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Numerical Homotopies to Compute Generic Points on Positive Dimensional Algebraic Sets

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Many applications modeled by polynomial systems have positive dimensional solution components (e.g., the path synthesis problems for four-bar mechanisms) that are challenging to compute numerically by homotopy continuation methods. A procedure of A. Sommese and C. Wampler consists in slicing the components with linear subspaces in general position to obtain generic points of the components as the isolated solutions of an auxiliary system. Since this requires the solution of a number of larger overdetermined systems, the procedure is computationally expensive and also wasteful because many solution paths diverge. In this article an embedding of the original polynomial system is presented, which leads to a sequence of homotopies, with solution paths leading to generic points of all components as the isolated solutions of an auxiliary system. The new procedure significantly reduces the number of paths to solutions that need to be followed. This approach has been implemented and applied to various polynomial systems, such as the cyclic n-roots problem. © 2000 Academic Press

Key Words: polynomial system; numerical homotopy continuation; components of solutions; numerical algebraic geometry; generic points; embedding.

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1. INTRODUCTION

Let

$$f(x) := \begin{bmatrix} f_1(x_1, ..., x_n) \\ \vdots \\ f_N(x_1, ..., x_n) \end{bmatrix} \tag{1}$$

denote a system of N polynomials on $\mathbb{C}^n$. Positive dimensional components of the solution set of $f(x) = 0$ are a common occurrence, even when $N = n$. Sometimes they are an unpleasant side show that happens with a system generated using a model, for which only the isolated nonsingular solutions are of interest; and sometimes, the positive dimensional solution components are of primary interest. In either case, dealing with positive dimensional components, is usually computationally difficult. In [54], Sommese and Wampler presented a numerical algorithm, which uses auxiliary systems to numerically find sets of solutions of the original system. These sets of solutions, which let one numerically decide the dimension of the zero set of the original system, include at least the isolated solutions of the original system, plus "generic points" of each positive dimensional irreducible component of the solutions of the original solutions. Generic points are the basic numerical data which we are using to investigate the positive dimensional solution components.

The algorithm from [54], which is based on slicing with general linear spaces of different dimensions, leads to n auxiliary systems, which must be dealt with. In this paper we present an embedding of the system $f(x) = 0$ into a family of systems of polynomials depending on $2n$ variables $(x, z) \in \mathbb{C}^{2n}$, and a large space of parameters. We then single out $n+1$ of the systems, $\mathscr{E}_i(x, z)$ for i from n to 0 (here $\mathscr{E}_0(x, z) = 0$ is equivalent to the system $f(x) = 0$) obtained by choosing particular values of the parameters, plus a homotopy H_i going from $\mathscr{E}_i$ to $\mathscr{E}_{i-1}$. For simplicity we assume that f is not identically zero. The system $\mathscr{E}_n = 0$ has isolated nonsingular solutions. We use polynomial continuation [1, 2, 37, 43] to implement the following algorithm

ALGORITHM 1. Cascade of homotopies between embedded systems.

Input: f, n. *system with solution in* $\mathbb{C}^n$
Output: $(\mathscr{E}_i, \mathscr{X}_i, \mathscr{Z}_i)_{i=0}^n$. *embeddings with solutions*
$\mathscr{E}_0 := f$; *initialize embedding sequence*
for i from 1 up to n do *slice and embed*
$\mathscr{E}_i := \text{Embed}(\mathscr{E}_{i-1}, z_i)$; $z_i =$ *new added variable*
end for; *homotopy sequence starts*
$\mathscr{Z}_n := \text{Solve}(\mathscr{E}_n)$; *all roots are isolated, nonsingular, with* $z_n \neq 0$
for i from $n-1$ down to 0 do *countdown of dimensions*
$H_{i+1} := t\mathscr{E}_{i+1} + (1-t)\begin{pmatrix} \mathscr{E}_i \\ z_{i+1} \end{pmatrix}$; $\mathscr{Z}_{i+1}$ *are start solutions in homotopy continuation* t: $1 \to 0$ *to remove* z_{i+1}
$\mathscr{X}_i :=$ limits of solutions of H_{i+1}
as $t \to 0$ with $z_i = 0$; *on component*
$\mathscr{Z}_i := H_{i+1}(x, z_i \neq 0, t = 0)$; *not on component: these solutions are isolated and nonsingular*
end for.

The routine Embed will be defined in the third section. In Section 4 we present a worked out example of the algorithm. Section 5 contains the mathematical background needed to prove our main results:

1. if i is the largest integer with $\mathscr{X}_i$ nonempty, then the dimension of $f^{-1}(0)$ is i; and
2. given any irreducible component W of $f^{-1}(0)$ of dimension i, then, counting multiplicities, $\mathscr{X}_i$ contains deg (W) generic points of W.

The applications described in Section 6 illustrate the performance of the new procedure. We end this paper with some directions for future research.

2. BACKGROUND MATERIAL AND RELATED WORK

Before addressing related work in this section

1. we give a brief discussion of generic points, a notion, which is basic in this paper; and
2. we give a discussion of the technique from [54] that allows us to reduce to square systems, i.e., to systems with the same number of equations as variables.

2.1. *Generic Points*

For both topics a fuller discussion will be found in [54]. We orient the discussion around the question of how we decide if a polynomial is zero on an algebraic set.

Let X be an irreducible and reduced algebraic set in $\mathbb{C}^N$. Geometrically, this means that the smooth points of X are dense and connected, and the ideal associated to X is all the polynomials on $\mathbb{C}^N$ vanishing on the set underlying X. Let $p(x)$ be some polynomial on $\mathbb{C}^N$. How do we decide if p vanishes identically on X? We know that if $p(x)$ does not vanish identically, then $p(x)$ vanishes only on a proper algebraic subset of X, which is very "thin" in that it is of real codimension two (and thus of Lesbesgue measure zero). Thus it is natural to check the condition by choosing a "random" point x^* on X and checking whether $p(x^*)=0$. If it is not, then of course $p(x)$ is not identically zero, and if $p(x^*)=0$, then we conclude correctly that $p(x)$ is identically zero unless we choose x^* from the measure zero set on X where $p(x)$ vanishes. Of course, with machine numbers there is more than a zero probability that things go wrong, but probabilistic algorithms speed up calculations just as the use of random numbers (which are of course not random) speed up calculations at the expense of some certainty. It is a typical situation in scientific computing, that we have a model we can analyze mathematically and use to construct algorithms, but when we use the finite (but large) set of double precision complex numbers on a modern computer, there is a question of how well the model fits the computations on the machine. We say more about this below.

In [54], it was noted that the notion of a random point on an irreducible algebraic variety is so close to the classical notion of a generic point from algebraic geometry, that it is very reasonable to model the idea of a random point on an irreducible variety using this concept. In [54], there was a discussion of the different ways that algebraic geometers have made this concept precise. At one extreme, which we do not find useful, there is the scheme theory approach of treating a generic point on an irreducible variety X as that nonclosed point whose closure is the set of all points on X. This "generic point," which has the function field of X as the stalk of the structure sheaf of X at the point, finesses away the difficulties of working with actual points. At the other extreme there is a classical notion rooted in fields of definition of the variety. For example, consider $x_1 - x_2 = 0$, the equation of a line L in $\mathbb{C}^2$. If we have any polynomial with rational coefficients $q(x_1, x_2)$ which vanishes at the point $(\pi, \pi) \in L$, then q vanishes on L. The third major variant is to use the language as shorthand for points in some set where none of some specified set of "conditions" are satisfied. For example, if we have some finite set of algebraic functions $f_1, ..., f_k$ on $\mathbb{C}^N$ with no common zeroes on $\mathbb{C}^N$, then a special case of Bertini's theorem [31] says that there is a Zariski open and dense set $U \subset \mathbb{C}^k$, such that, if $(\lambda_1, ..., \lambda_k) \in U$ then the zero set of $\lambda_1 f_1 + \cdots + \lambda_k f_k$ is smooth. Often this is phrased as saying that for a generic choice of $(\lambda_1, ..., \lambda_k) \in \mathbb{C}^N$, the zero set of $\lambda_1 f_1 + \cdots + \lambda_k f_k$ is smooth. There are

other variants of general that use other classes of dense sets than Zariski open sets, e.g., using complements of countable unions of proper algebraic subsets is sometimes useful.

Mathematically it is possible to translate between the languages of the first and third approach. Moreover, theorems about results holding for dense Zariski open sets, immediately imply the existence of a generic point in the middle approach, but not necessarily conversely. For this reason, in theorems and proofs we use the third approach with dense Zariski open sets, though in practice we think conceptually of generic points along the lines of the middle approach.

Finally, the real numbers come into algorithms in a number of places. This leads us to use Zariski open set of the underlying irreducible algebraic sets X, which are dense in the usual Euclidean topology. The complements of these sets have real codimension one. In this paper, this only happens at the point where we choose explicit forms of the homotopy parameter for the homotopy continuation.

We are currently working on algorithms to process the information contained in the points arising from the algorithm in this paper. Using these "generic points," we have an algorithm, which with probability one, computes the degrees and dimensions of all irreducible components of a reduced algebraic set, and says which of the "generic points" lie on which components of the algebraic set.

To understand the relation of the model we use to the actual numerical varieties that are computed on machines, let us try to analyze using a "generic" point to check whether a polynomial of degree d is identically zero. Of course, given a nontrivial polynomial, we know it is not identically zero, but let us try to analyze the probability that we will get a wrong answer using generic points. Then let us critique this heuristic calculation. For concreteness let $p(z) := \prod_{i=1}^{d} (z - z_i)$ be a monic polynomial of degree d with all its zeroes in a disk of radius R. Let z^* be a "generic point" contained in this disk. In practice we have chosen the disk containing z^* to have radius $R \approx 10^8$. Assume further that we use the criterion that we decide that $p(z)$ is identically zero if $|p(z^*)| < \varepsilon$, for some constant ε dependent on the hardware and software we use. Even though we are in the case of one variable, we take $\varepsilon := 10^{-8}$. When working with double precision, we usually take this value in problems to allow the condition number of Jacobian matrices to be up to 10^8. In this situation a rough estimate of the probability that we will mistakenly decide that the polynomial is zero is

$$\frac{\text{area}(\{x \in \mathbb{C} \mid |\prod_{i=1}^{d} (x - z_i)| < \varepsilon\})}{\pi R^2}. \tag{2}$$

It can be checked that the worst case happens when the roots coincide. We get the upper bound

$$\frac{\varepsilon^{2/d}}{R^2}. \tag{3}$$

Thus we end up with $10^{-16-(16/d)}$.

Putting aside the obvious criticism of this calculation, that it assumes a uniformity of distribution of machine numbers, which is far from satisfied, the calculation is reassuring. For low degree polynomials, it is very reassuring. In practice, we deal with relatively low degree polynomials. Indeed, if you use only double precision, high degree polynomials are difficult numerically. If you use higher precision, then we would also decrease ε and increase R.

Similar calculations are very difficult in higher dimensions. Even for the continuation methods people traditionally use, realistic estimates have not been worked out. For example, consider a traditional homotopy $H(t, x) := \gamma(1-t)\, f(x) + tg(x) = 0$ where γ is a random complex number of the form $e^{\sqrt{-1}\,\theta}$, f is a polynomial system of n polynomials on $\mathbb{C}^n$ that we wish to solve, and $g(x)$ is a polynomial system of n polynomials on $\mathbb{C}^n$ that we know the solutions of. For simplicity, assume that $f(x) = 0$ and $g(x) = 0$ both have $\prod_{i=1}^{n} d_i$ nonsingular solutions, where for $i = 1, ..., n$, $d_i = \deg(f_i) = \deg(g_i)$. In infinite precision, the bad γ would be those with a θ so that the "great circle" followed as t varies between 1 and 0, contains the image under the map $(t, x) \to t$ of a singular point of a fiber of the map $H^{-1}(0) \to \mathbb{C}$ induced by $(t, x) \to t$. Here we know that there are only a finite set of bad θ. So it is a probability zero event that we hit a bad point. The interesting papers of Yomdin [59] and Briskin and Yomdin [15] consider some related problems that arise with trying to apply Sard's theorem in the smooth case.

Being willing to give up complete mathematical certainty in order to be able to settle problems from engineering and science is part of the spirit of numerical analysis. We very much agree with the following statement in [34, p. 6], taken from Trefethen's definition [56] of numerical analysis:

> Rounding errors and instability are important, and numerical analysts will always be the experts in these subjects and at pains to ensure that the unwary are not tripped up by them. But our central mission is to compute quantities that are typically uncomputable, from an analytic point of view, and to do it with lightning speed.

In this quotation, "uncomputable" means that approximations are unavoidable. We strive to obtain those directly by floating-point computations.

Unfortunately, the consideration of solving polynomial systems has been ostensibly neglected by the mainstream numerical analysis so far [55]. In [55] the discrete model of computer algebra is contrasted to the paradigm of numerical algebra where the problems live in a continuous world. The embedding of an algebraic problem into analysis leads to additional meaningful results such as condition numbers that tell us how relevant the computed numbers are. In many cases it does not make much sense to solve a problem exactly whose input data is known only with low accuracy. We find a similar clash in paradigms between the discrete Turing model and continuous models of computation [13].

2.2. *Reduction to Square Systems*

Let

$$f(x) := \begin{bmatrix} f_1(x_1, ..., x_n) \\ \vdots \\ f_N(x_1, ..., x_n) \end{bmatrix} \tag{4}$$

be a system of N polynomials on $\mathbb{C}^n$. We carry out this discussion for polynomials on Euclidean space for simplicity: it equally well goes for systems of algebraic functions on a connected algebraic submanifold of Euclidean space. In this paper we deal for the most part with square systems, i.e., systems where $N = n$. We would like to discuss the results from [54] that allow us to reduce to this case.

Assume first that $N < n$. In this case we know that every irreducible component of the zero set $\{f = 0\}$ is at least $n - N$ dimensional. To reduce to a square system of N equations in N unknowns, we restrict the system to a "generic" linear subspace $L \subset \mathbb{C}^n$ of dimension N. For $k > n - N$, here is a one-to-one correspondence of the k-dimensional irreducible components of $\{f = 0\}$ and the $k + N - n$ dimensional irreducible components of $\{f_{\mathbb{C}^N} = 0\}$, gotten by associating the irreducible component $X \cap L \subset \{f_{\mathbb{C}^N} = 0\}$ to all irreducible component $X \subset \{f = 0\}$. Moreover, the multiplicity of such all X as a component of $\{f = 0\}$ is the same as the multiplicity of such an $X \cap L$ as a component of $\{f_{\mathbb{C}^N} = 0\}$. When k takes on the minimum possible value $n - N$, the correspondence takes each component X to degree X points $X \cap L$. Moreover, the multiplicity of each point of the intersection is equal to the multiplicity of X. Also, because L is general, the intersections have many "generic" properties. For example, if the singular set $\mathrm{Sing}(X)$, of the reduction of a component X of dimension k has codimension k' in X, then the singular set $\mathrm{Sing}(X \cap L)$ equals $\mathrm{Sing}(X) \cap L$ and still has codimension k' (and is in particular empty if $k' > k + N - n$).

Assume now that $N > n$. In this case the procedure leading to square systems is to replace the system f with n random linear combinations of the equations of the system. It turns out to be equivalent to work with the "Gaussian elimination" form of the system

$$F(x) := \begin{bmatrix} f_1(x_1, ..., x_n) + \sum_{j=1}^{N-n} \lambda_{1,n+j} f_{n+j} \\ \vdots \\ f_n(x_1, ..., x_n) + \sum_{j=1}^{N-n} \lambda_{n,n+j} f_{n+j} \end{bmatrix}. \tag{5}$$

It is shown in [54] that for λ chosen in a dense Zariski open subset of $\mathbb{C}^{n(N-n)}$,

1. the positive dimensional irreducible components of $\{f = 0\}$ are the same as the positive dimensional components of the zero set $\{F = 0\}$ of the randomized system;

2. the isolated zeroes of $\{f = 0\}$ are contained among the isolated zeroes of $\{F = 0\}$; and

3. a reduced and irreducible component of $\{f = 0\}$, e.g., a nonsingular isolated solution of $\{f = 0\}$, is a reduced and irreducible component of $\{F = 0\}$.

Unfortunately, if the multiplicity of an irreducible component X of $\{f = 0\}$ is $\mu > 1$, then the multiplicity of X in $\{F = 0\}$ can be $> \mu$. This happens in simple examples, e.g., the example at the end of Section 2 of [54]. From the point of view of many problems where we are interested in the reduction of a component, or about whether a component is nonsingular, this is not a serious problem. It is also a question for zero sets defined by more equations than codimension, whether the multiplicity information is of that much interest in applied problems. For example, for overdetermined systems, there is no geometric interpretation of isolated singular points in terms of the coalescing of smooth solutions. Nevertheless, the loss, of some of the multiplicity information, is a deficit of the method of passing to randomized square systems.

2.3. *Previous and Related Work*

Resultants may find explicit formulas for the solutions in terms of parameters. In this sense, they also are an effective way to deal with components of solutions. Resultants that exploit sparsity are described in [21] and [22]; see [23] for a survey. Recent papers on how to adapt sparse resultants to deal with degenerate situations are [50] and [52].

The standard tool in computer algebra is the Gröbner basis. As a recent conference celebrating 33 years of Gröbner bases [16] shows, this is still a very active and exciting research field. Gröbner bases or triangular sets [3, 4] are used to compute a primary decomposition of an ideal [19]. Singular [30] is a freely available package that does its Gröbner basis computations based on Hilbert series. We used Singular on the cyclic 7-roots problem on a Pentium II 400 Mhz machine with 256 Mb internal memory and .5 Gb swap space. Singular reported "out of memory." While this experience is anecdotal and better implementations [25] exists, the general criticism towards such computations is that the size of the Gröbner basis (its space complexity is surveyed in [41]) is too large to handle. Even if some term order may speedup the practical computations, the conversion of orders is still of doubly exponential complexity [36]. To overcome this problem, an approach with straight-line programs was proposed in [6]. We refer to [29] for a recent practical implementation and comparisons. In [40] we read about Kronecker's philosophy (implemented by those straight-line programs) and the approach of Gröbner for algebraic information on multiplicities of solutions of zero-dimensional systems.

Our approach uses deformations from a solved system to the system we wish to solve. It is interesting to contrast the almost completely reversed use of deformations in the Gröbner basis/elimination theory methods with homotopy continuation methods. A term order is, in a natural way [7], a prescription for a deformation from the system of interest represented by a fiber over a general point of $\mathbb{C}$ to the special degenerate fiber over 0 defined by monomials. Calculations over the special fiber are "lifted" to the general fiber. In homotopy continuation, we also use a deformation over $\mathbb{C}$, but the system we start from is, with probability one, a "general" fiber, and the system we are interested in is the special fiber that we degenerate to.

We refer to [48] for a continuation method to deal with solution manifolds. Rather than discussing curve tracing techniques, we here focus on homotopy methods, that is on methods to embed the given problem into a family of systems. Traditional homotopies on projective space give points on each connected component, e.g., [46, Theorem 7], but the points do not have to be generic and no information is given about dimensions or degrees of components. This precludes further processing of the points.

3. AN EMBEDDING OF A POLYNOMIAL SYSTEM

Throughout this paper we work with algebraic functions. This allows the possibility of using rational functions, and not just polynomials. This extra flexibility will be needed in a sequel, where, even though we start with a system of polynomials on $\mathbb{C}^n$, it becomes necessary to work with rational

functions on a Zariski open set of an associated Euclidean space. We assume in what follows that locally we have the same number of equations as unknowns. A procedure for reducing to this case is presented in [54].

Given a system of algebraic functions

$$f(x) := \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \tag{6}$$

on a connected algebraic manifold X of dimension n embedded into a Euclidean space $\mathbb{C}^A$ we have the following basic embedding into a family of systems. First we restrict n linear functions on $\mathbb{C}^A$ to X. Thus for j from 1 to n we have

$$L_j(x) := a_j + a_{j,1}x_1 + \cdots + a_{j,A}x_A, \tag{7}$$

where x_i is the restriction of the ith coordinate function of $\mathbb{C}^A$ to X. By abuse of notation we let $L_j \in \mathbb{C}^{A+1}$ denote

$$(a_j \quad a_{j,1} \quad \cdots \quad a_{j,A}). \tag{8}$$

We fix linear coordinates $z_1, ..., z_n$ on a complex Euclidean space $\mathbb{C}^n$. Then we have the system of equations

$$\mathscr{E}_i(f)(x, z, \Lambda_1, ..., \Lambda_i, L_1, ..., L_i) := \begin{bmatrix} f_1(x) + \sum_{j=1}^{i} \lambda_{1,j} z_j \\ \vdots \\ f_n(x) + \sum_{j=1}^{i} \lambda_{n,j} z_j \\ a_1 + a_{1,1}x_1 + \cdots + a_{1,A}x_A + z_1 \\ \vdots \\ a_i + a_{i,1}x_i + \cdots + a_{i,A}x_A + z_i \end{bmatrix}, \tag{9}$$

where we let

$$\Lambda_i := \begin{bmatrix} \lambda_{1,i} \\ \vdots \\ \lambda_{n,i} \end{bmatrix}. \tag{10}$$

We often refer to $\mathscr{E}_n(f)(x, z_1, ..., z_n, \Lambda_1, ..., \Lambda_n, L_1, ..., L_n)$ by $\mathscr{E}_n$ or $\mathscr{E}_n(f)$. Further we let $\mathscr{E}_i$ or $\mathscr{E}_i(f)$ denote $\mathscr{E}_i(f)(x, z_1, ..., z_i, \Lambda_1, ..., \Lambda_i, L_1, ..., L_i)$ on $\mathbb{C}^{n+i}$. Note that $\mathscr{E}_0$ is just f and that the solutions $(x, z_1, ..., z_i) \in X \times \mathbb{C}^i$ of

$$\mathscr{E}_i(x, z_1, ..., z_i, \Lambda_1, ..., \Lambda_i, L_1, ..., L_i) = 0 \tag{11}$$

are naturally identified with the solutions $(x, z_1, ..., z_i, 0, ..., 0) \in X \times \mathbb{C}^n$ of the system

$$\mathscr{E}_n(x, z_1, ..., z_n, \Lambda_1, ..., \Lambda_i, 0, ..., 0, L_1, ..., L_i, 0, ..., 0) = 0 \tag{12}$$

We let Y denote the space $\mathbb{C}^{n \times (A+1)} \times \mathbb{C}^{n \times n}$ of parameters

$$\begin{bmatrix} a_1 & a_{1,1} & \cdots & a_{1,A} & \lambda_{1,1} & \cdots & \lambda_{n,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_n & a_{n,1} & \cdots & a_{n,A} & \lambda_{1,n} & \cdots & \lambda_{n,n} \end{bmatrix} \in \mathbb{C}^{n \times (A+1)} \times \mathbb{C}^{n \times n} \tag{13}$$

for these systems. We have used the transpose of the $\lambda_{i,j}$ for convenience in describing the stratification $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ of the space Y given by defining $Y_n := Y$; and Y_i, for $i = 0, ..., n-1$, as the subset of Y with the coordinates

$$\begin{bmatrix} a_{i+1} & a_{i+1,1} & \cdots & a_{i+1,A} & \lambda_{1,i+1} & \cdots & \lambda_{n,i+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n,1} & \cdots & a_{n,A} & \lambda_{1,n} & \cdots & \lambda_{n,n} \end{bmatrix}$$

$$\in \mathbb{C}^{(n-i) \times (A+1)} \times \mathbb{C}^{(n-i) \times n} \tag{14}$$

set equal to 0. Thus using the identification (12) above, we can regard Y_i as the parameter space of the system of equations

$$\mathscr{E}_i(x, z_1, ..., z_i, \Lambda_1, ..., \Lambda_i, L_1, ..., L_i) = 0. \tag{15}$$

We consider homotopies H_i, for i from n to 1 and t from 1 to 0, defined by

$$H_i(x, z_1, ..., z_i, t) := \begin{bmatrix} f_1(x) + \sum_{j=1}^{i-1} \lambda_{1,j} z_j + t\lambda_{1,i} z_i \\ \vdots \\ f_n(x) + \sum_{j=1}^{i-1} \lambda_{n,j} z_j + t\lambda_{n,i} z_i \\ L_1(x) + z_1 \\ \vdots \\ L_{i-1}(x) + z_{i-1} \\ tL_i(x) + z_i \end{bmatrix}, \tag{16}$$

with the convention that if $i = 1$, we mean

$$H_1(x, z_1, t) := \begin{bmatrix} f_1(x) + t\lambda_{1,1} z_1 \\ \vdots \\ f_n(x) + t\lambda_{n,1} z_1 \\ tL_1(x) + z_1 \end{bmatrix}. \tag{17}$$

Thus $H_i(x, z_1, ..., z_i, 1) = \mathscr{E}_i$ and $H_i(x, z_1, ..., z_i, 0) = 0$ is

$$\begin{cases} \mathscr{E}_{i-1}(x, z_1, ..., z_{i-1}) = 0 \\ z_i = 0. \end{cases} \tag{18}$$

Note that using this convention, H_i can be rewritten as $t\mathscr{E}_i + (1-t)\binom{\mathscr{E}_{i-1}}{z_i}$.

LEMMA 2. *There is a nonempty Zariski open set U of points*

$$\begin{bmatrix} a_1 & a_{1,1} & \cdots & a_{1,A} & \lambda_{1,1} & \cdots & \lambda_{n,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n,1} & \cdots & a_{n,A} & \lambda_{1,n} & \cdots & \lambda_{n,n} \end{bmatrix} \in \mathbb{C}^{n \times (A+1)} \times \mathbb{C}^{n \times n} \tag{19}$$

such that for each $i = 1, ..., n$,

1. *the solutions of $\mathscr{E}_i(x, z_1, ..., z_i) = 0$ with $(z_1, ..., z_i) \neq 0$ are isolated and nonsingular*;

2. *given any irreducible component W of $f^{-1}(0)$ of dimension i, the set of isolated solutions of $\mathscr{E}_i(x, z_1, ..., z_i) = 0$ with $(z_1, ..., z_i) = 0$, contain* $\deg(W_{\text{red}})$ *generic points of W_{red}, where W_{red} is the reduction of W; and*

3. *the solutions of* $\mathscr{E}_i(x, z_1, ..., z_i) = 0$ *with* $(z_1, ..., z_i) \neq 0$ *are the same as the solutions of* $\mathscr{E}_i(x, z_1, ..., z_i) = 0$ *with* $z_i \neq 0$.

Proof. For a fixed $i \in \{1, ..., n\}$ consider the system of equations

$$\begin{cases} \sum_{j=1}^{n} \alpha_{1,j} f_j + \sum_{j=1}^{i} \beta_{1,j} z_j = 0 \\ \vdots \\ \sum_{j=1}^{n} \alpha_{n,j} f_j + \sum_{j=1}^{i} \beta_{n,j} z_j = 0 \\ \gamma_1 + \sum_{j=1}^{A} \gamma_{1,j} x_j + \sum_{j=1}^{i} \delta_{1,j} z_j = 0 \\ \vdots \\ \gamma_i + \sum_{j=1}^{A} \gamma_{i,j} x_j + \sum_{j=1}^{i} \delta_{i,j} z_j = 0. \end{cases} \tag{20}$$

By Bertini's theorem (see Section 5 for a convenient form of this result), there is a nonempty Zariski open set U of

$$(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times i} \times \mathbb{C}^{i \times (A+1)} \times \mathbb{C}^{i \times i}, \tag{21}$$

where

$$\alpha := \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{bmatrix}; \qquad \beta := \begin{bmatrix} \beta_{1,1} & \cdots & \beta_{1,i} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \cdots & \beta_{n,i} \end{bmatrix} \tag{22}$$

and

$$\gamma := \begin{bmatrix} \gamma_1 & \gamma_{1,1} & \cdots & \gamma_{1,A} \\ \vdots & \vdots & \iddots & \vdots \\ \gamma_i & \gamma_{i,1} & \cdots & \gamma_{i,A} \end{bmatrix}; \qquad \delta := \begin{bmatrix} \delta_{1,1} & \cdots & \delta_{1,i} \\ \vdots & \iddots & \vdots \\ \delta_{i,1} & \cdots & \delta_{i,i} \end{bmatrix} \tag{23}$$

such that, if not empty, the zero set Z of the above system of equations on $X \times \mathbb{C}^i$ minus the set of common zeroes of $f_1, ..., f_n, z_1, ..., z_i$ is smooth and of dimension 0. Left multiplying this $n+i$ vector of equations with an invertible $(n+i) \times (n+i)$ matrix G of the form

$$\begin{bmatrix} \mathscr{A} & 0 \\ 0 & \mathscr{B} \end{bmatrix}, \tag{24}$$

where $\mathscr{A}$ is an $n \times n$ matrix and $\mathscr{B}$ is a $i \times i$ matrix, results in the equivalent system of equations

$$\begin{aligned} \mathscr{A} \cdot \alpha \cdot f + \mathscr{A} \cdot \beta \cdot \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix} &= 0 \\ \mathscr{B} \cdot \gamma \cdot \begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_A \end{bmatrix} + \mathscr{B} \cdot \delta \cdot \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix} &= 0. \end{aligned} \tag{25}$$

Thus we can assume that the set V is invariant under this action by the matrices G. Thus we have a nonempty Zariski open set U of

$$\begin{bmatrix} a_1 & a_{1,1} & \cdots & a_{1,A} & \lambda_{1,1} & \cdots & \lambda_{n,1} \\ \vdots & \vdots & \ddots & \vdots & \iddots & \vdots & \vdots \\ a_n & a_{n,1} & \cdots & a_{n,A} & \lambda_{1,n} & \cdots & \lambda_{n,n} \end{bmatrix} \in \mathbb{C}^{n \times (A+1)} \times \mathbb{C}^{n \times n} \tag{26}$$

such that for each i the equivalent system is of the form

$$\begin{cases} f_1(x) + \sum_{i=1}^{i} \lambda_{1,i} z_i = 0 \\ \qquad \vdots \qquad\qquad \vdots \\ f_n(x) + \sum_{i=1}^{i} \lambda_{n,i} z_i = 0 \\ a_1 + \sum_{j=1}^{A} a_{1,j} x_j + z_1 = 0 \\ \qquad \vdots \qquad\qquad \vdots \\ a_i + \sum_{j=1}^{A} a_{i,j} x_j + z_i = 0 \end{cases} \tag{27}$$

and has smooth nonsingular zeroes when $(z_1, ..., z_i) \neq 0$. Thus we have the first assertion of the lemma.

To see the second assertion of the lemma, note that the solutions of $\mathscr{E}_i = 0$ with $(z_1, ..., z_i) = 0$ are naturally identified with the solutions of the system

$$\begin{cases} f_1(x) = 0 \\ \vdots \quad \vdots \\ f_n(x) = 0 \\ a_1 + \sum_{j=1}^{A} a_{1,j} x_j = 0 \\ \vdots \quad \vdots \\ a_i + \sum_{j=1}^{A} a_{i,j} x_j = 0 \end{cases} \tag{28}$$

The second assertion follows now from the Algorithm in [54, Sect. 3.1].

We prove the third assertion of the lemma by induction on i. If $i=1$, then it is a tautology that the solutions of $\mathscr{E}_i(x, z_1, ..., z_i)=0$ with $(z_1, ..., z_i) \neq 0$ are the same as the solutions of $\mathscr{E}_i(x, z_1, ..., z_i)=0$ with $z_i \neq 0$.

So we can assume that the result is true for $k<i$ where $i>1$. Note that a solution of $\mathscr{E}_i=0$ with $z_i=0$ but $(z_1, ..., z_i) \neq 0$ is a solution of the system

$$\begin{cases} \mathscr{E}_{i-1}(x, z_1, ..., z_{i-1}) = 0 \\ L_i(x) = 0 \\ z_i = 0 \\ \vdots \\ z_n = 0. \end{cases} \tag{29}$$

Since the solutions of $\mathscr{E}_{i-1}=0$ with $z_{i-1} \neq 0$ are isolated and nonsingular, a generic choice of $L_i(x)$ will not be zero on any of the solutions. But this means that the solutions of $\mathscr{E}_i=0$ for generic $L_i(x)$ will have no solutions with $z_i=0$. ∎

Note that if we choose y generically in Y, then we have chosen the associated y_i generically in Y_i for each i from n to 1. Thus we can assume that with an initial generic choice of parameters y, the behavior for each of the systems $\mathscr{E}_i=0$ is the behavior we expect with a generic choice of parameters on Y_i. One minor point remains. Given a generic choice of parameters $y \in Y$, generic behavior might not occur for the homotopy H_i with $t \in (0, 1]$. This is easily dealt with by a trick of Morgan and Sommese [44–46].

Assume that we have chosen the parameters y for the systems in the nonempty Zariski open set U of

$$\begin{bmatrix} a_1 & a_{1,1} & \cdots & a_{1,A} & \lambda_{1,1} & \cdots & \lambda_{n,1} \\ \vdots & \vdots & \ddots & \vdots & \iddots & \vdots & \vdots \\ a_n & a_{n,1} & \cdots & a_{n,A} & \lambda_{1,n} & \cdots & \lambda_{n,n} \end{bmatrix} \in \mathbb{C}^{n\times(A+1)} \times \mathbb{C}^{n\times n} \tag{30}$$

We want our homotopies $H_i(x, z_1, ..., z_i, t) = 0$ to define algebraic sets that are flat over a Zariski open set containing $(0, 1]$. Numerically this means that we want to have generic behavior for each $t \in (0, 1]$, i.e., the same number of isolated points, no components that do not correspond to components in any other fiber. Exploiting generic flatness in this way is the underlying approach of the work of Morgan and Sommese [44–46], to which we refer for more details. Since for all but a finite number of values of $t \in \mathbb{C}$ generic behavior occurs, we can conclude that given a set of parameters $y \in U$ as above, the homotopy

$$H_{\eta,i}(x, z_1, ..., z_i, t) := (x, z_1, ..., z_i, \eta t)$$
$$= \begin{bmatrix} f_1(x) + \sum_{j=1}^{i-1} \lambda_{1,j} z_j + t\eta\lambda_{1,i} z_i \\ \vdots \\ f_n(x) + \sum_{j=1}^{i-1} \lambda_{n,j} z_j + t\eta\lambda_{n,i} z_i \\ L_1(x) + z_1 \\ \vdots \\ L_{i-1}(x) + z_{i-1} \\ t\eta L_i(x) + z_i \end{bmatrix} \tag{31}$$

defines an algebraic set flat over a Zariski open set of $\mathbb{C}$ containing $(0, 1]$, for all but a finite set of $\eta \in \mathbb{C}$ with $|\eta| = 1$. Using openness of flatness, we can absorb the η into the parameters we use. We still have a dense open set of general parameters, but the set is only Zariski open in the underlying real algebraic structure.

By Lemma 2 if the L_i, Λ_i are chosen randomly, then with probability one, $\mathscr{E}_n(x, z) = 0$ has a solution with $z = 0$ only if $f(x)$ is identically zero on $\mathbb{C}^n$.

Recall from the introduction that for i from n to 1,

1. $\mathscr{Z}_i$ denotes the solutions to $\mathscr{E}_i = 0$ with $z_i \neq 0$; and

2. $\mathscr{X}_{i-1}$ denote the limits with $z_{i-1} = 0$ of the paths of the homotopy $H_i(t)$, from $t = 1$ to $t = 0$, starting at points of $\mathscr{Z}_i$. By convention, the condition $z_{i-1} = 0$ is empty when $i = 1$.

THEOREM 3. *Let f be as above. Assume that f is not identically zero and that the Λ_i are chosen generically. If i is the largest integer with $\mathscr{X}_i$ nonempty, then the dimension of $f^{-1}(0)$ is i. Moreover given any irreducible component W of $f^{-1}(0)$ of dimension i, then, the finite set, $\mathscr{X}_i$ contains* $\deg(W_{\text{red}})$ *generic points of W_{red}, where W_{red} is the reduction of W. The multiplicities of any of the points of $\mathscr{X}_i$ that lie on W are equal, and are greater than or equal to the multiplicity of W; and are equal to one if and only if W is reduced.*

Thus our algorithm achieves the same numerical goal of the algorithm of [54], but much more efficiently. As one piece of evidence of its optimality, note that as a consequence, we obtain the classical upper bound [27, 12.3.1] that $\sum_{i\in I}\mu_i \deg(W_i) \leqslant d_1 \cdots d_n$, where, $f_1, \ldots, f_n$ is a system of n polynomials on $\mathbb{C}^n$; d_j is the total degree of f_j; the irreducible components of the reduction of the zero set of the system is $\sum_{i\in I} W_i$; and μ_i is the multiplicity of W_i as a component of the zero set.

Proof of Theorem 3. We use the notation of the proof of Lemma 2. By Lemma 2, it follows that there is a nonempty Zariski open set of points $y \in Y$, such that for each i from n to 1, all elements of the set $\mathscr{Z}_i$ of solutions of $\mathscr{E}_i(x, z_1, \ldots, z_i) = 0$ with $z_i \neq 0$ are nonsingular and isolated. Thus for a dense set (Zariski open in the underlying real algebraic structure of $y \in Y$) the paths over $t \in (0, 1]$ are smooth if they start at nonsingular isolated solutions of $\mathscr{E}_i = 0$. By Lemma 6 and our random choice of $y \in Y$, each nonsingular isolated solutions $(x^*, z_1^*, \ldots, z_{i-1}^*, 0)$ of $\mathscr{E}_{i-1} = 0$ must start from a nonsingular isolated solution $(x', z_1', \ldots, z_i')$ of $\mathscr{E}_i = 0$.

If $z_i' = 0$ for a solution, then $(z_1', \ldots, z_i') = 0$ by Lemma 2. Thus $f(x') = 0$ and hence $H_i(x', 0, \ldots, 0, t) = 0$ identically in t; and hence $(x^*, z_1^*, \ldots, z_{i-1}^*, 0) = (x', 0, \ldots, 0)$. Thus we know from Lemma 2 that for a Zariski open set of Y_i the solutions of the system $\mathscr{E}_i = 0$ with $z_i = 0$ (and hence $(z_1, \ldots, z_i) = 0$) are on irreducible components of the algebraic set $f(x) = 0$ of dimension i. Thus $(x^*, z_1^*, \ldots, z_{i-1}^*, 0) = (x', 0, \ldots, 0)$ is on an irreducible component of the algebraic set of points with $\mathscr{E}_{i-1} = 0$, which has dimension $\geqslant 1$. This contradicts $(x^*, z_1^*, \ldots, z_{i-1}^*)$ being an isolated nonsingular solution of $\mathscr{E}_{i-1} = 0$. Thus the isolated nonsingular solutions of $\mathscr{E}_{i-1} = 0$ must be endpoints of paths of homotopy H_i starting at points of $\mathscr{Z}_i$.

Using Lemma 6 the above argument can be modified to show that isolated, but possibly singular, solutions of $(x^*, z_1^*, \ldots, z_{i-1}^*, 0)$ of $\mathscr{E}_{i-1} = 0$ must start from a nonsingular isolated solution $(x', z_1', \ldots, z_i')$ of $\mathscr{E}_i = 0$ with $(z_1', \ldots, z_i') \neq 0$ (and hence by Lemma 2 with $z_i' \neq 0$). ∎

The property of a generic system of the form $\mathscr{E}_n(f)$ that it has isolated nonsingular solutions is useful. In this direction, see [38].

4. A WORKED OUT EXAMPLE

Consider the polynomial system $f(x)=0$ and start system $g(x)=0$.

$$f(x)=\begin{cases} x_1^2x_2=0 \\ x_1^2(x_2^2+x_1)=0 \end{cases} \qquad g(x)=\begin{cases} x_1^3-c_1=0 \\ x_2^4-c_2=0 \end{cases} \tag{32}$$

To solve $f(x)=0$ we trace $D=3\times 4=12$ solution paths starting at the solutions of $g(x)=0$. One path diverges to infinity, three paths converge to $(0, 0)$, and the remaining eight paths end at the solution component $x_1=0$, for some $x_2\neq 0$. The condition numbers at the end of the paths do not allow us to decide which solution is isolated.

To embed $f(x)=0$ we take a random hyperplane $L(x)=a_0+a_1x_1+a_2x_2=0$ and choose two random complex constants, λ_1 and λ_2:

$$\mathscr{E}_1(x, z)=\begin{cases} x_1^2x_2+\lambda_1 z=0 \\ x_1^2(x_2^2+x_1)+\lambda_2 z=0 \\ a_0+a_1x_1+a_2x_2+z=0. \end{cases} \tag{33}$$

We can solve this system by tracing $D=12$ solution paths, using a standard linear homotopy with the equations $g(x)=0$, $z-1=0$ as start system, with $g(x)=0$ as in (32). Five paths diverge to infinity. Two paths go to the same solution with $z=0$, which reveals the degree of the solution component $x_1^2=0$. Note that geometrically this component corresponds to a pair of lines. The five remaining paths go to regular solutions with $z\neq 0$.

To compute the possible remaining isolated solutions, we trace five solution paths starting at the five regular solutions of the system (33). In going with t from 1 to 0, we use the homotopy

$$H_1(x, z, t)=t\left(\begin{cases} x_1^2x_2+\lambda_1 z=0 \\ x_1^2(x_2^2+x_1)+\lambda_2 z=0 \\ a_0+a_1x_1+a_2x_2+z=0 \end{cases}\right)+(1-t)\left(\begin{cases} x_1^2x_2=0 \\ x_1^2(x_2^2+x_1)=0 \\ z=0 \end{cases}\right) \tag{34}$$

Three paths converge to $(0, 0)$, two paths go to solutions on the component with $x_1=0$ and $x_2\neq 0$. End games are still needed to decide whether the solutions are isolated.

With our new method, 17 solution paths instead of 24 were traced, as 24 would have been the number of paths with an iteration of the procedure in [54].

Polyhedral root counting methods provide a generically sharp root count for polynomial systems. In particular, the mixed volume of the Newton polytopes of the system equals the number of roots in $(\mathbb{C}^*)^n$, $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$, for a system with generic coefficients. When the system has only few monomials with nonzero coefficients, then the mixed volume provides a much lower root count than the Bézout bounds based on the degrees of the polynomials.

For our type of applications, the distinction between solutions with $z=0$ and $z\neq 0$ is instrumental in identifying components of solutions. This difference does depend on the values of the coefficients of the original system and is not neglected by the ordinary mixed volume. So we will not miss any solutions with $z=0$, but we may miss solution components for which some $x_i=0$. Fortunately, extensions of the polyhedral methods that allow to count and compute all affine roots (that is in $\mathbb{C}^n$ instead of $(\mathbb{C}^*)^n$) are covered amply in the literature (see [24, 28, 35, 39, 49, 51, 53]). The key idea [39] is to add a random constant to every equation to shift the roots with zero components away from the coordinate axes. In removing these constants by continuation, all affine roots lie at the end of some path that starts at a root in $(\mathbb{C}^*)^n$.

Consider again the polynomial system $f(x)=0$ in (32). Because of the first equation we immediately see that there cannot be any solution with all components different from zero. The direct application of polyhedral methods does not yield anything, since the mixed volume for $f(x)=0$ equals zero. With affine polyhedral methods, we consider the system

$$f^{(0)}(x) = \begin{cases} x_1^2 x_2 + \gamma_1 = 0 \\ x_1^2(x_2^2 + x_1) + \gamma_2 = 0, \end{cases} \tag{35}$$

where γ_1 and γ_2 are randomly chosen complex numbers. Here the mixed volume equals five. Note the difference with the total degree $D=12$. Letting the γ's go to zero, three of the five paths converge to the origin, and the other two paths go to other solutions on the component.

To obtain information about the components, with polyhedral methods we consider the embedding

$$\mathscr{E}_1(x,z) = \begin{cases} x_1^2 x_2 + \gamma_1 + \lambda_1 z = 0 \\ x_1^2(x_2^2 + x_1) + \gamma_2 + \lambda_2 z = 0 \\ a_0 + a_1 x_1 + a_2 x_2 + z = 0. \end{cases} \tag{36}$$

This system has mixed volume equal to seven. We use this as start system to solve it with $\gamma_1=0$ and $\gamma_2=0$, following the paths that start at the seven

solution paths of $\mathscr{E}_1(x, z) = 0$. From the seven paths, five paths go to solutions with $z \neq 0$, and the other two paths go to the component ending with $z = 0$.

Observe again the gain in efficiency compared to the procedure in [54]. We now have to trace only 12 instead of 24 paths.

5. BERTINI'S THEOREM AND A LOCAL EXTENSION THEOREM

Here is a weak, but convenient form of Bertini's theorem, e.g., Fulton [27, Example 12.1.11]. For a further discussion of Bertini theorems, see also [8, Sect. 1.7].

THEOREM 4 (Bertini). *Let X be an algebraic manifold of dimension n, e.g., complex Euclidean space or complex projective space. Let $\mathscr{L}_1, ..., \mathscr{L}_n$ be line bundles on X. For i from 1 to n, let $\{s_{i,j} \mid j = 1, ..., r_i\}$ be a set of sections of $\mathscr{L}_i$. Let B_i denote the set of common zeroes of the sections $\{s_{i,j} \mid 1 \leqslant j \leqslant r_i\}$; and let $B := \bigcup_i B_i$. Then given general real or complex numbers $\{\lambda_{i,j} \mid j = 1, ..., r_i;\ i = 1, ..., n\}$, all solutions on $X - B$, of the system of equations*

$$\begin{cases} \sum_{j=1}^{r_1} \lambda_{1,j} s_{1,j} = 0 \\ \qquad\qquad \vdots \\ \sum_{j=1}^{r_n} \lambda_{n,j} s_{n,j} = 0 \end{cases} \tag{37}$$

are isolated and nonsingular.

In the proof of Theorem 3, we need an extension theorem, giving conditions ensuring that if we have a system of equations $f(x, y) = 0$ depending on parameters y with a multiplicity μ isolated solution x^* of the system for some point y^* in the parameter space, then there is a neighborhood U of x^* such that for points y near y^* in the parameter space, there are μ solutions counting multiplicities of the system $f(x, y) = 0$. Special cases are well known in the context of all polynomial systems, but we do not know a general reference in the numerical analysis literature. This sort of result is standard for algebraic geometers in the algebraic context, or within the German school of several complex variables in the complex analytic context. So we are simply explaining why this sort of result follows immediately from standard results in these fields. We work locally. All open sets are in the usual Euclidean topology, i.e., not in the Zariski topology.

A possibly nonreduced complex analytic space, $\mathscr{X}$, is said to be a local complete intersection if given any point $x \in \mathscr{X}$, there is a set $\mathscr{U} \subset \mathscr{X}$, open in the usual complex topology, that contains x, and such that

1. there is an embedding $\phi: \mathscr{U} \to B$ of $\mathscr{U}$ into an open ball in $B \subset \mathbb{C}^N$ for some $N := n + m > 0$;

2. the ideal of the complex space $\phi(\mathscr{U})$ is defined by holomorphic functions $g_1, ..., g_n$; and

3. the dimension of the maximal dimensional irreducible component of $\mathscr{X}$ through x is m.

Local complete intersections are very well behaved. One elementary, but important fact about them is that all the irreducible components of $\mathscr{X}$ through a given $x \in \mathscr{X}$ are equal dimensional. Local complete intersections are among the simplest examples after manifolds of spaces with Cohen–Macaulay local rings. Cohen–Macaulay local rings are discussed in many places, e.g., [20, 26].

We record the following less elementary fact, which we will use in Lemma 6. It holds equally for algebraic spaces and algebraic morphisms. Recall that a continuous map is said to be proper if the inverse image of any compact set is compact.

LEMMA 5. *Let $\pi: X \to Y$ be a proper surjective holomorphic map with zero dimensional fibers from a possibly nonreduced complex analytic space X with Cohen–Macaulay local rings, e.g., a local complete intersection, onto a complex manifold Y. Then π is flat.*

Proof. Since X has Cohen–Macaulay local rings, all irreducible components are equal dimensional. Thus, since π has zero dimensional fibers and is surjective, π is open. Thus it follows from Fischer [26, Proposition on p. 158] that π is flat. ∎

Let X be a connected n-dimensional complex manifold. Let Y be a connected m-dimensional complex manifold. Let

$$f(x, y) = \begin{bmatrix} f_1(x, y) \\ \vdots \\ f_n(x, y) \end{bmatrix} = 0 \tag{38}$$

be a system of n holomorphic functions. Let x^* be an isolated solution of the system $f(x, y^*) = 0$ for a fixed value $y^* \in Y$, i.e., assume that there is an open set $\mathscr{O} \subset X$ containing x^* with x^* the only solution of $f(x, y^*) = 0$

on $\mathcal{O}$. Assume that the multiplicity of x^* is μ. This number, which is 1 exactly when the Jacobian of $f(x, y^*)$ is invertible at x^*, is equal to

$$\dim_{\mathbb{C}} \mathcal{O}_{X|x^*}/\mathcal{J}(f_1(x, y^*), ..., f_n(x, y^*)), \tag{39}$$

where $\mathcal{O}_{X|x^*}$ is the local ring of convergent power series on X centered at the point x^*, and $\mathcal{J}(f_1(x, y^*), ..., f_n(x, y^*))$ is the ideal in $\mathcal{O}_{X|x^*}$ generated by the functions $f_1(x, y^*), ..., f_n(x, y^*)$.

LEMMA 6 (Local extension lemma). *Let X, Y, f, x^*, and y^* be as above. There are open neighborhoods U of $x^* \in X$ and V of $y^* \in Y$ such that for any $y \in V$ there exist μ isolated solutions (counting multiplicities) of $f(x, y) = 0$ on U.*

Proof. Let $\dim_{\{(x^*, y^*)\}} Z$ denote the dimension of an analytic set $Z \subset X \times Y$ at the point (x^*, y^*). Let X' denote the zero set of $f_1, ..., f_n$ on $X \times Y$. Since there are n functions, the dimension of each irreducible component of X' is at least m, and in particular

$$\dim_{(x^*, y^*)} X' \geqslant m. \tag{40}$$

Since x^* is an isolated solution of $f(x^*, y) = 0$ on $X \times \{y^*\}$, we have that $\dim_{(x^*, y^*)} (X \times \{y^*\}) \cap X' = 0$. Since $X \times Y$ is smooth, we have

$$\begin{aligned} \dim_{(x^*, y^*)} (X \times \{y^*\}) \cap X' &\geqslant \dim_{(x^*, y^*)} (X \times \{y^*\}) \\ &\quad + \dim_{(x^*, y^*)} X' - \dim_{(x^*, y^*)} X \times Y \\ &= n + \dim_{(x^*, y^*)} X' - n - m. \end{aligned}$$

Thus we conclude that

$$\dim_{(x^*, y^*)} X' = m. \tag{41}$$

Thus the union X'' of the components of X' passing through (x^*, y^*) has pure dimension m. It follows, e.g., use Gunning [32, Theorem 16], that there are neighborhoods U of $x^* \in X$ and V of $y^* \in Y$ such that the projection π of $X'' \cap (U \times V)$ to V is proper and finite. By shrinking U and V we can assume that $X'' \cap (U \times V) = X' \cap (U \times V)$ and that $X' \cap (U \times \{y^*\}) = (x^*, y^*)$.

Now $X' \cap (U \times V)$ is a local complete intersection, and since the map $\pi_{X' \cap (U \times V)}: X' \cap (U \times V) \to V$ is proper with finite fibers, we conclude by Lemma 5, that this map is flat. Thus the direct image $\mathscr{E} := \pi_*(\mathcal{O}_{X' \cap (U \times V)})$ of the structure sheaf of X' is locally free, e.g., Fischer [26,

Proposition 3.13]. On $U \times V$, X' is defined by the functions $f_i(x, y) = 0$, for $i = 1, ..., n$. By definition, at a point $(x', y') \in (U \times V) \cap X'$ we have $\mathcal{O}_{X' \mid \{(x', y')\}}$ is

$$\mathcal{O}_{(U \times V) \mid \{(x', y')\}} / \mathcal{J}(f_1(x, y), ..., f_n(x, y)), \tag{42}$$

where $\mathcal{O}_{(U \times V) \mid \{(x', y')\}}$ is the local ring of convergent power series on $U \times V$ centered at $\{(x', y')\}$ and $\mathcal{J}(f_1(x, y), ..., f_n(x, y))$ denotes the ideal in $\mathcal{O}_{(U \times V) \mid \{(x', y')\}}$ generated by $f_1, ..., f_n$.

The statement that $\mathscr{E}$ is locally free, is equivalent to ranks of $\mathscr{E}$ at different points of V being equal. The rank of $\mathscr{E}$ at a point $y' \in V$ is by definition

$$\dim_{\mathbb{C}} \mathscr{E}_{y'} := \mathscr{E} \Big/ \left(\mathfrak{m}_{y'} \underset{\mathcal{O}_{Y \mid y'}}{\otimes} \mathscr{E} \right), \tag{43}$$

where $\mathfrak{m}_{y'}$ is the maximal ideal generated $\mathcal{O}_{Y \mid y'}$ consisting of convergent power series vanishing at y'. Thus comparing to the definition of the multiplicity of the solution x^* of $f(x, y^*)$ being μ, we see that the rank of $\mathscr{E}$ is μ. Unwinding the definition of π_*, $\mathscr{E}_{y'}$ for any fixed $y' \in V$ is the direct sum of the modules

$$\mathcal{O}_{X \mid x'} / \mathcal{J}(f_1(x, y'), ..., f_n(x, y')), \tag{44}$$

with index set the set of distinct points $x' \in U$ with $f(x', y') = 0$. This proves the assertion of the lemma. ∎

6. APPLICATIONS AND COMPUTATIONAL EXPERIENCES

We have conducted a systematic set of experiments with PHCpack [57] on three case studies of familiar polynomial systems. Little symbolic manipulation of polynomials is needed to set up homotopies derived from the embedding presented in this paper.

To avoid wasting paper, we have omitted the algebraic formulations of the systems, which can either be found electronically in the database maintained on the web sites of the second author, or can be consulted in the cited literature. Although the systems are academic examples and only interesting for benchmarking purposes, we try to indicate the relevance to their application fields.

Unless stated otherwise, all reported timings concern a 450 MHz Intel Pentium machine with 1 Gb of main memory and 1 Gb swap space,

running Debian GNU/Linux. We observe that dealing with components of solutions is a much harder problem than just approximating the isolated solutions.

With a faster computer one can attack larger problems, but also the quality of the software matters. Concerning this latter aspect we want to point out that no special-purpose software has been developed that exploits the structure of the embedded systems. The continuation could for instance go faster if one eliminated explicitly some variables using the linear hyperplanes that were introduced in the slicing.

6.1. *A Planar Four-Bar Mechanism*

Four-bar mechanisms are ubiquitous in mechanical design. The 4-variable polynomial system that was derived and solved in [47] has total degree 256 and lowest multi-homogeneous Bézout bound 96. The mixed volume equals 80, avoiding the calculation of zero component solutions. There is a solution component of dimension two, with sum of degrees equal to two. The lowest multi-homogeneous Bézout bound of $\mathscr{E}_2$ equals 240 whereas the mixed volume equals 96.

We summarize the computational experiments in Table 1, whose format goes as follows. The cascade of homotopies starts with solving the start system g by polyhedral methods since mixed volumes are sharper than the bounds based on the degrees. To solve the system $\mathscr{E}_k$ we must trace # paths solution curves. This number is partitioned into ones on the component ($z=0$), regular finite solutions ($z\neq 0$), and diverging ones ($\rightarrow\infty$). Observe the take over of the ($z\neq 0$)-solutions to the next row. The number in the column with heading $z\neq 0$ on the $\mathscr{E}_0$-row is the number of isolated solutions of the original system. In the last column we list cpu times.

The information in Table 1 is useful to make some rough comparisons with other homotopy methods. For instance, if we are only interested in the isolated roots, the cost would be of the same magnitude as on the $\mathscr{E}_0$-row. To compare with the procedure of Sommese and Wampler [54],

TABLE 1

The Number of Paths and Timings for a Planar Four-Bar Mechanism

System	Number of paths	$z=0$	$z\neq 0$	$\rightarrow\infty$	cpu time
g, start	96	0	96	0	10 s 880 ms
$\mathscr{E}_2$	96	2	68	26	4 min 47s 830 ms
$\mathscr{E}_1$	68	12	56	0	18 s 370 ms
$\mathscr{E}_0$	56	20	36	0	11 s 10 ms
Total	316	34	256	26	5 min 28 s 90 ms

we could add up the entries in the columns with headings $z = 0$ and $\to \infty$ to the # paths in the next row, simulating the process of no solution recycling.

We must note that there exists an isotropic formulation of this problem [58] for which the mixed volume is an exact root count for the 36 isolated solutions. The black-box solver of PHC needs only 11s 710ms to solve that system.

6.2. *Constructing Runge-Kutta formulas*

The following system is due to Butcher [17] and was used as test problem in [14]. The system has seven variables and a three-dimensional component. Its total degree is 4608 and mixed volume equals 24. After slicing and embedding thrice, the resulting 10-variable system has mixed volume 247. We summarize the running statistics in Table 2.

6.3. *On Fourier Transforms*: *The Cyclic n-roots Problem*

In [11] an example was given which stems from the problem of finding all "bi-equimodular" vectors $\mathbf{x} \in \mathbb{C}^n$, i.e.: all $\mathbf{x}$ with coordinates of constant absolute value such that the Fourier transform of $\mathbf{x}$ is a vector with coordinates of constant absolute value, see also [9] and [10]. This system was popularized in [18], and is by far the most notorious benchmark problem in polynomial system solving. Other references are [5], [12] and [22].

In [42], the following conjectures of Ralf Fröberg are mentioned. If n has a quadratic divisor, then there are infinitely many solutions. If the number of solutions is finite, then this number equals all possible combinations of $n-1$ elements from a set of $2n-2$ elements. Uffe Haagerup [33] has proven that for n prime, the number of roots is always finite, and equals indeed $(2n-2)!/(n-1)!^2$.

In Table 3 we summarize the results of the embedding for the cyclic 8-roots problem, which has a solution component of dimension one. The

TABLE 2

The Number of Paths and Timings for Butcher's System

System	Number of paths	$z = 0$	$z \neq 0$	$\to \infty$	cpu time
g, start	247	0	247	0	17 min 13 s 770 ms
$\mathscr{E}_3$	247	3	193	51	9 min 57 s 830 ms
$\mathscr{E}_2$	193	15	161	17	1 min 59 s 470 ms
$\mathscr{E}_1$	161	11	72	78	4 min 57 s 600 ms
$\mathscr{E}_0$	72	0	4	68	2 min 24 s 650 ms
Total	920	29	677	214	34 min 32 s 550 ms

TABLE 3

The Number of Paths and Timings for Cyclic 8-Roots

System	Number of paths	$z=0$	$z\neq 0$	$\to\infty$	cpu time
g, start	4176	0	4176	0	1 h 30 min 26 s 800 ms
$\mathscr{E}_1$	4176	144	3975	57	1 h 10 min 3 s 930 ms
$\mathscr{E}_0$	3975	495	1152	2328	7 h 7 min 55 s 580 ms
Total	12327	639	9303	2385	9 h 48 min 26 s 310 ms

original system has mixed volume equal to 2560. The embedded system has mixed volume 4176. See Table 3 for the summary of the runs. We remark that if one is only interested in the isolated roots, one only has to trace 2560 solution paths.

Before developing the embedding method, we attacked the cyclic 8-roots problem first with the slicing method of Sommese and Wampler. The computation of generic points on the solution component required the tracing of 10,940 solution paths (which takes now 4176 paths). For this task, the black-box solver of PHC needed 4 days and nights, 15 h 6 min 49 s 392 ms cpu time on a SUN workstation. Although the Linux machine we used is somewhat faster than that SUN workstation, it certainly does not speed up things 50 times!

We end this section with another illustration on the importance of having good polynomial equations for the same problem. Thanks to a trick of John Canny, described in [21], there exists a reduced version of the cyclic n-roots problem. This version reduces the number of roots by eight and yields a significant saving in the path tracking. For instance, the mixed volume of the reduced cyclic 8-roots problem equals 320, instead of 2560 for the original problem. Table 4 summarizes the computational experiments.

TABLE 4

The Number of Paths and Timings for Reduced Cyclic 8-Roots

System	Number of paths	$z=0$	$z\neq 0$	$\to\infty$	cpu time
g, start	775	0	775	0	10 min 1 s 860 ms
$\mathscr{E}_1$	775	18	744	13	11 min 23 s 680 ms
$\mathscr{E}_0$	744	445	144	155	14 min 52 s 340 ms
Total	2294	463	1663	168	36 min 17 s 860 ms

Finally, this reduction also allows us to treat the cyclic 9-roots system that has a two-dimensional component of solutions. The sum of the degrees of the reduced cyclic 9-roots problem equals two. The mixed volume of embedded system $\mathscr{E}_2$ equals 4044, computed in cpu time 2 h 48 min 4 s 320 ms. The polyhedral continuation for the start system requires 1 h 4 m 5 s 130 ms and the path following to solve $\mathscr{E}_2$ takes 2 h 15 min 4 s 54 ms. In total, the black-box solver of PHCpack needs 6 h 7 min 45 s 270 ms to solve $\mathscr{E}_2$.

7. CONCLUSIONS AND FUTURE DIRECTIONS

In the paper we have presented an embedding of polynomial systems to compute generic points on components of solutions. The homotopies recycle solutions while moving down to the isolated solutions. The major advantage compared to the method in [54] is that fewer solution paths diverge. We provide practical evidence for the efficiency of the embedding technique.

Besides the reported progress on homotopy methods, we want to point out that our embedding technique might be of independent interest, in the sense that it allows us to focus on one component of a particular dimension. The main novelty of the embedding consists in the fact that the solutions of the embedded system that do not lie on the focused component provide meaningful information about the components of lower dimension. We hope that besides homotopy methods, other solvers for polynomial systems could benefit from our embedding technique.

An important problem is to measure the amount of randomness that is needed to guarantee the success of our algorithms. Allowing singular solutions, taking into account the numerical precision being used, the "problem dependent" system numbers, e.g., degrees, Bézout numbers, number of variables, some sort of compactification, etc. the task is to quantify the actual probabilities involved for the measure zero events that occur in solving polynomial systems by numerical continuation.

It is sometimes said that every scientific problem solved generates three other questions. Here we briefly indicate three different directions for future research. First of all, the embedding leads to higher-dimensional systems with special structure. It would be worthwhile to exploit this structure to obtain a more efficient mixed-volume computation to alleviate the complexity of the first stage. Secondly, the computed generic points on the components are an interesting starting point for future computational explorations of the solution components. Note that continuation is a very natural tool to scan the component while wiggling the added hyperplanes. Lastly, in the final stage of the cascade of embedded homotopies, it remains

difficult to separate solution paths converging to components from other possibly ill-conditioned isolated solutions. End games that produce certificates in the form of a random hyperplane through a nonisolated solution will have to be developed.

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