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Cross-shaped and degenerate singularities in an unstable elliptic free boundary problem

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Abstract

We investigate singular and degenerate behavior of solutions of the unstable free boundary problem

$$\Delta u = -\chi_{\{u>0\}}.$$

First, we construct a solution that is not of class $C^{1,1}$ and whose free boundary consists of four arcs meeting in a *cross*-shaped singularity. This solution is completely unstable/repulsive from above and below which would make it hard to get by the usual methods, and even numerics is nontrivial. We also show existence of a degenerate solution. This answers two of the open questions in the paper [R. Monneau, G.S. Weiss, An unstable elliptic free boundary problem arising in solid combustion, http://arxiv.org/abs/math.AP/0507315, submitted for publication].

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1. Introduction

We will investigate singular and degenerate behavior of solutions of the unstable elliptic free boundary problem

$$\Delta u = -\chi_{\{u>0\}} \quad \text{in } \Omega. \tag{1.1}$$

The problem (1.1) is related to traveling wave solutions in solid combustion with ignition temperature (see the introduction of [7] for more details).

An equation similar to (1.1) arises in the composite membrane problem (see [1,3,4]). Another application is the shape of self-gravitating rotating fluids describing stars (see [2, Eq. (1.26)]).

This problem has been investigated by R. Monneau and G.S. Weiss in [7]. Their main result is that local minimisers of the energy

$$\int_{\Omega} |\nabla u|^2 - 2\max(u, 0)$$

are $C^{1,1}$ and that their free boundaries are locally analytic. They also establish partial regularity for *second order nondegenerate* solutions of (1.1) (cf. Definition 3.1). More precisely they show that the singular set has Hausdorff dimension less than or equal to $n-2$, and that in two dimensions the free boundary consists close to singular points of four Lipschitz graphs meeting at right angles. However they left open the question of the existence of cross-shaped singular points and of degenerate singularities (cf. [7, Sections 9 and 10]).

In this paper we will construct both singular points where the free boundary consists of four arcs meeting in a cross (see Corollary 4.2 and Fig. 1) and solutions that are degenerate of second order at a free boundary point (see Corollary 4.4). At this time we do not know whether the shape of the singularity is that of an asterisk or a product of even higher disconnectivity (see Fig. 2).

In particular, the cross-example is a counter-example to regularity of the solution since the solution is not of class $C^{1,1}$.

In [7] it has been shown that the second variation of the energy takes the value $-\infty$ at the function $x_1^2 - x_2^2$. That means that the cross-solution is completely unstable/repulsive. Moreover it cannot be approximated from above or below. This makes it hard to construct it by methods like the implicit function theorem or comparison methods.

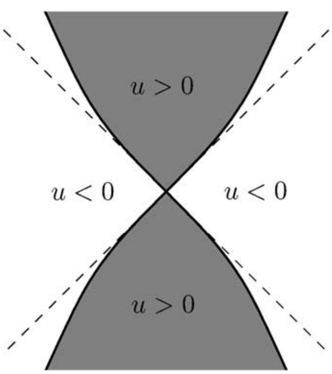

Fig. 1. A cross-shaped singularity.

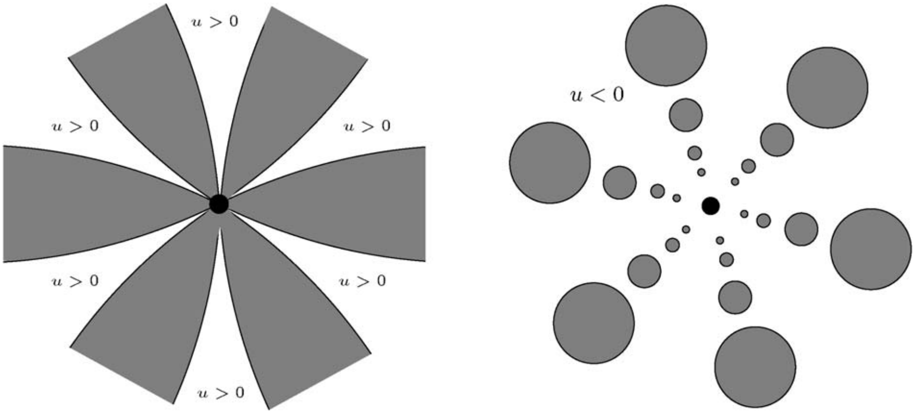

Fig. 2. Asterisk-shaped singularity or pulse accumulation?

Our approach is simple. We construct an operator T such that each fixed point of T, *when adding a certain constant*, satisfies Eq. (1.1) *and the origin is a point of the* 0*-level set*! By reflection and results from [7] it is then possible to show that the origin is nondegenerate of second order and to obtain the cross.

The construction of degenerate solutions is similar but simpler.

2. Notation

Throughout this article $\mathbf{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$. We define e_i as the ith unit vector in $\mathbf{R}^n$, and $B_r(x^0)$ will denote the open n-dimensional ball of center x^0, radius r and volume $r^n\omega_n$. When not specified, x^0 is assumed to be 0. We shall often use abbreviations for inverse images like $\{u > 0\} := \{x \in \Omega\colon\ u(x) > 0\}$, $\{x_n > 0\} := \{x \in \mathbf{R}^n\colon\ x_n > 0\}$, etc. and occasionally we shall employ the decomposition $x = (x_1, \dots, x_n)$ of a vector $x \in \mathbf{R}^n$. When considering a set A, χ_A shall stand for the characteristic function of A, while ν shall typically denote the outward normal to a given boundary.

3. Preliminaries

In this section we state some of the definitions and tools from [7].

Definition 3.1 *(Nondegeneracy).* Let u be a solution of (1.1) in Ω, satisfying at $x^0 \in \Omega$

$$\liminf_{r \to 0} r^{-2} \left(r^{1-n} \int\limits_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} > 0. \tag{3.1}$$

Then we call u "nondegenerate of second order at x^0." We call u "nondegenerate of second order" if it is nondegenerate of second order at each point in Ω.

Remark 3.2. In [7, Section 3] it has been shown that the maximal solution and each local energy minimiser are nondegenerate of second order.

A powerful tool, that we will use in Corollary 4.2, is the monotonicity formula introduced in [8] by one of the authors for a class of semilinear free boundary problems. For the sake of completeness let us state the unstable case here:

Theorem 3.3 *(Monotonicity formula). Suppose that u is a solution of* (1.1) *in Ω and that $B_\delta(x^0) \subset \Omega$. Then for all $0 < \rho < \sigma < \delta$ the function*

$$\Phi_{x^0}(r) := r^{-n-2} \int\limits_{B_r(x^0)} \big(|\nabla u|^2 - 2\max(u,0)\big) - 2r^{-n-3} \int\limits_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1},$$

defined in $(0, \delta)$, satisfies the monotonicity formula

$$\Phi_{x^0}(\sigma) - \Phi_{x^0}(\rho) = \int\limits_\rho^\sigma r^{-n-2} \int\limits_{\partial B_r(x^0)} 2\left(\nabla u \cdot \nu - 2\frac{u}{r}\right)^2 d\mathcal{H}^{n-1}\, dr \geqslant 0.$$

The following proposition has been proven in [7, Section 5].

Proposition 3.4 *(Classification of blow-up limits with fixed center). Let u be a solution of* (1.1) *in Ω and let us consider a point $x^0 \in \Omega \cap \{u = 0\} \cap \{\nabla u = 0\}$.*

(1) *In the case $\Phi_{x^0}(0+) = -\infty$, $\lim_{r\to 0} r^{-3-n} \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1} = +\infty$, and for $S(x^0, r) := (r^{1-n} \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1})^{\frac{1}{2}}$ each limit of*

$$\frac{u(x^0 + rx)}{S(x^0, r)}$$

as $r \to 0$ is a homogeneous harmonic polynomial of degree 2.

(2) *In the case $\Phi_{x^0}(0+) \in (-\infty, 0)$,*

$$u_r(x) := \frac{u(x^0 + rx)}{r^2}$$

is bounded in $W^{1,2}(B_1(0))$, and each limit as $r \to 0$ is a homogeneous solution of degree 2.

(3) *Else $\Phi_{x^0}(0+) = 0$, and*

$$\frac{u(x^0 + rx)}{r^2} \to 0 \quad \text{in } W^{1,2}\big(B_1(0)\big) \text{ as } r \to 0.$$

Remark 3.5.

(1) As shown in [7, Lemma 5.2], the case (2) is not possible in two dimensions.
(2) Case (3) is equivalent to u being degenerate of second order at x^0.

4. Main results

From now on we assume the space dimension n to be 2. Let $\pi/\phi_0 \in \mathbf{N}$ and let us define the disk sector $K = K_{\phi_0} = \{r(\cos\phi, \sin\phi)\colon\ 0 < r < 1,\ 0 < \phi < \phi_0\}$. For $g \in C^\alpha(\partial B_1 \cap \partial K)$, $C^\alpha_g(\bar{K})$ will denote the subspace of $C^\alpha(\bar{K})$ consisting of all the functions with boundary values g on $\partial B_1 \cap \partial K$.

Consider now the operator $T = T_{\epsilon,g} : C^\alpha_g(\bar{K}) \to C^\alpha_g(\bar{K})$ defined by

$$\begin{aligned} &\Delta T(u) = -f_\epsilon\big(u - u(0)\big) && \text{in } K,\\ &T(u) = g && \text{on } \partial B_1 \cap \partial K,\\ &\frac{\partial(T(u))}{\partial\nu} = 0 && \text{on } \partial K - \partial B_1; \end{aligned}$$

here $f_\epsilon \in C^\infty(\mathbf{R})$, $f_\epsilon(z) \geqslant \chi_{\{z>0\}}$ in $\mathbf{R}$ and $f_\epsilon \downarrow \chi_{\{z>0\}}$ as $\epsilon \downarrow 0$.

Since there exists for $F \in L^\infty(K)$ a $W^{1,2}(K)$-solution v of

$$\begin{aligned} &\Delta v = F && \text{in } K,\\ &v = g && \text{on } \partial B_1 \cap \partial K,\\ &\frac{\partial v}{\partial \nu} = 0 && \text{on } \partial K - \partial B_1, \end{aligned}$$

we obtain after even reflection a $W^{1,2}(B_1)$-function that solves $\Delta v = F$ in $B_1 - \{0\}$, where F means the reflected function defined on B_1. As the origin is a set of vanishing capacity (cf. [5]), v is a weak solution of $\Delta v = F$ in B_1. Applying the regularity theory for elliptic equations (see, for example, [6, Lemma 9.29]), we see that T is for small α a continuous compact operator from $C^\alpha_g(\bar{K})$ into itself, and that

$$\big\|T_{\epsilon,g}(w)\big\|_{C^\alpha(\bar{K})} \leqslant C \quad \text{for every } \sigma \in [0,1] \text{ and every solution } w \text{ of } \Delta w = -\sigma f_\epsilon\big(w - w(0)\big),$$

where C is a constant depending only on g.

From Schauder's fixed point theorem (see, for example, [6, Chapter 11]) we infer that $T_{\epsilon,g}$ has a fixed point $u_\epsilon \in C^\alpha_g(\bar{K}) \cap \{\|\cdot\|_{C^\alpha(\bar{K})} \leqslant C\}$. Alternatively, we could also show existence of a fixed point in a class of symmetric functions.

Reflecting and applying L^p-estimates we obtain a sequence $\epsilon_m \to 0$ such that the reflected $u_{\epsilon_m} - u_{\epsilon_m}(0) \to u$ strongly in $C^{1,\beta}(\overline{B_{1-\delta}})$ and weakly in $W^{2,p}(B_{1-\delta})$ for each $\delta \in (0,1)$ as $m \to \infty$. At a.e. point of $\{u > 0\} \cup \{u < 0\}$, u satisfies the equation $\Delta u = -\chi_{\{u>0\}}$. At a.e. point of $\{u = 0\}$, the weak second derivatives of the $W^{2,2}$-function u are 0, so that we obtain:

Proposition 4.1 *(Existence of a fixed point). For each $g \in C^\alpha(\partial B_1 \cap \partial K)$ there exists a constant κ such that the boundary value problem*

$$\begin{aligned} &\Delta u = -\chi_{\{u>0\}} && \textit{in } K,\\ &u = g - \kappa && \textit{on } \partial B_1 \cap \partial K,\\ &\frac{\partial u}{\partial \nu} = 0 && \textit{on } \partial K - \partial B_1 \end{aligned}$$

has a solution $u \in \bigcap_{\delta\in(0,1)} C^{1,\beta}(\bar{K} \cap \overline{B_{1-\delta}})$ such that $u(0) = 0$.

We will use Proposition 4.1 to prove the existence of singular and degenerate solutions:

Corollary 4.2 *(Construction of a cross-shaped singularity). There exists a solution u of*

$$\Delta u = -\chi_{\{u>0\}} \quad \text{in } B_1$$

that is not of class $C^{1,1}$, such that each limit of

$$\frac{u(rx)}{S(0,r)}$$

as $r \to 0$ is after rotation the function $(x_1^2 - x_2^2)/\|x_1^2 - x_2^2\|_{L^2(\partial B_1(0))}$.

Proof. By Proposition 4.1 there exists for each $M \in \mathbf{R} - \{0\}$ a constant $\kappa \in \mathbf{R}$ and a solution in $K_{\pi/2}$ with boundary values $g = M(x_1^2 - x_2^2) - \kappa$ on $\partial B_1 \cap \partial K_{\pi/2}$ satisfying the homogeneous Neumann boundary condition on $\partial K_{\pi/2} - \partial B_1$. Using the homogeneous Neumann boundary condition and the fact that $u \in C^{1,\beta}(\overline{K_{\pi/2} \cap B_{1-\delta}})$ we can reflect this solution twice at the coordinate axes to obtain a solution in the unit ball B_1, called again u.

Also by Proposition 4.1, we know that $u(0) = 0$. Thus $u(0) = 0$ and $\nabla u(0) = 0$ so that Proposition 3.4 applies. What remains to be done is to exclude case (3) of Proposition 3.4 (see Remark 3.5(1)). That done, it follows from the statement in case (1) that u is not of class $C^{1,1}$.

To this end we use the monotonicity formula Theorem 3.3. If $\lim_{r\to 0} \Phi_0(r) = 0$, then $\Phi_0(r) \geqslant 0$ for all $r > 0$. Therefore we only need to show that $\Phi_0(1) < 0$.

For $h = M(x_1^2 - x_2^2)$ and $g = h$ let us write $u = v + h - \kappa$: the function v satisfies

$$\Delta v = \Delta u \quad \text{in } B_1 \quad \text{and} \quad v = 0 \quad \text{on } \partial B_1.$$

Notice that $-1 \leqslant \Delta v \leqslant 0$ implies that $0 < v < C_1$ and $|\nabla v| < C_1$ where C_1 is a universal constant. In particular C_1 is independent of M. We also know that $\kappa = v(0) \in (0, C_1)$ since $u(0) = 0$. Now we calculate the energy $\Phi_0(1)$ of u:

$$\begin{aligned}
\Phi_0(1) &= \int_{B_1} |\nabla u|^2 - 2u^+ - 2\int_{\partial B_1} u^2\, d\mathcal{H}^1 \\
&= \int_{B_1} \left|\nabla(v+h)\right|^2 - 2(v+h-\kappa)^+ - 2\int_{\partial B_1} (v+h-\kappa)^2\, d\mathcal{H}^1 \\
&= \int_{B_1} |\nabla v|^2 + 2\nabla v\cdot\nabla h + |\nabla h|^2 - 2(v+h-\kappa)^+ - 2\int_{\partial B_1} (h-\kappa)^2\, d\mathcal{H}^1,
\end{aligned}$$

where we have used that κ is a constant and that $v = 0$ on ∂B_1. Integrating by parts and using the specific form of h shows that

$$\Phi_0(1) = \int_{B_1} |\nabla v|^2 - 2(v+h-\kappa)^+ - 2\int_{\partial B_1} \kappa^2\, d\mathcal{H}^1$$

$$< \int_{B_1} |\nabla v|^2 - 2(v + h - \kappa)^+ < \int_{B_1} C_1^2 - 2(h - C_1)^+$$

$$= \int_{B_1} C_1^2 - 2\bigl(M\bigl(x_1^2 - x_2^2\bigr) - C_1\bigr)^+.$$

The last integral is negative if M is large. We have thus shown that $\Phi_0(1) < 0$ for sufficiently large M. $\square$

Remark 4.3. To calculate the just obtained solution numerically would—because of the severe instability—not be easy.

The next corollary establishes the existence of degenerate solutions of second order.

Corollary 4.4 *(Construction of a degenerate point). There exists a nontrivial solution u of*

$$\Delta u = -\chi_{\{u>0\}} \quad \text{in } B_1$$

that is degenerate of second order at the origin.

Proof. This is also a direct consequence of Proposition 4.1. The proposition yields a solution in $K_{\pi/4}$ with boundary data $\cos(4\phi) - \kappa$ on $\partial K_{\pi/4} \cap \partial B_1$. Let us reflect this solution three times to get a solution u in the unit ball B_1. As in the previous corollary $0 = u(0) = |\nabla u(0)|$. We only have to show that u is degenerate of second order. Suppose towards a contradiction that this is not true: then by Remark 3.5(1), case (1) of Proposition 3.4 has to apply. We obtain after a rotation a blow-up limit of the form $(x_1^2 - x_2^2)/\|x_1^2 - x_2^2\|_{L^2(\partial B_1(0))}$. But there is no rotation for which that blow-up limit could be symmetric with respect to the two axes $x_1 = 0$ and $x_1 = x_2$, yielding a contradiction. $\square$

5. Open questions

Concerning the set of degenerate singular points there remains the question whether *large* degenerate singular sets are possible. Also it would be nice to know the precise shape of isolated degenerate singularities, and whether infinite order vanishing is possible or not.

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