Recursively Unsolvable Word Problems of Modular Lattices and Diagram-Chasing

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The unsolvability of the word problem for modular lattices is demonstrated, using the known unsolvability of the word problem for semigroups. More generally, suppose that \( A \) is a nontrivial ring, \( N \) is a denumerably infinite set, and \( \Gamma(A^n) \) is the lattice of submodules of the left \( A \)-module of all functions \( N \to A \). Then a lattice presentation with a finite number of generators and relations is constructed that has a recursively unsolvable word problem in any quasivariety of modular lattices that contains \( \Gamma(A^n) \).

Given a finite commutative abelian category diagram with specified exactness conditions, it may be that other exactness conditions are implied. For example, consider the five-lemma. It is proved that no algorithm can compute every exactness implication in every finite commutative diagram with specified exactness conditions. The result is obtained essentially by expressing an unsolvable modular lattice word problem as a family of diagram-chasing problems.

I. INTRODUCTION

There has been considerable study of the word problem for the most important lattice varieties (see the survey article [12]). Word problems for modular lattices have proved less tractable than the corresponding problems for all lattices or for distributive lattices. From our main result, we see that the word problem is unsolvable for the variety of modular lattices and for the related variety of Arguesian lattices considered by B. Jónsson [6]. The word problem for free modular lattices is still an open problem.

Since the word problem for semigroups is unsolvable, it is easily seen that commutativity relations are not always recursively decidable for a finitely presented abelian category diagram. The deeper result is that exactness implications are also not computable by a general method, even in finite diagrams that commute everywhere.
2. AN UNSOLVABLE WORD PROBLEM FOR MODULAR LATTICES

Following the notations of universal algebra [2], we shall be concerned with two kinds of algebra. A "groupoid" is an algebra with one binary operation \( \times \). A "lattice algebra" is an algebra with two binary operations \( \vee \) and \( \wedge \). We shall write groupoid monomials and lattice polynomials as usual, with the understanding that the parenthesis-free notation of [2, pp. 116-117] is actually used.

Let \( \mathcal{V} \) be the variety of semigroups (associative groupoids). A "modular lattice quasivariety" \( \mathcal{M} \) is the class (or category) of all lattice algebras satisfying all the formulas of some set \( \Sigma \), where:

(1) Every member of \( \Sigma \) is a lattice Horn formula, of form

\[
(e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1} \wedge e_n) = e_{n+1} \cdots e_n,
\]

for lattice polynomials \( e_1, e_2, \ldots, e_n \) in some denumerable set of variables. (Identities \( e_1 = e_2 \) are permitted also.)

(2) The modular lattice axioms (commutativity, associativity, absorption, and modularity laws) are contained in \( \Sigma \) (or are consequences of formulas in \( \Sigma \)).

Given a quasivariety \( \mathcal{V} \) of either type, let \( W \) be the word algebra of monomials (polynomials) on some set of generators \( X \). If \( \Xi \) is a set of relations (a subset of \( W \times W \)), then \( V = \mathcal{V}(X \mid \Xi) \) is the algebra of \( \mathcal{V} \) generated by \( X \) subject to the relations \( \Xi \). That is, there is a canonical epimorphism \( \mu: W \to V \) with the universal property that any homomorphism \( v: W \to V' \) for \( V' \) in \( \mathcal{V} \) such that \( v(x_1) = v(x_2) \) for all \( (x_1, x_2) \in \Xi \) can be uniquely factored through \( \mu \). (For a unique homomorphism \( v^*: V \to V' \), we have \( v = v^* \mu \).) Observe that \( \mathcal{V} \) is SP-closed by [2, Theorem 4.3, p. 235 and Theorem 2.8, p. 226], and so is R-closed by [2, Proposition 8.2, p. 106]. Therefore, \( V = \mathcal{V}(X \mid \Xi) \) always exists in \( \mathcal{V} \) by [2, Theorem 8.2, p. 152].

The word problem for \( \mathcal{V}(X \mid \Xi) \) is recursively unsolvable if there is no recursive function that computes the predicate \( \mu(x_1) \cdots \mu(x_2) \) for all \( (x_1, x_2) \) in \( W \times W \).

**Theorem 1.** Let \( A \) be a nontrivial ring, let \( N \) be a denumerably infinite set, and let \( \Gamma(A^N) \) be the lattice of submodules of the left \( A \)-module of all functions \( N \to A \). If \( \mathcal{M} \) is a quasivariety of modular lattices containing \( \Gamma(A^N) \), then there is a finitely presented lattice \( \mathcal{M}(Y \mid \Psi) \) with a recursively unsolvable word problem.

**Proof.** By [3, Theorem 4.6, p. 98], there exists a finitely presented semigroup \( S = \mathcal{V}(s, t \mid \Delta) \) with two generators \( s \) and \( t \), having a recursively
unsolvable word problem. We are going to construct within a suitable finitely presented lattice \( M(Y,Y) \) a semigroup recursively isomorphic to \( S \). This is done by imitating the classical construction of the ring multiplication in the coordinate ring of a complemented modular lattice with a homogeneous basis of order four [11, Part II, Chap. VI]. Alternatively, we can view the construction as a lattice representation of a diagram of \( A \)-linear transformations, using the methods of [5]. The theorem is then proved by contradiction: A solution for the word problem of \( M(Y,Y) \) also solves the unsolvable word problem of \( S \).

Assume the hypotheses. Our first goal is to model \( S \) within \( \Gamma(A^N) \). Let \( T \) be the monoid obtained by adjoining a unit element to \( S \). More precisely, \( T \) is a disjoint union \( S \cup \{1\} \), \( S \) is a subsemigroup of \( T \), and \( 1 \times x = x \times 1 = x \) for all \( x \in T \). Since \( S \) is finitely generated and has an unsolvable word problem, both \( S \) and \( T \) are denumerably infinite. Let \( M \) denote the \( A \)-module of all functions \( T \to A \), and let \( \text{End},(M) \) denote the semigroup under composition of all \( A \)-linear transformations \( T \to M \). Define \( H: S \to \text{End},(M) \) by

\[
(H(x)(m))(z) = m(x \cdot z) \quad (x \in S, m: T \to A, z \in T).
\]

That is, \( H(x) \) permutes the coordinates of \( m \) in \( M \), using the (right) action of \( x \) in \( T \). Then \( H(x \cdot y) = H(x)H(y) \) for \( x, y \) in \( S \), and \( H \) is one-to-one because \( A \) is nontrivial and \( T \) has a unit. So, \( H \) is a monomorphism of semigroups.

Composition in \( \text{End},(M) \) can be represented by lattice operations in \( M \otimes \Gamma(M) \). Further analysis shows that associativity results can be proved if a fourth copy of \( M \) is available (see [11, Theorem 6.1, p. 132; 5, 3.6, p. 165]). So, let \( M^4 \) denote the \( A \)-module \( M \otimes M \otimes M \otimes M \), and let \( \Gamma(M^4) \) denote the lattice of submodules of \( M^4 \). Since \( \Gamma(M^4) \) is isomorphic to \( M(\Gamma(M)) \), it belongs to \( M \). Let \( c_i: M \to M^4 \) be the \( i \)-th injection, let \( M_i = c_i[M] \), and let \( d_i: M \to M_i \) be the isomorphism induced by \( c_i \), for \( i = 1, 2, 3, 4 \). Now we can identify \( f \) in \( \text{End},(M) \) with \( f^* \cdot d_4^{-1} \cdot d_4^{-1}: M_4 \to M_1 \). Let \( h_j = d_{j+1}^{-1}d_{j+1}^{-1}: M_j \to M_{j+1} \) for \( j = 1, 2, 3 \). Note that \((fg)^* \) for \( f, g \) in \( \text{End},(M) \) equals the composite,

\[
M_4 \xrightarrow{g^*} M_1 \xrightarrow{h_1} M_2 \xrightarrow{h_2} M_3 \xrightarrow{h_3} M_4 \xrightarrow{f^*} M_1.
\]

Then, the above maps are modeled in \( \Gamma(M^4) \) using negative graphs (see [11, p. 135] or [5, Section 2]). More precisely, the negative graph of \( f^* \cdot d_4^{-1} \) for \( f \) in \( \text{End},(M) \) is given by

\[
Ff = \{c_4(m) - c_1 f(m): m \in M\}.
\]
This defines a function $F: \text{End}_M(M) \to \Gamma(M^4)$ since $Ff$ is a submodule of $M^4$. The negative graph of $h_j = d_j, d_j^{-1}$ is given by
\[ P_j = \langle c_j(m) : c_{j+1}(m), m \in M_4 \rangle \quad (j = 1, 2, 3). \]

Define a binary operation $\Omega$ on $\Gamma(M^4)$ as follows:
\[
\begin{align*}
K(E) &= (E \vee P_3) \wedge (M_3 \vee M_4), \\
K'(E') &= (E' \vee P_1 \vee P_2) \wedge (M_4 \vee M_3), \\
E \Omega E' &= (K(E) \vee K'(E')) \wedge (M_4 \vee M_3),
\end{align*}
\]
for $E, E'$ in $\Gamma(M^4)$. Given $f, g$ in $\text{End}_M(M)$, $K(Ff)$ is the negative graph of $f^*h_3$ and $K'(Fg)$ is the negative graph of $h_3g^*$, that is
\[
\begin{align*}
K(Ff) &= \langle c_3(m) : c_1f(m), m \in M_4 \rangle, \\
K'(Fg) &= \langle c_3(m) : c_3g(m), m \in M_4 \rangle.
\end{align*}
\]

So $F \Omega Fg$ is the negative graph $F(fg)^*$ of $(fg)^*$, and, therefore, $F$ is a groupoid homomorphism from $\text{End}_M(M)$ into $\Gamma(M^4)$ under $\Omega$. Since $F$ is clearly one-to-one, $FH: S \to \Gamma(M^4)$ is a monomorphism of groupoids.

Consider the diagram of Fig. 1. Let $W_1$ be the free groupoid of monomials with generators $s$ and $t$, and let $\alpha: W_1 \to S$ in Fig. 1 be the canonical groupoid homomorphism from $W_1$ onto $S : = \mathcal{A}\{s, t, A\}$. The significant elements of $\Gamma(M^4)$ for us are $M_1, M_2, M_3, M_4, P_1, P_2, P_3, FH\alpha(s)$ and $FH\alpha(t)$. So, we consider the free lattice algebra $W_2$ of lattice polynomials on the set $Y$ of nine generators,
\[ Y = \{m_1, m_2, m_3, m_4, p_1, p_2, p_3, a_1, a_2\}. \]

Let $\xi: W_2 \to \Gamma(M^4)$ in Fig. 1 be the unique homomorphism of lattice algebras such that $\xi(m_i) = M_i$ for $i \leq 4$, $\xi(p_j) = P_j$ for $j \leq 3$, $\xi(a_k) = FH\alpha(x)$.
and $\xi(a_2) = FH\alpha(t)$. Now make $W_2$ into a groupoid under the operation $\omega$ formally similar to $\Omega$. That is, for $e, e'$ in $W_2$

$$k(e) = (e \lor p_3) \land (m_3 \lor m_4),$$

$$k'(e') = ((e' \lor p_1) \lor p_2) \land (m_4 \lor m_3),$$

$$e \omega e' = ((k(e) \lor k'(e'))) \land (m_4 \lor m_1).$$

Clearly, $\xi$ is a groupoid homomorphism of $W_2$ under $\omega$ into $\Gamma(M^4)$ under $\Omega$. Let $\rho: W_1 \to W_2$ in Fig. 1 be the groupoid homomorphism from $(W_1, \ast)$ into $(W_2, \omega)$ such that $\rho(s) = a_1$ and $\rho(t) = a_2$. Since $FH\alpha(s) = \xi p(s)$, $FH\alpha(t) = \xi p(t)$, and $W_1$ is the free groupoid with generators $s$ and $t$, we have $FH\alpha = \xi p$, as groupoid homomorphisms.

We are now prepared to define the finitely presented lattice $L = \mathcal{M}(Y, \Psi)$ having a recursively unsolvable word problem. The set $Y$ was given above. Let $0$ denote $m_1 \land m_2 \land m_3 \land m_4$ in $W_2$, and let $\Psi_0$ be the set containing the following thirteen relations in $W_2 \times W_2$

$$(m_1 \land m_2, 0), \quad ((m_1 \lor m_2) \land m_3, 0), \quad ((m_1 \lor m_2 \lor m_3) \land m_4, 0);$$

$$(p_i \lor m_1, m_j \lor m_i, 0) \quad \text{and} \quad (p_i \land m_i, 0) \quad \text{for} \quad j = 1, 2, 3;$$

$$(a_i \lor m_1, m_4 \lor m_1) \quad \text{and} \quad (a_i \land m_i, 0) \quad \text{for} \quad i = 1, 2.$$

Let $\Psi = \Psi_0 \cup \rho(\Delta)$, where $\Delta$ is the set of relations of $S$, and

$$\rho(\Delta) = \{(\rho(x_1), \rho(x_2)): (x_1, x_2) \in \Delta \} \subset W_2 \times W_2.$$

Since $\Delta$ is finite, so is $\Psi$. Let $\beta: W_2 \to L$ in Fig. 1 be the canonical homomorphism of lattice algebras from $W_2$ onto the finitely presented lattice $L = \mathcal{M}(Y, \Psi)$. Now $\xi(e_1) = \xi(e_2)$ for $(e_1, e_2)$ in $\Psi_0$ by direct examination. Also, for $(e_1, e_2)$ in $\rho(\Delta)$, there exists $(x_1, x_2)$ in $\Delta$ such that $(e_1, e_2) = (\rho(x_1), \rho(x_2))$, and so $\xi(e_1) = \xi(e_2) = FH\alpha(x_1) \cdot FH\alpha(x_2) = \xi p(x_2) = \xi p(x_1)$ because $\alpha(x_1) = \alpha(x_2)$. Then by the universal property of $\mathcal{M}(Y, \Psi)$, there exists a homomorphism $\xi^*: L \to \Gamma(M^4)$ as in Fig. 1, satisfying $\xi^* \beta = \xi$. Given $y_1, y_2$ in $L$, choose $e_1, e_2$ in $W_2$ such that $y_1 = \beta(e_1)$ and $y_2 = \beta(e_2)$, and let $y_1 \omega^* y_2 = \beta(e_1 \omega e_2)$. Then $\omega^*$ is a well-defined binary operation on $L$ (independent of the choices of $e_1$ and $e_2$), and $\beta$ and $\xi^*$ are groupoid homomorphisms,

$$(W_2, \omega) \to (L, \omega^*) \to (\Gamma(M^4), \Omega).$$

Now $\beta \rho[W_1]$ is a subgroupoid of $L$ under $\omega^*$, and we shall verify that it is a semigroup. The argument is essentially the same as that used by von Neumann to prove the closure and associativity of the coordinate ring. However, we will use [5] for convenience. The notations $2, T, x/y, f^-$,
$S(A, B)$ and $g \circ f$ are taken from [5, pp. 159–165]. Let $A_i : 2 \to L$ be given by $A_i = \beta(i) \beta(0)$ for $i = 1, 2, 3, 4$. Using the 13 relations of $\Psi$, and the hypothesis that $L$ is a modular lattice, we have:

1. $A_1, A_2, A_3, A_4$ is a mixed sequence.

2. There exist maps $h_j : T \to L$ such that $h_j = \beta(p_j)$ and $h_j$ is in $S(A_j, A_{j+1})$ for $j = 1, 2, 3$.

3. There exist maps $b_i : T \to L$ in $S(A_4, A_1)$ such that $b_i = \beta(a_i)$ for $i = 1, 2$.

For $f_1, f_2$ in $S(A_4, A_1)$, let $f_1 \uparrow f_2$ denote $(f_1 \circ h_0) \circ (h_2 \circ h_1 \circ f_3)$ in $S(A_4, A_1)$. From the above and [5, 3.5, 3.6, pp. 164–165], we have $f_1 \omega^* f_2 \Rightarrow (f_1 \uparrow f_2)^{-1}$ in $L$. By (3) and the formula for $f_1^* \omega^* f_2^*$, an induction on monomial length shows that for every $x$ in $W_1$, there exists a unique map in $S(A_4, A_1)$, denoted $f_x$, such that $f_x = \beta_p(x)$. For $f_1, f_2, f_3$ in $S(A_4, A_1)$, let $f_1 \uparrow (f_2 \uparrow f_3) = ((f_1 \circ h_0) \circ (h_2 \circ h_1 \circ f_3)) \circ (h_2 \circ h_1 \circ f_3) = (f_1 \circ h_0) \circ (h_2 \circ (h_1 \circ f_3)) \circ (h_2 \circ h_1 \circ f_3)$, by several applications of [5, 3.6]. But then for $x_1, x_2, x_3$ in $W_1$

$$\beta_p(x_1) \omega^* (\beta_p(x_2) \omega^* \beta_p(x_3)) \Rightarrow (f_1 \uparrow (f_2 \uparrow f_3))^{-1} \Rightarrow ((f_1 \uparrow f_2 \uparrow f_3))^{-1} \Rightarrow (\beta_p(x_1) \omega^* \beta_p(x_2) \omega^* \beta_p(x_3)).$$

This proves that $\beta_p[W_1]$ is associative under $\omega^*$.

Now $\rho(\Delta) \subset \Psi$, so $\beta_p(x_1) = \beta_p(x_2)$ for $(x_1, x_2) \in \Delta$. Then, by the defining universal property of $\S\setminus t^\Delta$, there exists a homomorphism $\rho^* : S \to \beta_p[W_1]$ such that the following diagram commutes.

We can regard $\rho^*$ as a groupoid homomorphism $S \to L$, and obtain $\beta_p = \rho^* \alpha$, as in Fig. 1.

We can now chase the diagram of Fig. 1, as follows:

$$FH \alpha = \xi \rho = \xi^* \beta_p - \xi^* \rho^* \alpha.$$

Since $\alpha$ is onto, $FH = \xi^* \rho^*$. Since $FH$ is one-to-one, so is $\rho^*$. Since $\rho^*$ is one-to-one and $\rho^* \alpha - \beta_p$, the predicates $\alpha(x_1) - \alpha(x_2)$ and $\beta_p(x_1) - \beta_p(x_2)$
are equivalent for all \((x_1, x_2)\) in \(W_1 \times W_1\). To see that \(\rho\) is recursive, note that \(x_1\) and \(x_2\) can be computed from \(x_1 \times x_2\) by \([2, \text{III.2.3, p. 118}]\), and then we can recursively substitute into the formula for \(\omega\). This is quite easy in list-processing languages; the author has constructed and verified an algorithm computing \(\rho\) in LISP \([8, 9]\). To construct a Turing machine to compute \(\rho\) is then a lengthy but straightforward task. If \(R(e_1, e_2)\) is a recursive function computing the predicate \(\beta(e_1) = \beta(e_2)\) for all \((e_1, e_2)\) in \(W_2 \times W_2\), then \(R(\rho(x_1), \rho(x_2))\) is a recursive function computing the predicate \(\alpha(x_1) = \alpha(x_2)\) for all \((x_1, x_2)\) in \(W_1 \times W_1\). But this would contradict the choice of \(\mathcal{P}(s, t \mid \mathcal{A})\), and, therefore, \(\mathcal{A}(\mathcal{P}, \mathcal{P})\) has a recursively unsolvable word problem. This proves Theorem 1.

As already noted, Theorem 1 shows that the varieties of modular lattices and of Arguesian lattices have unsolvable word problems. B. Jónsson gave a quasivariety characterization of all lattices representable by commuting equivalence relations \([7, \text{Theorem 2, p. 457}]\); this quasivariety also has an unsolvable word problem. It is also possible to construct lattice Horn formulas that are satisfied in \(I(A^n)\) for some but not every ring \(A\). For example, the projective geometry addition formulas \([11, \text{pp. 148-150}]\) can be used to construct a Horn formula that is satisfied in \(I(A^n)\) iff \(A\) has characteristic two.

A lattice \(L\) is "representable by \(A\)-modules" if it can be embedded in the lattice of submodules of some \(A\)-module. The class \(\mathcal{L}(A)\) of all lattices representable by \(A\)-modules is clearly \(\mathcal{SP}\)-closed. If \(\mathcal{L}(A)\) is a quasivariety, then it has an unsolvable word problem.

3. The unsolvability of the family of commutative exactness problems

In the naive view, an abelian category diagram is a pictorial representation that certain variables represent objects and morphisms in some fixed abelian category. The conventional use of arrows specifies the domain and range of each morphism variable, and special notations or descriptive phrases may indicate other relationships (zero objects, commutativity, exactness, direct sums, additivity relations, etc.).

Precise definitions of diagram-scheme and diagram were offered by Grothendieck (see \([10, \text{p. 42}]\)) and Freyd \([4, \text{pp. 95-96}]\). In Grothendieck's formulation, the diagram-scheme may be regarded as a set of points and arrows generating a category subject to given commutativity relations. A diagram is then a suitable function from the points and arrows into some abelian category, determining a functor from the category generated by the
diagram scheme into the given abelian category. Without going into detail, we remark that the existence of the semigroup monomorphism $H: S \to \text{End}_\mathcal{A}(M)$ eliminates the possibility that a general method for computing commutativity relations in an Abelian category diagram can be found. That is, there is an Abelian category diagram-scheme with one point and two endomorphic arrows subject to a finite number of commutativity relations such that the associated decision problem for commutativity relations is recursively undecidable.

To consider commutative exactness problems like the five-lemma, a modification of Freyd's definition is used. A "finite commutative diagram scheme" is a finite category $\mathcal{D}$ with at most one map $A \to B$ for any objects $A$ and $B$ of $\mathcal{D}$. Finite commutative diagram schemes correspond to reflexive and transitive relations on finite sets. The unique map $A \to A$ is the identity, for each object $A$. If $Z_0$ is a finite set and $P \subset Z_0 \times Z_0$, then the finite commutative diagram scheme "generated" by $P$ corresponds to the smallest reflexive and transitive relation on $Z_0$ that contains $P$. A "composing pair" of $\mathcal{D}$ is an ordered pair $(\alpha_1, \alpha_2)$ of maps of $\mathcal{D}$ such that $\alpha_2\alpha_1$ is defined. $E$ of "exactness conditions" on $\mathcal{D}$ is an arbitrary set of composing pairs of $\mathcal{D}$. A functor $G: \mathcal{D} \to \mathcal{C}$, $\mathcal{C}$ a small abelian category, "satisfies" $E$ if $(G\beta_1, G\beta_2)$ is exact in $\mathcal{C}$ for every $(\beta_1, \beta_2)$ in $E$. If $\mathcal{D}$ is a finite commutative diagram scheme, $E$ is a set of exactness conditions on $\mathcal{D}$, and $(\alpha_1, \alpha_2)$ is a composing pair of $\mathcal{D}$, then $Q(\mathcal{D}, E, (\alpha_1, \alpha_2))$ is the predicate: For every small Abelian category $\mathcal{C}$ and functor $G: \mathcal{D} \to \mathcal{C}$ satisfying $E$, $(G\alpha_1, G\alpha_2)$ is exact in $\mathcal{C}$. In the naive sense, $Q(\mathcal{D}, E, (\alpha_1, \alpha_2))$ is the assertion that the exactness of $(\alpha_1, \alpha_2)$ is a consequence of the exactness conditions $E$ on $\mathcal{D}$. The family of commutative exactness problems is represented by the predicate $Q$.

Now $Q$ is certainly not a recursive predicate, since its domain is a proper class. To repair this defect, we introduce a code by which each argument triple $(\mathcal{D}, E, (\alpha_1, \alpha_2))$ of $Q$ can be represented (up to isomorphism) by a word in an appropriate denumerable alphabet. A word predicate $Q^*$ corresponding to $Q$ can then be constructed, and the recursiveness or non-recursiveness of $Q^*$ is then a well posed problem.

Let $Z = \{z_1, z_2, z_3, \ldots\}$ be a denumerable set of variables. A delimiting symbol "" is also needed. The variables represent objects of $\mathcal{D}$, a juxtaposition $z_iz_j$ represents a morphism $z_i \to z_j$ of $\mathcal{D}$, and a juxtaposition $z_i z_j z_k$ represents a composing pair $z_j \to z_k \to z_i$ of $\mathcal{D}$. As an example, consider the diagram:
Suppose that the exactness of \( z_1 \rightarrow z_2 \rightarrow z_3 \) is specified, and we ask whether the exactness of \( z_1 \rightarrow z_4 \rightarrow z_5 \) follows. One word on \( Z \cup \{ b \} \) representing this commutative exactness problem is:

\[
\overline{z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8 z_9} \, .
\]

In general, a succession of two variable juxtapositions specifying a finite commutative diagram scheme is given, followed by a succession of one or more three variable juxtapositions representing composing pairs of the given diagram scheme. The two variable juxtapositions must determine a reflexive and transitive relation on some finite set of the variables. The final three variable juxtaposition corresponds to \((\alpha_1, \alpha_2)\); the others form the set of specified exactness conditions \( E \).

Let \( Z^* \) denote the set of words on \( Z \cup \{ b \} \); that is, the finite strings of variable symbols and delimiting symbols. Clearly, we can recursively decide whether \( u \) in \( Z^* \) represents an argument \((D, E, (\alpha_1, \alpha_2))\) for \( Q \), and can compute the argument triple from \( u \). Also, given \((D, E, (\alpha_1, \alpha_2))\), we can label the objects of \( D \) with distinct symbols \( z_1, z_2, \ldots, z_n \), and then construct a word \( u \) of \( Z^* \) representing \((D, E, (\alpha_1, \alpha_2))\) up to isomorphism. For all \( u \) in \( Z^* \), let \( Q^*(u) \) denote the following predicate: The word \( u \) represents an argument triple \((D, E, (\alpha_1, \alpha_2))\) for \( Q \), and \( Q(D, E, (\alpha_1, \alpha_2)) \) is true. We can now state:

**Theorem 2.** The family of commutative exactness problems is recursively unsolvable. That is, the predicate \( Q^*(u) \) for \( u \) in \( Z^* \) is not recursively computable.

**Proof.** Let \( C \) be a small abelian category. The subobjects of an object \( A \) of \( C \) form a modular lattice (see [4, Example A, p. 103] or [5, Theorem 4.3, p. 182]), denoted \( \Gamma(A; C) \) here. Recall that \( W_2 \) is the free lattice algebra of lattice polynomials on the set \( Y \) of nine generators, and \( \Psi \) is a certain finite subset of \( W_2 \times W_2 \). If \( m: W_2 \rightarrow \Gamma(A; C) \) is a lattice algebra homomorphism for some \( A \), such that \( m(d_1) = m(d_2) \) for all \((d_1, d_2)\) in some subset \( \mathcal{E} \) of \( W_2 \times W_2 \), then \( m \) is called a "\( C \)-model of \((Y, \mathcal{E})\)." We intend to construct a recursive function \( \tau: W_2 \times W_2 \rightarrow Z^* \) such that \( Q^*(\tau(e_1, e_2)) \) for \( e_1, e_2 \) in \( W_2 \) is equivalent to the predicate \( Q_0(e_1, e_2) \) next: If \( C \) is a small Abelian category and \( m \) is a \( C \)-model of \((Y, \Psi)\), then \( m(e_1) = m(e_2) \).

Let us first show that the existence of \( \tau \) proves Theorem 2, by proving that \( \alpha(w_1) = \alpha(w_2) \) and \( Q^*(\tau(\rho(w_1), \rho(w_2))) \) are equivalent predicates on \( W_1 \times W_1 \). Here, \( \alpha \) and \( \rho \) are defined as in Fig. 1. Let \( w_1, w_2 \) be in \( W_1 \). Also, let \( \mathcal{M} \) be the variety of modular lattices, and let \( \beta: W_2 \rightarrow L \) for \( L = \mathcal{M}(Y, \Psi) \) as in Fig. 1. Supposing that \( m: W_2 \rightarrow \Gamma(A; C) \) is a \( C \)-model of \((Y, \Psi)\), we see that \( m = m^* \beta \) for a unique lattice homomorphism \( m^*: L \rightarrow \Gamma(A; C) \) by the universal property of \( \mathcal{M}(Y, \Psi) \). But \( \alpha(w_1) = \alpha(w_2) \) implies
\[ \beta p(\omega_1) = \beta p(\omega_2) \] as in the proof of Theorem 1, and \( \beta p(\omega_1) \neq \beta p(\omega_2) \) implies \( mp(\omega_1) = mp(\omega_2) \) because \( m = m^* \) \( \beta \). Therefore, \( \alpha(\omega_1) \neq \alpha(\omega_2) \) implies \( Q_0(\rho(\omega_1), \rho(\omega_2)) \).

Now suppose that \( \alpha(\omega_1) \neq \alpha(\omega_2) \). We must construct a \( C \)-model \( m \) of \( (Y, \Psi) \) such that \( mp(\omega_1) \neq mp(\omega_2) \). Consider \( I(A^N; A-\text{Mod}) \), where \( A-\text{Mod} \) is the category of all left \( A \)-modules. That lattice is isomorphic to \( I(M^1) \) in Fig. 1. Now \( A-\text{Mod} \) is not small, but this defect is easily remedied. Let \( A \) be the field of rationals, for example, and let \( C_0 \) be the category of all vector subspaces of \( A^N \). Then \( C_0 \) is equivalent to the category of all rational vector spaces of dimension no greater than the dimension of \( A^N \). So, \( C_0 \) is a small Abelian category which is an exact full subcategory of \( A-\text{Mod} \), and \( I(A^N; C_0) \) is also isomorphic to \( I(M^1) \). Identifying these isomorphic lattices, we see that \( \xi \) in Fig. 1 is a \( C_0 \)-model of \( (Y, \Psi) \). But \( FHIx(\omega_1) \neq FHIx(\omega_2) \) because \( FHI \) is one-to-one, and so \( \xi p(\omega_1) \neq p(\omega_2) \) because \( FHIx \neq \xi p \). Therefore, \( \text{not}(\alpha(\omega_1) \neq \alpha(\omega_2)) \) implies \( \text{not}(Q_0(\rho(\omega_1), \rho(\omega_2))) \).

We have proved that the existence of a recursive \( \tau \) as described above implies the equivalence of the predicates \( \alpha(\omega_1) = \alpha(\omega_2) \) and \( Q^*(\tau(\rho(\omega_1), \rho(\omega_2))) \) on \( W_1 \times W_1 \). If \( R \) is a recursive function on \( Z^* \) computing \( Q^* \), then \( R(\tau(\rho(\omega_1), \rho(\omega_2))) \) for \( \omega_1, \omega_2 \) in \( W_1 \) is a recursive function solving the unsolvable word problem of \( \Psi \) in \( A \). Given \( \tau \), this contradiction proves Theorem 2.

To construct \( \tau \), we use the kernel cokernel isomorphism between the subobject lattice \( I(A; C) \) and the lattice of quotient objects of \( A \) in \( C \) [4, Theorem 2.11, p. 36]. That is, words of \( W_2 \) will correspond to short exact sequences,

\[ z_2 \to z_1 \to z_\rho \to z_{\rho+1} \to z_3 \quad (p \text{ even, } p > 4). \]

Here \( z_1 \) corresponds to an arbitrary object \( A \) of \( C \), and \( z_2 \) and \( z_3 \) will be zero objects. Then exactness of the sequence implies that the monomorphism \( z_\rho \to z_1 \) represents a subobject of \( A \) and the epimorphism \( z_1 \to z_{\rho+1} \) represents the corresponding quotient object.

Let \( D_0 \) and \( E_0 \) be a finite commutative diagram scheme with exactness conditions generated by nine short exact sequences,

\[ z_0 \to z_{2i-4} \to z_i \to z_{2i-3} \to z_0 \quad (i = 1, 2, ..., 9). \]

More precisely, \( D_0 \) is the finite commutative diagram scheme corresponding to the reflexive and transitive relation on \( \{ z_1, z_2, ..., z_{36} \} \) generated by the 36 pairs in those nine sequences. In all, \( D_0 \) has 159 morphisms. Let \( E_0 \) be the set of pairs \( z_0 \to z_2 \to z_3 \) and \( z_3 \to z_0 \to z_3 \) plus the 27 composing pairs specifying the exactness of the given nine sequences. By [4, p. 96], the first
two exactness conditions given for $E_n$ are equivalent to requiring that $x_2$ and $x_n$ be zero objects.

Functors $G: D_0 \to C$ satisfying $E_n$ correspond to arbitrary choices of nine not necessarily distinct subobjects of $Gx_1$ and, hence, induce $C$-models of $(Y, \mathcal{X})$ if each element of $Y$ is identified with a subobject. To make this concept precise, some definitions are given. An ordered sextuple $(D, E, \mathcal{X}, X, a, b)$ is called a “scheme” if $D$ is a finite commutative diagram scheme with objects in $Z$, $E$ is a set of exactness conditions on $D$, $\mathcal{X}$ is a subset of $W_2 \times W_2$, $X$ satisfies $Y \subseteq X \subseteq W_2$, and $a$ and $b$ are functions from $X$ into the set of morphisms of $D$ such that for every $e \in X$ there exists an even integer $p, p > 4$, with

$$\begin{array}{cccccccc}
x_2 & \rightarrow & x_p & \rightarrow & x_1 & \rightarrow & x_{p+1} & \rightarrow & x_3 \\
\end{array}$$

a sequence of morphisms of $D$ specified to be a short exact sequence by $E$.

Given a functor $G: D \to C$ satisfying $E$, say that $a$ and $G$ “induce” the $C$-model $m: W_2 \to \Gamma(Gx_1; C)$ of $(Y, \mathcal{X})$ if $m(e) = [Ga(e)]$ for every $e \in X$. Here, the subobject $[f]$ of $A$ corresponds to the monomorphism $f: I! + A$. Note that $a$ and $G$ can induce at most one $C$-model of $(Y, \mathcal{X})$ since $Y \subseteq X$ and $Y$ generates $W_2$. A scheme $(D, E, \mathcal{X}, X, a, b)$ is called a “lattice scheme” if for every small Abelian category $C$, $a$ and any functor $D \to C$ satisfying $E$ induce a $C$-model of $(Y, \mathcal{X})$, and every $C$-model of $(Y, \mathcal{X})$ is induced by $a$ and some functor $D \to C$ satisfying $E$.

Number $Y$ arbitrarily, say $Y = \{y_1, y_2, ..., y_n\}$, and let $a_0$ and $b_0$ be functions on $Y$ such that $a_0(y_i)$ is the morphism $x_{2+i} \rightarrow x_i$ of $D_0$ and $b_0(y_i)$ is the morphism $x_1 \rightarrow x_{2+i}$ of $D_0$, for $i = 1, 2, ..., 9$. We will show that

$$(D_0, E_n, \mathcal{X}, Y, a_0, b_0)$$

is a lattice scheme.

Meets and joins in $\Gamma(D; C)$ can be represented by exactness conditions on a commutative diagram. Suppose $D$ and $E$ specify three short exact sequences interconnected as follows:

![Diagrams](image)

Given $G: D \to C$ satisfying $E$,

$$[G(x_n \rightarrow x_1)] = [G(x_n \rightarrow x_1)] \land [G(x_m \rightarrow x_1)]$$
is equivalent to the exactness of \((G(z_n \rightarrow z), G(z_n \rightarrow z_{m+1}))\) by [4, Theorem 2.13, pp. 37-38 and Proposition 2.22, p. 45]. Dually, suppose \(D\) and \(E\) specify the following:

\[
\begin{array}{c}
\text{Then } [G(z_n \rightarrow z)] = [G(z_n \rightarrow z)] \vee [G(z_n \rightarrow z)] \text{ is equivalent to the exactness of } (G(z_n \rightarrow z_{m+1}), G(z_n \rightarrow z_{m+1})).
\end{array}
\]

By the above considerations, we are motivated to define two standard methods of enlarging a scheme. Let \(r = (D, E, Z, X, a, b)\) be a scheme, and suppose that \(d_1\) and \(d_2\) are in \(Z\) but \(d_1 \wedge d_2\) is not in \(Z\). Assume that \(a(d_1): z_n \rightarrow z_1\) and \(a(d_2): z_m \rightarrow z_1\) in \(D\). Let \(\ell(D) \subseteq Z\) be the set of objects of \(D\), and suppose that \(\ell\) is an even integer, \(\ell > 4\), such that \(z_n\) and \(z_{m+1}\) are not in \(\ell(D)\). Then \(r = (D', E', Z', X', a', b')\) is a "meet augmentation" of \(r\) with respect to \(d_1 \wedge d_2\) if \(D'\) is the finite commutative diagram scheme on \(\ell(D) \cup \{z_n, z_{m+1}\}\) generated by the morphisms of \(D\) plus the following eight morphisms:

\[
\begin{align*}
&z_2, z_3, \\
z_{m+1}, z_{m+1} \\
&z_{m+1} \\
\end{align*}
\]

\(E'\) equals \(E\) with four additional exactness conditions: three pairs specifying that \((A_1)\) is an exact sequence plus \(z_n \rightarrow z_{m+1} \rightarrow z_{n+1}\).

\(X'\) equals \(X \cup \{d_1 \wedge d_2\}\).

For all \(e \in X\), \(a'(e) - a(e)\) and \(b'(e) - b(e)\).

In \(D'\), \(a'(d_1 \wedge d_2): z_n \rightarrow z_1\) and \(b'(d_1 \wedge d_2): z_1 \rightarrow z_{m+1}\).

To define a "join augmentation" \(r'\) of \(r\), make the following changes in the above definition: replace \(d_1 \vee d_2\) by \(d_1 \wedge d_2\) throughout, replace the exactness condition \(z_n \rightarrow z_{m+1}\) by \(z_n \rightarrow z_{m+1} \rightarrow z_{m+1}\), and replace \((A2)\) by

\[
\begin{align*}
&z_n, z_n \\
z_{m+1}, z_{m+1} \\
&z_{m+1} \\
\end{align*}
\]

\(A2''\)

Suppose \(r_0, r_1, r_2, \ldots, r_k\) is a sequence such that

\[\text{is a meet or join augmentation of } r_{i-1} \text{ for } i = 1, 2, \ldots, k. \text{ It is easily verified that } r_i \text{ is a scheme for } i = 0, 1, \ldots, k, \text{ since } r_0 \text{ is a scheme and a meet or join augmentation of a scheme is also a scheme. It can also be seen that } a_i \text{ and any functor } D_i \rightarrow C \text{ satisfying } E_i \text{ induce a } C\text{-model of } (Y, \varepsilon).\]

Let \(m\) be any
C-model of \((Y, \circlearrowleft)\). To prove that \(r_k\) is a lattice scheme, we must construct functors \(G_i : D_i \to C\) satisfying \(F_i\) such that \(a_i\) and \(G_i\) induce \(m_i\) for \(i = 0, 1, \ldots, k\).

Let \(Z_n\) be a finite subset of \(Z\) containing \(z_1\), and say that a binary relation \(P\) on \(Z_n\) is "special" if whenever \(p\) is even, \(q\) is odd and \(z_p, z_q\) are in \(Z_n\), then \((z_p, z_1)\) and \((z_1, z_q)\) are in \(P\) but \((z_q, z_p)\) is not in \(P\). Given a special relation \(P\) on \(Z_n\), a "partial functor" \(g\) from \(P\) to \(C\) is a function from \(P\) to the set of morphisms of \(C\) such that:

(PF1) For some function \(g' : Z_n \to C\), \(g'(z_n, z_{p}) : g'(z_n, z_{q}) \to g'(z_{p}, z_{q})\) in \(C\) for each \((z_n, z_{p}) \in P\).

(PF2) For any even \(p\) such that \(z_p \in Z_n\), \(g(z_p, z_1)\) is a monomorphism.

(PF3) For any odd \(p\) such that \(z_p \in Z_n\), \(g(z_1, z_p)\) is an epimorphism.

(PF4) If \((z_p, z_q) \in P\), then \(g(z_p, z_1)g(z_p, z_q) = g(z_p, z_q)\) if \(p\) and \(q\) are even, \(g(z_p, z_q)g(z_1, z_p) = g(z_1, z_q)\) if \(p\) and \(q\) are odd, and \(g(z_p, z_q) = g(z_1, z_n)g(z_p, z_{1})\) if \(p\) is even and \(q\) is odd.

Given a functor \(G : D \to C\), the "morphism function" \(h\) of \(G\) is given by \(h(z_{p}, z_q) = G(z_p \to z_q)\) for every pair \((z_{p}, z_q)\) such that \(z_p \to z_q\) is in \(D\).

We have the following:

**Lemma.** If \(P\) is a special relation on \(Z_n\), \(g\) is a partial functor from \(P\) to \(C\), and \(D\) is the finite commutative diagram scheme generated by \(P\), then there exists a functor \(G : D \to C\) such that \(G(z_p \to z_q) = g(z_p, z_q)\) for all \((z_p, z_q) \in P\). The morphism function of \(G\) is also a partial functor defined on a special relation on \(Z_n\).

That is, a partial functor can be extended to a functor. Observe that the reflexive and transitive relation generated by a special relation is also special. So, we can define \(G\) equal to \(g'\) on the objects and let \(G(z_p \to z_q)\) be given by:

the unique morphism satisfying \(g(z_q, z_1) G(z_p \to z_q) = g(z_p, z_1)\) for \(p\) and \(q\) even.

the unique morphism satisfying \(G(z_p \to z_q) g(z_1, z_p) = g(z_1, z_q)\) for \(p\) and \(q\) odd, and

\(G(z_p \to z_q) = g(z_1, z_q) g(z_p, z_1)\) for \(p\) even and \(q\) odd.

(There are no morphisms \(z_p \to z_q\) of \(D\) with \(p\) odd and \(q\) even.) By applying the definition of partial functor and the monomorphism and epimorphism properties, we can show that \(G\) is a functor. Also, \(G\) extends \(g\), and the morphism function of \(G\) is a partial functor.

We can now construct the functors \(G_0, G_1, \ldots, G_k\) by induction. Let
$P_0$ be the binary relation on $\{z_1, z_2, \ldots, z_{21}\}$ containing the 36 pairs corresponding to the nine short exact sequences of $D_0$, plus the three pairs $(z_2, z_3), (z_1, z_4)$ and $(z_1, z_5)$. Clearly, $P_0$ is special and generates $D_0$. By choosing a short exact sequence for each subobject $m(y_i), i \leq 9$, we can define a partial functor $g_0$ on $P_0$ such that $m(y_i) = [g_0(z_{2i-2}, z_{1i})]$ for $i \leq 9$. By the lemma, $g_0$ can be extended to a functor $G_0 : D_0 \to C$. Clearly $G_0$ satisfies $E_0$, and $a_0$ and $G_0$ induce $m$. The morphism function of $G_0$ is a partial functor defined on a special relation on $C(D_0)$.

Assume the induction hypothesis, that a functor $G_i : D_i \to C$ satisfying $E_i$ exists such that $a_i$ and $G_i$ induce $m$, and that the morphism function of $G_i$ is a partial functor defined on a special relation on $C(D_i)$. For the first case, suppose that $r_{i-1}$ is a meet augmentation of $r_i$ by $d_1 \land d_2$. Let $P_{i-1}$ denote the binary relation on $C(D_i) \cup \{z_p, z_{p+1}\}$ corresponding to the morphisms of $D_i$ plus the right morphisms of $(A1)$ and $(A2)$. Then $P_{i-1}$ is special and generates $D_{i-1}$. We can define a partial functor $g_{i-1}$ on $P_{i-1}$ to $C$ by setting $g_{i-1}(f) = G_i(f)$ for morphisms $f$ of $D_i$, and choosing a short exact sequence such that $m(d_1 \land d_2) = [g_{i-1}(z_p \to z_1)]$ for the eight morphisms of $(A1)$ and $(A2)$. By the lemma, $g_{i-1}$ can be extended to a functor $G_{i-1} : D_{i-1} \to C$. Again we can show that $G_{i-1}$ satisfies $E_{i-1}$, and $a_{i-1}$ and $G_{i-1}$ induce $m$. Also, the morphism function of $G_{i-1}$ is a partial functor defined on a special relation on $C(D_{i-1})$. A similar method is used when $r_{i-1}$ is a join augmentation of $r_i$. This completes the induction, proving that $r_i$ is a lattice scheme.

Hereafter, let $e_1$ and $e_2$ be fixed elements of $W$. Let $(D_E, E, \Xi, X, a, b)$ be a lattice scheme such that $X$ contains $e_1$ and $e_2$ and also $d_1$ and $d_2$ whenever $(d_1, d_2) \in \Psi$. Since $\Psi$ is finite and every lattice polynomial has finite length, this lattice scheme can be obtained by a finite number of meet and join augmentations of $(D_0, E_0 \cup Y, a_0, b_0)$.

Given a lattice scheme $(D, E, \Xi, X, a, b)$ and $c_1, c_2 \in X$, we have two short exact sequences:

\[
\begin{array}{c}
G_{z_2} \\
G_{z_m} \\
G_{z_n}
\end{array}
\quad \begin{array}{c}
G_2 (c_1) \\
G_2 (c_2) \\
G_{z_n+1}
\end{array}
\quad \begin{array}{c}
G_1 (c_1) \\
G_1 (c_2) \\
G_{z_{m+1}}
\end{array}
\quad \begin{array}{c}
G_{z_3}
\end{array}
\]

for any $G : D \to C$ satisfying $E$. But $[G_2(c_1)] = [G_2(c_2)]$ in $\Gamma(J; C)$ iff $(G_2(c_1), G_2(c_2))$ is exact in $C$ [4, pp. 42–44]. If $m$ is the $C$-model induced by $a$ and $G$, then $m(c) = m(c')$ iff $(G_2(c_1), G_2(c_2))$ is exact. So, $(D_E, E_{n+1}, \Psi, X, a, b)$ is a lattice scheme if

$$E_{n+1} = E_n \cup \{(a_n(d_1), b_n(d_2)); (d_1, d_2) \in \Psi\}.$$
Finally, let $\tau(e_1, e_2)$ be a word of $Z^*$ corresponding to

$$(D_1, E_1, (a_1(e_1), b_1(e_2))),$$

a proper argument triple for $Q$. By the definition of lattice scheme and the equivalence between exactness conditions and lattice polynomial equalities, we can verify that the predicate $Q_{\alpha}(e_1, e_2)$ previously given is equivalent to $Q^*(\tau(e_1, e_2))$ on $W_2 \times W_2$.

We will omit further details needed to give a precise definition of $\tau$ as a recursive function $W_2 \times W_2 \to Z^*$. The author has constructed and verified an algorithm computing $\tau$, again using LISP [8, 9]. This completes the proof of Theorem 2.

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