \mathcal{M} \) is a flat \( R \)-module for which \( S \otimes_R \mathcal{M} \) is a projective left \( S \)-module, \( _R \mathcal{M} \) is a projective left \( R \)-module.

\textit{Proof.} This is Proposition 3.13 of [3].

We can now apply the above to finite normalizing extensions as follows.

Corollary 3.3. Suppose that \( S \) is a finite normalizing extension of \( R \). If \( S \otimes_R \mathcal{M} \) is a projective \( S \)-module then \( _R \mathcal{M} \) is a projective \( R \)-module.

\textit{Proof.} If \( S \otimes_R \mathcal{M} \) is projective then it is flat and so by \((P_r), \) \( _R \mathcal{M} \) is flat. Because \((Q_r)\) is verified in this case, Proposition 3.1 and Theorem 3.2 together imply that \( _R \mathcal{M} \) is projective.

It is possible to generalize descent of flatness and projectivity to flat and projective dimensions. We use \( \text{fd} \ _R \mathcal{M} \) (resp. \( \text{pd} \ _R \mathcal{M} \)) to designate the flat dimension (resp. projective dimension) of \( _R \mathcal{M} \). Also, \( \text{lgldim} \ R \) (resp. \( \text{lgldim} \ R \)) designates the left weak global dimension (resp. the left global dimension) of \( R \).

Corollary 3.4. Suppose that \( S \) is a finite normalizing extension of \( R \) for which \( S_R \) is flat.

1. If \( \text{fd} \ _R \mathcal{M} < \infty \) then \( \text{fd} \ _S (S \otimes_R \mathcal{M}) = \text{fd} \ _R \mathcal{M} \).
2. If \( \text{pd} \ _R \mathcal{M} < \infty \) then \( \text{pd} \ _S (S \otimes_R \mathcal{M}) = \text{pd} \ _R \mathcal{M} \).
3. If \( \text{lgldim} \ R < \infty \) then \( \text{lgldim} \ R \leq \text{lgldim} \ S \).
4. If \( \text{lgldim} \ R < \infty \) then \( \text{lgldim} \ R \leq \text{lgldim} \ S \).

\textit{Proof.} The proof uses standard techniques of induction on \( \text{fd} \ _R \mathcal{M} \) and \( \text{pd} \ _R \mathcal{M} \), together with descent of flatness and projectivity. Statements (3) and (4) are immediate consequences of (1) and (2).

References