Differential equations for principal series Whittaker functions on $SU(2, 2)$

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ABSTRACT

Let $SU(2, 2)$ be the special unitary group of index $(2, 2)$. In this paper, two explicit linear partial differential equations are obtained: one from the Casimir operator and the other from the Schmid operator by taking their radial parts. Whittaker functions belonging to an irreducible principal series representation of $SU(2, 2)$ satisfy the system of differential equations and furthermore this system becomes holonomic when the dimension of a minimal $K$-type of the representation is one or two.

1. INTRODUCTION

A Whittaker model belonging to the principal series is a realization of the principal series representation in an induced representation related to a maximal unipotent subgroup. Its element is called a Whittaker function of principal series. The understanding of Whittaker functions greatly helps us to study automorphic $L$-functions. In [6], Shalika proved the multiplicity-one theorem for archimedean field and established the existence of a global Whittaker model for the general linear group of degree $n$, which helped us to study the $L$-functions associated to the automorphic cusp forms. Though Whittaker models in general is not in existence, Kostant and Vogan showed that the largeness of the representation is equivalent to the existence of the Whittaker model when the group is real semisimple and quasi-split ([3, 8]). Accordingly, in this paper, we deal principal series representations of $SU(2, 2)$, the special unitary group of signature $(2, 2)$, and obtain the differential equations for the Whittaker functions.
A remarkable method of getting differential equations for Whittaker functions is the usage of the Schmid operator. For the discrete series representations of $SU(2,2)$, Yamashita succeeded in the calculation of the differential equations using this method and thus in the realization of the discrete series ([9]). For the symplectic group $Sp(2;\mathbb{R})$, Oda obtained the differential equations also using a similar method, and further found an integral expression of Whittaker functions of the large discrete series ([5]). These differential equations for both groups $SU(2,2)$ and $Sp(2;\mathbb{R})$ are quite similar, which could be explained by the coincidence of the restricted root systems. In [4], on the other hand, Miyazaki and Oda obtained explicit formulae for the radial part of the principal series Whittaker functions of $Sp(2;\mathbb{R})$. Thus, by the same reason, we can expect similar formulae for the principal series Whittaker functions on $SU(2,2)$.

Here is a more precise description of the main results of this paper. Let $\pi$ be an irreducible principal series representation of $G = SU(2,2)$ induced from a character of a minimal parabolic subgroup $P$ of $G$. Let $\eta$ be a unitary character of a unipotent radical $N$ of $P$. The space of intertwining operators is called the space of the (algebraic) Whittaker vectors defined by the representation $\pi$, where $K$ is a maximal compact subgroup of $G$. Let $(\tau, V_\tau)$ be an irreducible representation of $K$ and assume that the contragredient representation $(\tau^*, V_{\tau}^*)$ of $\tau$ appears in $\pi|_K$ with multiplicity one; in this case $\tau^*$ is called a $K$-type of $\pi$. We fix an injection $\iota_{\tau^*}$ of $V_{\tau}^*$ to $H^K_{\pi}(N/G)$. Then, by irreducibility of $\pi$, the mapping $\iota_{\tau^*}$ gives rise to a natural injection

$$\text{Hom}_{[\pi, \kappa]}(H^K_{\pi}, C^{\infty}_\eta(N\setminus G)) \longrightarrow \text{Hom}_K(V_{\tau^*}, C^{\infty}_\eta(N\setminus G)).$$

Given an element $\phi$ in $\text{Hom}_K(V_{\tau^*}, C^{\infty}_\eta(N\setminus G))$, we define the $V_\tau$-valued function $\phi_\tau \in C^{\infty}_{\eta, \tau}(N\setminus G/K)$ (§4.1) by

$$\phi(v^*(g)) = \langle v^*, \phi_\tau(g) \rangle, \quad (v^* \in V_{\tau^*}, g \in G).$$

Here $\langle , \rangle$ is the pairing of $V_{\tau^*}$ and $V_\tau$. Thus, given a Whittaker vector $\Phi_\pi$, we get the function $\Phi_{\pi, \tau}$ called the Whittaker function of $\pi$ with $K$-type $\tau^*$.

We next derive the differential equations for Whittaker functions by utilizing the Schmid operator $\nabla$, which is a mapping from $C^{\infty}_{\eta, \tau}(N\setminus G/K)$ to $C^{\infty}_{\eta, \tau \otimes \text{Ad}}(N\setminus G/K)$ (§6). Let $(\tau', V_{\tau'})$ be an irreducible representation of $K$ that appears in $\tau \otimes \text{Ad}$ as an irreducible constituent and $P'$ the projector to $V_{\tau'}$. By composition, we get a shift operator from $C^{\infty}_{\eta, \tau}(N\setminus G/K)$ to $C^{\infty}_{\eta, \tau'}(N\setminus G/K)$. For any $v'^{\tau'} \in V_{\tau'}^*$, we have

$$\langle v'^{\tau'}, P' \circ \nabla \Phi_{\pi, \tau}(g) \rangle = (\text{const.}) \Phi_\pi(\iota_{(\tau')^*}(v'^{\tau'}))(g), \quad (g \in G)$$

where $\iota_{(\tau')^*}$ is an injection of $V_{(\tau')^*}$. In other words, the shift operator maps a Whittaker function with $K$-type $\tau^*$ to a Whittaker function with $K$-type $\tau^{(\tau')^*}$. Since we can find a series of representations starting from $\tau^*$ and coming back again to $\tau^*$ itself by iteration of the shift operators, we can construct a differ-
ential operator on $C_{\pi_r}^\infty(N\backslash G/K)$. Its radial part defines a differential equation for $\Phi_{\pi_r}$. As we will see in Theorems 4.2 and 4.3, the differential equations obtained in this way are less complicated than the ones obtained by the generators of the center of the universal enveloping algebra $Z(\mathfrak{g}_C)$. Furthermore, the system derived becomes holonomic when the dimension of $K$-type is either one or two. In both cases, the Whittaker functions can be characterized by the two operators: one is the Casimir operator and the other is a shift operator (§6).

The contents of this paper are as follows.

In §2, we fix several notations for $SU(2,2)$ and its Lie algebra. In §3, we recall the basic facts on the $K$-types of the principal series representations of $G$. Giving the definition of principal series representations of $G$, we review the parametrization of the irreducible representations of $K$ and fix the standard bases in these representation spaces. We calculate the multiplicity of $K$-types and specify the standard basis as functions on $K$ in the representation space of the principal series representation. In terms of these explicit realizations, we compute the eigenvalues of shift operators. In §4, recalling the definition of Whittaker vectors and Whittaker functions, we state the main results (Theorems 4.2 and 4.3). Proofs are given in §7. In §5, we calculate the radial part of the Casimir operator; the radial part gives directly one of the differential equations in the main theorems. In §6, we introduce the Schmid operator and construct shift operators between two $K$-types. We also calculate their radial parts when the dimension of $K$-type is one or two. The other differential operator in the main theorems is obtained by composing these operators.

2. THE STRUCTURE OF $SU(2,2)$ AND ITS LIE ALGEBRA

In this section, we fix several notations for the group $SU(2,2)$, its subgroups and the corresponding Lie algebras.

2.1. $SU(2,2)$ and its Lie algebra

Let $G$ be the special unitary group of signature $(2,2)$ defined by

$$G = SU(2,2) = \{ g \in SL_4(\mathbb{C}) \mid {}^t\bar{g}I_{2,2}g = I_{2,2}\}$$

where $I_{2,2} = \text{diag}(1,1,-1,-1)$. We denote by $^t\bar{g}$, $\bar{g}$, and $I_2$, the transpose of the matrix $g$, the complex conjugation of $g$ and the identity matrix of size two, respectively. This group $G$ is semisimple, of hermitian type and of real rank two. A maximal compact subgroup $K$ of $G$ is given by the following:

$$K = S(U(2) \times U(2)) = \left\{ \begin{pmatrix} u & \\ \bar{v} & \end{pmatrix} \mid u,v \in U(2), \det(u)\det(v) = 1 \right\}. $$

Hereafter we use the convention that blank entries in matrices are zero. Note that $G/K$ is a hermitian upper-half plane.

The Lie algebra $\mathfrak{g}$ of $G$ is expressed as follows:

$$\mathfrak{g} = \left\{ X = \begin{pmatrix} X_1 & X_{12} \\ ^t\bar{X}_{12} & X_2 \end{pmatrix} \mid ^t\bar{X}_i = -X_i \ (i = 1,2), \ X_{12} \in M_2(\mathbb{C}), \ tr(X_1 + X_2) = 0 \right\}. $$
It is of real dimension 15. We write its complexification by \( \mathfrak{g}_C = \mathfrak{sl}_4(\mathbb{C}) \).

The Lie algebra of \( K \) is:

\[
\mathfrak{f} = \left\{ X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mid \mathsf{tr} X = 0 \right\}.
\]

We set

\[
\mathfrak{p} = \left\{ X = \begin{pmatrix} X_{12} \\ X_{12} \end{pmatrix} \mid X_{12} \in \mathcal{M}_2(\mathbb{C}) \right\}.
\]

Then we have a Cartan decomposition: \( \mathfrak{g} = \mathfrak{f} + \mathfrak{p} \). We also denote the complexification of \( \mathfrak{f} \) and \( \mathfrak{p} \) by \( \mathfrak{f}_C \) and \( \mathfrak{p}_C \) whose dimensions are 7 and 8, respectively.

Next we describe a restricted root system of \( \mathfrak{g} \). Put

\[
H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

We fix a maximal abelian subalgebra contained in \( \mathfrak{p} \), \( \mathfrak{a} = \{ H_1, H_2 \} \). The Lie group \( A = \exp(\mathfrak{a}) \) is the identity component of a maximal \( \mathbb{R} \)-split torus in \( G \). We identify \( A \) with \( \mathbb{R}^2_{\geq 0} \) by,

\[
A = \{ a = (a_1, a_2) = (e^s, e^t) = \exp(sH_1 + tH_2) \mid s, t \in \mathbb{R} \}.
\]

Choose a basis \( \{ \lambda_1, \lambda_2 \} \) of the dual space \( \mathfrak{a}^* \) of \( \mathfrak{a} \) such that \( \lambda_j(H_j) = \delta_{ij} \). Then we can describe the restricted root system of \( \mathfrak{g} \) associated to \( \mathfrak{a} \) as

\[
\Delta = \Delta(\mathfrak{g}, \mathfrak{a}) = \{ \pm \lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2 \}.
\]

It is of type \( C_2 \). We take a positive root system \( \Delta_+ = \{ \lambda_1 \pm \lambda_2, 2\lambda_1, 2\lambda_2 \} \), and a fundamental root system \( \Delta_{\text{fund}} = \{ \lambda_1 - \lambda_2, 2\lambda_2 \} \). We set,

\[
H_{12} = \text{diag}(\sqrt{-1}, -\sqrt{-1}, 0, 0), \quad H_{34} = \text{diag}(0, 0, \sqrt{-1}, -\sqrt{-1}),
\]

\[
E_1 = \sqrt{-1} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \sqrt{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

\[
E_3 = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad E_4 = \frac{\sqrt{-1}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},
\]

\[
E_5 = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad E_6 = \frac{\sqrt{-1}}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.
\]

Also fix a basis of \( \mathfrak{f}_C \):

\[
(1) \quad h^1 = \left( \begin{array}{c} h \\ 1 \end{array} \right), \quad h^2 = \left( \begin{array}{c} 1 \\ h \end{array} \right), \quad e^1_\pm = \left( \begin{array}{c} e_\pm \\ 1 \end{array} \right), \quad e^2_\pm = \left( \begin{array}{c} 1 \\ e_\pm \end{array} \right), \quad I_{2,2},
\]

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where \( h = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) are 2 \( \times \) 2 matrices. Using this notation, one has \( \mathfrak{g} = c(\alpha) + \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda \) where

\[
\begin{align*}
\mathfrak{g}(\alpha) &= \text{the centralizer of } \alpha = \{H_1, H_2, H_{12} + H_{13}\}_R, \\
\mathfrak{g}_{2\lambda_1} &= \{E_1\}_R, \quad \mathfrak{g}_{2\lambda_2} = \{E_2\}_R, \quad \mathfrak{g}_{\lambda_1 + \lambda_2} = \{E_3, E_4\}_R, \\
\mathfrak{g}_{\lambda_1 - \lambda_2} &= \{E_5, E_6\}_R, \quad \mathfrak{g}_\mu = 'g_\mu = \{X \mid X \in \mathfrak{g}_\mu\}.
\end{align*}
\]

The Weyl group \( W \) with respect to \( (\mathfrak{g}, \alpha) \) is the semi-direct product of the symmetric group of degree 2 and two copies of \( \mathbb{Z}/2\mathbb{Z} \), thus its order is 8.

### 2.2. A minimal parabolic subgroup

Next we consider a minimal parabolic subgroup \( P \) of \( G \) with Langlands decomposition \( P = MAN \). The split component \( A \) is already defined.

Let \( \mathfrak{n} \) be the subalgebra defined by \( \mathfrak{n} = \sum_{\lambda \in \Delta_+} \mathfrak{g}_\lambda \). This is two-step nilpotent, that is, \( [\mathfrak{n}, \mathfrak{n}] \) is the center of \( \mathfrak{n} \). We also find that \( \mathfrak{n} \) is 6-dimensional. The Lie group \( N = \exp(\mathfrak{n}) \) is the unipotent radical of \( P \) and a maximal unipotent subgroup of \( G \);

\[
N = \exp \mathfrak{n} = \left\{ \kappa^{-1} \begin{pmatrix} 1 & \alpha \\ & 1 \\ & & 1 \\ & & -\bar{\alpha} \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar{S} \\ & 1 \\ & & 1 \\ & & & 1 \end{pmatrix} \kappa \mid \alpha \in \mathbb{C}, 'S = S \right\}
\]

where,

\[
\kappa = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -\sqrt{-1} & 1 \\ \sqrt{-1} & 1 \end{pmatrix}.
\]

By definition, a Levi subgroup \( M \) of \( P \) is, \( M = \{e^{\sqrt{-1}\theta} \gamma_j \mid \gamma_j \in G, \theta \in \mathbb{R}, j = \pm 1\} \) with,

\[
\begin{align*}
\gamma &= \text{diag}(1, -1, 1, -1) \in G, \\
I_0 &= \text{diag}(1, -1, 1, -1) \in \mathfrak{g}_C, \\
[e^{\sqrt{-1}\theta}] &= \exp(\sqrt{-1}\theta I_0) = \text{diag}(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}, e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}).
\end{align*}
\]

We denote by \( \mathfrak{m} \) the Lie algebra of \( M \).

### 2.3. Iwasawa decomposition of root vectors

Let \( X_{ij} = \{\delta_{ip}\delta_{jq}\}_{pq} \) and \( H_{ij} = \sqrt{-1}(X_{ii} - X_{jj}) \). Put,

\[
(3) \quad \mathfrak{p}_+ = \{X_{ij} \mid i = 1, 2, j = 3, 4\}_C, \quad \mathfrak{p}_- = \{X_{ij} \mid i = 3, 4, j = 1, 2\}_C.
\]

Then the Lie algebra \( \mathfrak{g}_C \) is decomposed as \( \mathfrak{g}_C = \mathfrak{p}_+ + \mathfrak{t}_C + \mathfrak{p}_- = \mathfrak{n}_C + \mathfrak{a}_C + \mathfrak{t}_C \).

Thus we obtain the following formulae by direct calculation.

\[
X_{13} = \tfrac{1}{2}(\sqrt{-1}E_1 + H_1 + \tfrac{1}{2}(I_{2,2} + h^t - h^2)) = \tfrac{1}{2}(\sqrt{-1}E_1 + H_1 - \sqrt{-1}H_{13}),
\]

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$X_{14} = \frac{1}{2} (E_3 + E_5 + \sqrt{-1}(E_4 - E_6) - 2e_4^2)$,

$X_{23} = \frac{1}{2} (E_3 + E_5 + \sqrt{-1}(E_4 + E_6) + 2e_4)$,

$X_{24} = \frac{1}{2} (\sqrt{-1}E_2 + H_2 + \frac{1}{2}(I_{2,2} - h^1 + h^2))$

$= \frac{1}{2} (\sqrt{-1}E_2 + H_2 - \sqrt{-1}H_{24})$,

$X_{31} = \frac{1}{2} (-\sqrt{-1}E_1 + H_1 - \frac{1}{2}(I_{2,2} + h^1 - h^2))$

$= \frac{1}{2} (-\sqrt{-1}E_1 + H_1 + \sqrt{-1}H_{13})$,

$X_{32} = \frac{1}{2} (E_3 + E_5 - \sqrt{-1}(E_4 + E_6) - 2e_4^2)$,

$X_{41} = \frac{1}{2} (-E_3 + E_5 - \sqrt{-1}(E_4 - E_6) + 2e_4)$,

$X_{42} = \frac{1}{2} (\sqrt{-1}E_2 + H_2 - \frac{1}{2}(I_{2,2} + h^1 + h^2))$

$= \frac{1}{2} (-\sqrt{-1}E_2 + H_2 + \sqrt{-1}H_{24})$.

3. PRINCIPAL SERIES REPRESENTATIONS AND THEIR $\mathcal{K}$-TYPES

In this section, we recall the definition of the principal series representations of $G$ and the parametrization of the irreducible representations of $K$. In addition, we determine the multiplicity of a $\mathcal{K}$-type in a given principal series representation and realize $\mathcal{K}$-types in the representation space by using their matrix coefficients. We fix notation for infinitesimal representations of $\mathfrak{f}$ and the contragredient representations. The tensor products between $\mathcal{K}$-types and the adjoint representation are also given here, to define the Schmid operator mainly. Furthermore, we construct certain operators in $U(\mathfrak{g}_C)$ and $M_2(U(\mathfrak{g}_C))$ closely related to the shift operators considered in §6.

3.1. Principal series representations

We review the definition of the principal series representations of $G$. We begin by constructing characters of the minimal parabolic subgroup $P = MAN$. Let $n$ be an integer and let $\epsilon$ be a character of the group $\{ \pm 1 \}$, which we also identify with an element of $\{ \pm 1 \}$. Define the unitary character of $M$ as follows:

$$\sigma_{n, \epsilon}(e^{\sqrt{-1} \theta} \gamma^j) = \epsilon(-1)^j e^{\sqrt{-1}n\theta}.$$

Denoting by $\rho = 3\lambda_1 + \lambda_2$, we define a (not necessarily unitary) character $e^{\mu + \rho}$ of $A$ by

$$e^{\mu + \rho}(a) = e^{(\mu + \rho)(\log a)}$$

for $\mu \in \alpha_C^*$. We extend it to a character of $AN$ so that the restriction to $N$ is trivial. We get an admissible character of $P$ by tensoring these characters.

Now we define an induced representation of $G$. Let $H^0_x$ be the space of continuous functions on $G$ satisfying the equation,

$$f(pg) = \sigma_{n, \epsilon} \otimes e^{\mu + \rho}(p)f(g)$$

for $p \in P$ and $g \in G$. This space is a pre-Hilbert space endowed with the inner
product \((\phi_1, \phi_2) = \int_K \phi_1(k) \phi_2(k) \, dk\). Then \(\pi = \text{ind}_{\nu}^{G}(\sigma_{n, r} \otimes e^{\mu + \rho})\) acts on \(H^0_\pi\) by right translation:
\[
\pi(g)\phi(x) = \phi(xg), \quad (g, x \in G, \phi \in H^0_\pi).
\]
We remark that \(\pi\) is unitary if and only if \(\mu\) is purely imaginary. We also denote the \(K\)-finite vectors by \(H^K_\pi\).

3.2. Parametrization of irreducible unitary representations of \(K\)

Firstly, we review the parametrization of the finite-dimensional irreducible representations of \(SL_2(\mathbb{C})\). Let \(\{f_1, f_2\}\) be a standard basis of the 2-dimensional vector space \(V = V_1\). Then \(SL_2(\mathbb{C})\) acts on \(V\) by matrix multiplication. We denote the symmetric tensor space by \(V_d = S_d(V)\) regarded as a \((d + 1)\)-dimensional vector space with the basis \(\{f^p_1 \otimes f^q_2\} 0 \leq p \leq d\) where \(f^p_1 \otimes f^q_2\) (symmetric tensor). Here \(V_0 = \mathbb{C}\). We consider \(V_d\) as an \(SL_2(\mathbb{C})\)-module by:
\[
\text{sym}^d(g)(v_1 \otimes v_2 \otimes \cdots \otimes v_d) = g(v_1 \otimes g(v_2 \otimes \cdots \otimes g(v_d).
\]
It is well known that all the finite-dimensional irreducible (polynomial) representations of \(SL_2(\mathbb{C})\) can be obtained in this way. By Weyl's unitary trick, all irreducible unitary representations of \(SU(2)\) are obtained by restriction. The irreducible representations of \(SU(2) \times SU(2) \times \mathbb{C}^{(1)}\) are obtained by tensoring irreducible representations of the simple components.

Let \(V_{rs} = V_r \otimes V_s\) and choose the standard basis of \(V_{rs}\) as
\[
\{f^p_{rs} = f^p_r \otimes f^q_s \mid 0 \leq p \leq r, 0 \leq q \leq s\}.
\]
Note that \(\text{dim}_C V_{rs} = (r + 1)(s + 1)\). Define
\[
(4) \quad \tau_{[r,s,u]}(g_1,g_2;e^{\sqrt{-1}\theta}) = \text{sym}^r(g_1) \otimes \text{sym}^s(g_2) \otimes e^{\sqrt{-1}u\theta}.
\]
Then \(\{\tau_{[r,s,u]}(V_{rs}) \mid r, s \in \mathbb{Z}_{\geq 0}, u \in \mathbb{Z}\}\) gives a complete system of representatives of the unitary dual of \(SU(2) \times SU(2) \times \mathbb{C}^{(1)}\).

Let us consider the representations of \(K\). The group \(SU(2) \times SU(2) \times \mathbb{C}^{(1)}\) can be regarded as a covering of \(K\) by the map \(pr\):
\[
pr(g_1,g_2;e^{\sqrt{-1}\theta}) = \begin{pmatrix} e^{\sqrt{-1}\theta}g_1 & e^{-\sqrt{-1}\theta}g_2 \end{pmatrix}.
\]
Since the kernel of \(pr\) is \(\{\pm(1, 1, 1)\}\), this is actually a two-fold cover. The representation \(\tau_{[r,s,u]}\) induces a representation of \(K\) if and only if the restriction of \(\tau_{[r,s,u]}\) to \(\ker(pr)\) is trivial, hence we can state the following proposition.

**Proposition 3.1.** \(\hat{K} = \{\tau_{[r,s,u]} \mid r, s \in \mathbb{Z}_{\geq 0}, u \in \mathbb{Z}, r + s + u \in 2\mathbb{Z}\}\).

3.3. The multiplicity of \(K\)-types

For the principal series representation \(\pi\) of \(G\), the restriction of \(\pi\) to \(K\) decom-
poses into the Hilbert space sum of irreducible representations of $K$ with finite multiplicities:

$$\pi| _K = \bigoplus_{\tau \in \hat{K}} [\pi| _K : \tau] \tau.$$ 

Here $[\pi| _K : \tau]$ is the multiplicity of $\tau$ in $\pi| _K$. If $[\pi| _K : \tau] \neq 0$, we call $\tau$ a $K$-type of $\pi$. The multiplicity is given by Frobenius reciprocity: (see [1, Theorem 1.14]).

**Lemma 3.2.** Let $\pi = \text{ind}_F^G(\sigma_{n,e} \otimes e^{\mu+\rho})$ and $\tau \in \hat{K}$. Then $[\pi| _K : \tau] = [\tau| _M : \sigma_{n,e}]$.

We proceed to calculate the multiplicity. Let $\tau_d = \tau_{[r,s,u]}$ with $d = [r,s,u]$ and let the representation space be $V_d = V_{r,s}$, given as,

$$V_d = \bigoplus_{0 \leq p \leq r, 0 \leq q \leq s} \mathbb{C}f^{(d)}_{pq}.$$

This can be regarded as a direct sum of 1-dimensional $M$-submodules $\mathbb{C}f^{(d)}_{pq}$.

**Lemma 3.3.** We have,

1. $\tau_d(\gamma)f^{(d)}_{pq} = \sqrt{-1}^{(r-2p+2q-s+u)}f^{(d)}_{pq},$
2. $\tau_d([e^{\sqrt{-1}\theta}])f^{(d)}_{pq} = e^{\sqrt{-1}(2p-r+s)}f^{(d)}_{pq}.$

**Proof.** We see that, by definition,

$$\text{pr}(\text{diag}(-\sqrt{-1}, \sqrt{-1}), \text{diag}(\sqrt{-1}, -\sqrt{-1}); \sqrt{-1}) = \gamma,$$

$$\text{pr}(\text{diag}(e^{\sqrt{-1}\theta}, e^{\sqrt{-1}\theta}), \text{diag}(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}); 1) = [e^{\sqrt{-1}\theta}].$$

Thus, from the values of $\tau_d$ at $\gamma$ and $[e^{\sqrt{-1}\theta}]$, we obtain the lemma. \qed

Here is a multiplicity formula.

**Proposition 3.4.** Let $\pi = \text{ind}_F^G(\sigma_{n,e} \otimes e^{\mu+\rho})$ and let $\tau_d$ be an irreducible representation of $K$ with parameter $d = [r,s,u]$. Assume that,

$$r + s \geq |n| \quad \text{and} \quad -2s + u = n + 1 - \epsilon(-1) \pmod{4}.$$

Then,

$$[\pi| _K : \tau_d] = \frac{1}{4} \{2(r+s) - |r-s| - |n| - ||r-s| - |n||\} + 1.$$

**Proof.** By Lemma 3.2, it is enough to calculate the multiplicity $[\tau_d| _M : \sigma_{n,e}]$.

First, $[\tau_d| _M : \sigma_{n,e}]$ is equal to the number of indices $(p,q)$ which satisfy

$$\tau_d(m)f^{(d)}_{pq} = \sigma(m)f^{(d)}_{pq} \quad (m \in M).$$

By comparing Equation (5) in Lemma 3.3 with the definition of $\sigma(\gamma)$, we get,

$$\sqrt{-1}^{(r-2p+2q-s+u)} = \epsilon(-1).$$

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or equivalently,
\[(10) \quad r - 2p + 2q - s + u \equiv 1 - \epsilon(-1) \pmod{4}.
\]

Similarly, from Equation (6) and the value of \(\sigma([e^{\sqrt{-1}\theta}])\), we have,
\[(11) \quad 2p - r + 2q - s = n.
\]

Adding Equations (10) and (11), we get the necessary condition for the multiplicity to be positive. This is exactly the assumption (7) in the statement. Now, to determine the multiplicity, we count the indices \((p, q)\) satisfying (11).

Assume \(r \geq s\). Then (see Fig. 1),

\[(i) \quad \text{If } n \geq 0 \text{ and } 2r \leq r + s + n, \text{ then,}
\]

\[\begin{align*}
(p, q) &= \left(\frac{r - s + n}{2}, s\right), \left(\frac{r - s + n}{2} + 1, s - 1\right), \ldots, \left(\frac{r - s + n}{2}, 0\right)
\end{align*}
\]
satisfy (11), thus the number of \((p, q)\)'s is \((r + s - n)/2 + 1\).

\[(ii) \quad \text{If } 2s \leq r + s + n < 2r, \text{ then,}
\]

\[\begin{align*}
(p, q) &= \left(\frac{r - s + n}{2}, s\right), \left(\frac{r - s + n}{2} + 1, s - 1\right), \ldots, \left(\frac{r + s + n}{2}, 0\right)
\end{align*}
\]
satisfy (11), thus the number of \((p, q)\)'s is \(s + 1\).

\[(iii) \quad \text{If } n < 0 \text{ and } r + s + n < 2s, \text{ then,}
\]

\[\begin{align*}
(p, q) &= \left(0, \frac{r + s + n}{2}\right), \left(1, \frac{r + s + n}{2} - 1\right), \ldots, \left(\frac{r + s + n}{2}, 0\right)
\end{align*}
\]
satisfy (11), thus the number of \((p, q)\)'s is \((r + s + n)/2 + 1\).

In conclusion, the number of \((p, q)\)'s satisfying (11) is
\[\min \left\{ s + 1, \frac{r + s - |n|}{2} + 1 \right\}.
\]

Similarly, in the case \(r < s\), we get the formula:
\[
\min \left\{ r + 1, \frac{r + s - |n|}{2} + 1 \right\}.
\]
Thus, the multiplicity of \( \tau_d \) in \( \pi \mid_K \) equals
\[
\min \left\{ s + 1, r + 1, \frac{r + s - |n|}{2} + 1 \right\},
\]
which coincides with the right-hand side of (8) in the proposition. \( \square \)

### 3.4. \( K \)-isotypic components of the principal series representations

Let \( \pi = \text{ind}_F^G(\sigma_{n,\epsilon} \otimes e^{\mu + \rho}) \) and let \((\tau_d, V_d) (d = [r, s; u])\) be an irreducible representation of \( K \) which occurs in \( \pi \mid_K = \text{ind}_F^G(\sigma_{n,\epsilon} \otimes e^{\mu + \rho}) \mid_K \). By definition, the \( \tau_d \)-isotypic component \( H_x(\tau_d) \) is isomorphic to a \([\pi \mid_K : \tau_d]\)-tuple direct sum of \( \tau_d \). In this subsection we describe its basis by using matrix coefficients of \( \tau_d \).

To begin with, define the functions \( a_{pq, lm} \) on \( K \) by the following rule:
\[
\tau_d(k) f_{pq}^{(d)}(x) = \sum_{0 \leq l \leq r, 0 \leq m \leq s} a_{pq, lm}^{(d)}(k) f_{lm}^{(d)}(k), \quad (k \in K).
\]

If we rewrite the equation:
\[
\tau_d(xy) f_{pq}^{(d)} = \tau_d(x) \tau_d(y) f_{pq}^{(d)}
\]
in terms of matrix coefficients, we get,
\[
a_{pq, lm}^{(d)}(xy) = \sum_{0 \leq \mu \leq r, 0 \leq \nu \leq s} a_{\mu \nu, lm}^{(d)}(x) a_{pq, \nu \mu}^{(d)}(y)
\]
for all \( x, y \in K \). We also remark that
\[
a_{pq, lm}^{(d)}(1) = \delta_{pq, lm}.
\]

Define the action of \( K \) on these matrix coefficients by
\[
R_k a_{pq, lm}^{(d)}(x) = a_{pq, lm}^{(d)}(xk) \quad (k \in K).
\]

We also define the \( K \)-module generated by \( a_{rs, lm}^{(d)} \) as \( W_{lm}^{(d)} = \{ R_k a_{rs, lm}^{(d)} \mid k \in K \} \). By Equation (12) with \( k \in K \) in place of \( x \), we see that
\[
W_{lm}^{(d)} \subset \bigoplus_{\mu, \nu} \mathbb{C} a_{\mu \nu, lm} \subset L^2(K).
\]

We can easily check that the vector \( a_{rs, lm}^{(d)} \) has the same properties as the highest weight vector \( f_{rs}^{(d)} \). So we have,

**Lemma 3.5.** For any \( 0 \leq l \leq r, 0 \leq m \leq s \), \( W_{lm}^{(d)} \) is \( K \)-isomorphic to \( V_d \).

Let \( \sigma = \sigma_{n,\epsilon} \in \hat{M} \), and put
\[
L^\sigma_{\epsilon}(K) = \{ \phi \in L^2(K) \mid \phi(mk) = \sigma(m) \phi(k) \text{ for all } m \in M, \text{ a.e. } k \in K \}.
\]

Through the restriction, the induced representation \( \pi = \text{ind}_F^G(\sigma_{n,\epsilon} \otimes e^{\mu + \rho}) \) can be realized on \( L^\sigma_{\epsilon}(K) \) by right translation.
Lemma 3.6. Assume \( r + s \geq |n| \) and \( u \equiv 2s + n + 1 - \epsilon(-1) \pmod{4} \). Then,
\[
H_{\pi}(\tau_\delta) = \bigoplus_l W_{l,([r+s+n]/2)-l}
\]
where \( l \) runs over integers satisfying,
\[
\begin{align*}
(r-s+n)/2 \leq l &\leq r, & \text{if } n \geq 0, \quad 2 \max(r,s) \leq r + s + n, \\
(r-s+n)/2 \leq l &\leq (r+s+n)/2, & \text{if } 2s \leq r + s + n < 2r, \\
0 \leq l &\leq r, & \text{if } 2r \leq r + s + n < 2s, \\
0 \leq l &\leq (r+s+n)/2, & \text{if } n < 0, \quad 2 \min(r,s) > r + s + n.
\end{align*}
\]

Proof. The element \( a^{(d)}_{pq,lm} \) belongs to \( L^2_{df}(K) \) if and only if,
\[
a^{(d)}_{pq,lm}(kx) = \sigma(k)a^{(d)}_{pq,lm}(x)
\]
for any \( k \in M \). By Lemma 3.3 and Equation (12), we get
\[
a^{(d)}_{pq,lm}(\gamma x) = \sqrt{-1}^{-1} a^{(d)}_{pq,lm}(x),
\]
\[
a^{(d)}_{pq,lm}([e^{-i\theta}]x) = e^{i\theta} a^{(d)}_{pq,lm}(x).
\]
Therefore as shown in Proposition 3.4, we see that the sum of the \( W_{l,([r+s+n]/2)-l} \)'s satisfying (15) is contained in the \( \tau_d \)-isotypic component of \( \pi \). They exhaust \( H_{\pi}(\tau_\delta) \) since its dimension is \( [\pi : \tau_\delta] \dim V_d \).

Corollary 3.7. Let \( \alpha = \alpha_{n,c} \in \hat{M} \) and assume that \( r + s = |n| \) and \(-2s + u \equiv n + 1 - \epsilon(-1) \pmod{4} \). Then the map defined by
\[
\begin{align*}
\{ f_{pq} &\mapsto a^{(d)}_{pq,rs} (0 \leq p \leq r, 0 \leq q \leq s), & \text{if } n \geq 0, \\
\} f_{pq} &\mapsto a^{(d)}_{pq,00} (0 \leq p \leq r, 0 \leq q \leq s), & \text{if } n < 0.
\end{align*}
\]
is up to a multiple the unique \( K \)-injection into \( L^2_{df}(K) \simeq H_{\pi} \).

Proof. By Proposition 3.4, we have \( [\pi : \tau_\delta] = 1 \) in this case, therefore there is a unique \( l \) such that \( W_{l,([|n|+n)/2)-l} = H_{\pi}(\tau_\delta) \). Lemma 3.6 tells for which \( l \) this is valid.

3.5. Infinitesimal representations of \( K \)

In this subsection, we collect explicit formulae for the action of standard generators of \( f_C \). These will be frequently used later, especially in the calculation of shift operators.

Let \( (\tau_{[r,s,u]}, V_{rs}) \) be a representation of \( K \). By definition (cf. (4)),
\[
\tau_{[r,s,u]} \left( \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \right) = d \ \text{sym}^r(Y_1) \otimes \text{id}_{V_1} + \text{id}_{V_2} \otimes d \ \text{sym}^s(Y_2) \quad (Y_1, Y_2 \in \hat{S}_{l}(\mathbb{C})),
\]
\[
\tau_{[r,s,u]}(I_{2,2}) = u \text{id}_{V_1},
\]

Using the basis of \( f_C \) defined by (1) in §2.1, we have,
Lemma 3.8. Let $\tau = \tau_{[r,s,u]}$ and $f_{pq} = f_{pq}^{(r,s)}$. Then,

\[
\begin{align*}
\tau(h^1)f_{kl} &= (2k - r)f_{kl}, & \tau(h^2)f_{kl} &= (2l - s)f_{kl}, \\
\tau(e^+_{1})f_{kl} &= (r - k)f_{k+1,l}, & \tau(e^2_{2})f_{kl} &= (s - l)f_{k,l+1}, \\
\tau(e^-_{-})f_{kl} &= k f_{k-1,l}, & \tau(e^2_{-})f_{kl} &= l f_{k,l-1}, \\
\tau(I_{2,2})f_{kl} &= u f_{kl}.
\end{align*}
\]

\[
(16)
\]

3.6. Contragredient representations of $K$

Let $(\tau_d^*, V_d^*)$ be the contragredient representation of $(\tau_d, V_d)$. Then, using the action of $h^1, h^2$ and $I_{2,2}$, we know that $f_{00}^*$ is a highest weight vector with highest weight $[r,s,-u]$. Therefore $\tau^*$ is isomorphic to $\tau_{[r,s,-u]}$; an isomorphism is induced by $f_{00}^* \mapsto f_{00}^{(r,s,-u)}$. Summing up,

**Proposition 3.9.** The contragredient representation $\tau^*$ of $\tau_{[r,s,u]}$ is isomorphic to $\tau_{[r,s,-u]}$. More precisely, let $f_{kl}^*$ be the dual basis of $f_{kl}^{([r,s,u])}$, i.e., $(f_{kl}^* f_{ij}^{([r,s,u])}) = \delta_{kl} \delta_{ij}$. Then the correspondence between the bases,

\[
f_{kl}^* \mapsto (-1)^{k+l} \begin{pmatrix} r \\ k \end{pmatrix} \begin{pmatrix} s \\ l \end{pmatrix} f_{r-k,s-l}^{([r,s,-u])}
\]

determines the unique isomorphism up to a constant multiple.

3.7. Adjoint representation of $K$

In this subsection, we consider the adjoint representation of $K$ restricted to $\mathfrak{p}_C$. Write simply $\text{Ad} = \text{Ad}_{\mathfrak{p}_C}$. Since both $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are $K$-invariant subspaces, $\text{Ad}_{\mathfrak{p}_C}$ decomposes into two representations $\text{Ad}_+ = \text{Ad}_{\mathfrak{p}_+}$ and $\text{Ad}_- = \text{Ad}_{\mathfrak{p}_-}$.

**Proposition 3.10 ([9, §4.2]).** The representation $\text{Ad}_+$ (resp. $\text{Ad}_-$) is irreducible and equivalent to $\tau_{[1,1,2]}$ (resp. $\tau_{[1,1,-2]}$) and the $K$-isomorphism is given by

\[
(17)
\]

We again begin with the case of $SL_2(\mathbb{C})$. If we take an $SL_2(\mathbb{C})$-module $V_r$, it has a decomposition $V_r \cong V_1 \oplus V_{r-1}$ as $SL_2(\mathbb{C})$-module (cf. [9, Lemma 4.1]). Let $P_r^\pm$ be the projectors from $V_r \otimes V_1$ to $V_{r \pm 1}$ defined by:

\[
(18)
\]

These conditions determine $P_r^\pm$ uniquely because they give the action on the highest weight vectors. We have,

**Lemma 3.11.** Let $0 \leq j \leq r$ and $e = 0, 1$. Then,
Proof. The $K$-equivariance of the projectors gives correspondences of other weight vectors. First, considering the weights of each vectors, we can put, for $e = 0, 1$,

$$
P^e_r(f_j^{(r)} \otimes f_e^{(1)}) = \alpha_j^{(e)} f_{j+e}^{(r+1)}, \quad P^{-e}_r(f_j^{(r)} \otimes f_e^{(1)}) = (r \delta_{e1} - j) f_{j-1+e},$$

where $\alpha_j^{(e)}$ and $\beta_j^{(e)}$ are constants. Equations (18) imply that $\alpha_j^{(1)} = 1$, $\alpha_j^{(0)} = 0$, $\beta_j^{(1)} = 0$ and $\beta_j^{(1)} - \beta_j^{(0)} = r + 1$. By applying $e_+$ or $e_-$ on both sides of (20), we have,

$$
(r-j)\alpha_j^{(1)} + \alpha_j = (r+1-j)\alpha_j^{(0)},
$$

$$
(r-j)\beta_j^{(1)} + \beta_j = (j-1)\beta_j^{(0)}.\]

These equations imply that $\alpha_j^{(1)} = 1$, $\beta_j^{(1)} = r-j$ and $\beta_j^{(0)} = -j$ for all $j$. □

We see that $V_{rs} \otimes p_\pm$ decomposes into four irreducible $[f,f]$-submodules $V_{rs \pm 1, s \pm 1}$ by the argument above. We define the projectors

$$
P^{(e_1, e_2)}_{rs} = P^{e_1}_r \otimes P^{e_2}_s : V_{rs} \otimes p_+ \rightarrow V_{r+e_1, s+e_2},$$

$$
P^{(e_1, e_2)}_{rs} = P^{e_1}_r \otimes P^{e_2}_s : V_{rs} \otimes p_- \rightarrow V_{r+e_1, s+e_2},$$

where $e_1, e_2$ are in $\{+, -\}$. Here $x + e_j$ means $x + 1$ if $e_j$ is $+$ and $x - 1$ if $e_j$ is $-$.

Lemma 3.12. Put $P^{(e_1, e_2)}_{rs}$ and $P^{(e_1, e_2)}_{rs}$, then we have,

$$
\begin{align*}
P^{(-,+)}(f_{kl} \otimes X_{13}) &= -P^{(-,+)}(f_{kl} \otimes X_{42}) = (k-r) f_{k,l-1}, \\
P^{(-,+)}(f_{kl} \otimes X_{24}) &= -P^{(-,+)}(f_{kl} \otimes X_{31}) = k(s-l) f_{k-1,l}, \\
P^{(-,+)}(f_{kl} \otimes X_{23}) &= P^{(-,+)}(f_{kl} \otimes X_{41}) = kl f_{k-1,l-1}, \\
P^{(-,+)}(f_{kl} \otimes X_{14}) &= P^{(-,+)}(f_{kl} \otimes X_{32}) = (k-r)(s-l) f_{kl}, \\
P^{(+,-)}(f_{kl} \otimes X_{13}) &= -P^{(+,-)}(f_{kl} \otimes X_{42}) = (-l) f_{k+1,l-1}, \\
P^{(+,-)}(f_{kl} \otimes X_{24}) &= -P^{(+,-)}(f_{kl} \otimes X_{31}) = (l-s) f_{k+1,l}, \\
P^{(+,-)}(f_{kl} \otimes X_{23}) &= P^{(+,-)}(f_{kl} \otimes X_{41}) = (-l) f_{k+1,l-1}, \\
P^{(+,-)}(f_{kl} \otimes X_{14}) &= P^{(+,-)}(f_{kl} \otimes X_{32}) = (l-s) f_{k+1,l}, \\
P^{(-,-)}(f_{kl} \otimes X_{13}) &= -P^{(-,-)}(f_{kl} \otimes X_{42}) = (r-k) f_{kl}, \\
P^{(-,-)}(f_{kl} \otimes X_{24}) &= -P^{(-,-)}(f_{kl} \otimes X_{31}) = k f_{k-1,l+1}, \\
P^{(-,+)}(f_{kl} \otimes X_{23}) &= -P^{(-,+)}(f_{kl} \otimes X_{41}) = (-k) f_{k-1,l}, \\
P^{(-,+)}(f_{kl} \otimes X_{14}) &= P^{(-,+)}(f_{kl} \otimes X_{32}) = (k-r) f_{k,l+1}.
\end{align*}
$$
Proof. By Proposition 3.10 and Lemma 3.11, we readily get these equations. □

3.8. An operator between one-dimensional \(K\)-types

From now on, we consider the principal series representation \(\pi = \text{ind}_F^G(\sigma_n, \epsilon \otimes e^{\mu + \rho})\) as a \((U(\mathfrak{g}_C), K)\)-module and identify \(H^K_\pi\) with \(L^2(K)^K\) through restriction (cf. (14)). We will construct certain elements \(Y\) of degree 2 in \(U(\mathfrak{g}_C)\) such that \(\pi(Y)\) maps each 1-dimensional \(K\)-types to another in \(\pi|_K\).

Put \(d_0 = [0, 0; 0, 0]\), \(d_{\pm 1} = [1, 1; u \pm 2]\) and \(d_{\pm 2} = [0, 0; u \pm 4]\). Write \(\tau_j = \tau_{d_j}\) for simplicity. We assume \(n = 0\) so that \(\pi = \text{ind}_F^G(\sigma_0, \epsilon \otimes e^{\mu + \rho})\) contains 1-dimensional \(K\)-types. Put \(\sigma = \sigma_0,\) and assume that \(u \equiv 1 - \epsilon(-1)\), then \([\pi|_K : \tau_0] = 1,\) \([\pi|_K : \tau_{\pm 1}] = 2\) and \([\pi|_K : \tau_{\pm 2}] = 1\) (cf. Proposition 3.4). We also write \(a_{pq}^{(j)} := a_{pq,00}\) (cf. Corollary 3.7).

Define

\[
\begin{align*}
g_{00}^{(1)}(14) &= X_{23} \cdot a_{00}^{(0)}, \\
g_{10}^{(1)}(14) &= X_{13} \cdot a_{00}^{(0)}, \\
g_{01}^{(1)}(14) &= -X_{24} \cdot a_{00}^{(0)}, \\
g_{11}^{(1)}(14) &= -X_{14} \cdot a_{00}^{(0)}, \\
g_{00}^{(-1)}(14) &= X_{41} \cdot a_{00}^{(0)}, \\
g_{10}^{(-1)}(14) &= -X_{32} \cdot a_{00}^{(0)}, \\
g_{01}^{(-1)}(14) &= X_{31} \cdot a_{00}^{(0)}, \\
g_{11}^{(-1)}(14) &= -X_{32} \cdot a_{00}^{(0)}.
\end{align*}
\]

Proposition 3.13. The space generated by \(g_{ij}^{(1)}\) (resp. \(g_{ij}^{(-1)}\)) \((0 \leq i \leq 1, 0 < j < 1)\), as a \(K\)-module, is isomorphic to \(\tau_1\) (resp. \(\tau_{-1}\)).

Proof. In fact, this space turns out to be isomorphic to \((\text{Ad}_+, \mathfrak{p}_+)\) (resp. \((\text{Ad}_-, \mathfrak{p}_-))\) if we restrict to the commutator \([\mathfrak{f}, \mathfrak{f}]\). We thus get the result by Proposition 3.10. □

Lemma 3.14. We have,

\[
\begin{align*}
g_{00}^{(1)}(14) &= g_{10}^{(1)}(14) = 0, & g_{00}^{(-1)}(14) &= g_{10}^{(-1)}(14) = 0, \\
g_{10}^{(1)}(14) &= (2\mu_1 + u + 6)/4, & g_{01}^{(-1)}(14) &= (2\mu_1 - u + 6)/4, \\
g_{01}^{(1)}(14) &= -(2\mu_2 + u + 2)/4, & g_{10}^{(-1)}(14) &= -(2\mu_2 - u + 2)/4.
\end{align*}
\]

Proof. Using §2.3,

\[
g_{00}^{(1)}(14) = \frac{1}{2}(E_3 + E_5 + \sqrt{-1}(E_4 + E_6) + 2e_+^\cdot) \cdot a_{00}^{(0)}(k)|_{k=14}.
\]

Note that \(X \cdot a_{00}^{(0)}(k) \equiv 0\) for any \(X \in \mathfrak{u}\). Thus we conclude that \(g_{00}^{(1)}(14) = 0\).

Similarly, we get \(g_{10}^{(1)}(14) = 0\).

Next, considering the action of the center of \(\mathfrak{f}\), we have,

\[
g_{10}^{(1)}(14) = \frac{1}{2}(\sqrt{-1}E_1 + H_1 + \frac{1}{2}(I_{2,2} + h^1 - h^2)) \cdot a_{00}^{(0)}(k)|_{k=14} \\
= \frac{1}{2}(H_1 + \frac{1}{2}I_{2,2}) \cdot a_{00}^{(0)}(k)|_{k=14} - (2\mu_1 + u + 6)/4.
\]

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Correspondingly,
\[ g_{01}^{(1)}(14) = -\frac{1}{2}(H_2 + \frac{1}{2} I_{2,2}) \cdot a_{00}^{(0)}(k) \bigg|_{k=14} = -(2\mu_2 + u + 2)/4. \]

By using exactly the same method, we find the relations for the \( g_{kl}^{-1}(14) \).

Define
\[
\begin{align*}
&g^{(2)} = X_{14} \cdot g_{00}^{(1)} + X_{13} \cdot g_{01}^{(1)} - X_{24} \cdot g_{10}^{(1)} - X_{23} \cdot g_{11}^{(1)}, \\
&g^{(-2)} = X_{32} \cdot g_{00}^{(-1)} + X_{31} \cdot g_{01}^{(-1)} - X_{42} \cdot g_{10}^{(-1)} - X_{41} \cdot g_{11}^{(-1)}. 
\end{align*}
\]

We can check that the weight of \( g^{(\pm 2)} \) is \( d_{\pm 2} \). Hence \( g^{(\pm 2)} \) generates a 1-dimensional \( K \)-module isomorphic to \( \tau_{\pm 2} \), respectively.

**Proposition 3.15.** We have,
\[
\begin{align*}
g^{(2)} &= 
\begin{pmatrix}
X_{14} & X_{13} \\
-X_{24} & X_{23}
\end{pmatrix}
\begin{pmatrix}
X_{24} & X_{23} \\
-X_{14} & X_{13}
\end{pmatrix} = -\left( \mu_1 + \frac{u}{2} + 1 \right) \left( \mu_2 + \frac{u}{2} + 1 \right) a_{00}^{(2)}, \\
g^{(-2)} &= 
\begin{pmatrix}
X_{32} & X_{31} \\
-X_{42} & X_{41}
\end{pmatrix}
\begin{pmatrix}
X_{41} & X_{31} \\
-X_{32} & X_{42}
\end{pmatrix} = -\left( \mu_1 - \frac{u}{2} + 1 \right) \left( \mu_2 - \frac{u}{2} + 1 \right) a_{00}^{(-2)}. 
\end{align*}
\]

**Proof.** We calculate the value at 14:
\[
g^{(2)}(14) = -e_{14} \cdot g_{00}^{(1)}(14) + \frac{1}{2} \left( H_1 + \frac{1}{2} (I_{2,2} + h^1 + h^2) \right) \cdot g_{01}^{(1)}(14)
\]
\[
-\frac{1}{2} (H_2 + \frac{1}{2} (I_{2,2} - h^1 + h^2)) \cdot g_{10}^{(1)}(14) - e_{14} \cdot g_{11}^{(1)}(14)
\]
\[
= -g_{01}^{(1)}(14) + \frac{1}{2} \left( \mu_1 + 3 + \frac{1}{2} (u + 2 - 1 - 1) \right) g_{10}^{(1)}(14)
\]
\[
-\frac{1}{2} \left( \mu_2 + \frac{1}{2} (u + 2 - 1 - 1) \right) g_{11}^{(1)}(14)
\]
\[
= -(\mu_1 + u/2 + 1)(\mu_2 + u/2 + 1).
\]

Similarly, we get,
\[
g^{(-2)}(14) = -(\mu_1 - u/2 + 1)(\mu_2 - u/2 + 1).
\]

Equation \( a_{00}^{(\pm 2)}(14) = 1 \) implies the proposition.

**Remark 3.16.** We know that \( \left[ \pi \right]_K : \tau_{\pm 1} = 2 \), that is, there are two independent \( K \)-injections of \( \tau_{\pm 1} \) into the induced representation \( L_2^\sigma(K) \) (cf. Lemma 3.6). Set,

\[
\begin{align*}
&\iota_{\pm 1, 1} : f_{pq}^{(\pm 1)} \longmapsto a_{pq,01}^{(\pm 1)}, \\
&\iota_{\pm 1, 2} : f_{pq}^{(\pm 1)} \longmapsto a_{pq,10}^{(\pm 1)},
\end{align*}
\]

By taking the value at 14 of the functions above, we get
\[
\begin{align*}
\iota_1 &= -\frac{1}{4} (2\mu_2 + u + 2) \mu_{1,1} + \frac{1}{4} (2\mu_1 + u + 6) \mu_{1,2}, \\
\iota_{-1} &= \frac{1}{4} (2\mu_1 + u + 6) \mu_{-1,1} - \frac{1}{4} (2\mu_2 + u + 2) \mu_{-1,2}.
\end{align*}
\]
3.9. An operator between two-dimensional $K$-types

If there are two $K$-types of dimension 2 in $\pi$ and if these are ‘close’, we can construct an element in $M_{2}(U(\mathfrak{g}_{\mathfrak{c}}))$ which maps one to the other.

Put $d=d_{0}=[r,s;u]$, $d_{-1}=[r+1,s-1;u \pm 2]$ and $d_{+2}=[r-1,s+1;u \pm 2]$. For $\pi=\operatorname{ind}_{P}^{G}(\sigma_{n,e} \otimes e^{u+\rho})$, assume that $r+s=n$, $-2s+u=n+1-\epsilon(-1) \pmod{4}$, then $\tau_{0}$ is in $\pi|_{\mathfrak{k}}$ with multiplicity one. Throughout this subsection we assume that all $\tau_{j}$ appear in $\pi|_{\mathfrak{k}}$ (cf. Proposition 3.4). Then, the injection of $\tau_{j}$ into $\pi|_{\mathfrak{k}}$ is unique up to a scalar. Corollary 3.7 says that $a^{(j)}_{pq,rs}$ (resp. $a^{(j)}_{pq,00}$) ($j=0, \pm 1, \pm 2$) are in $L_{\mathfrak{g}}^{2}(K)$ if $n \geq 0$ (resp. $n < 0$).

Define

$$
g^{(1)} = g^{(1)}_{r+1,s-1} = X_{13} \cdot a_{rs}^{(0)} + X_{14} \cdot a_{r,s-1}^{(0)},
$$

$$
g^{(2)} = g^{(2)}_{r-1,s+1} = X_{23} \cdot a_{rs}^{(0)} + X_{14} \cdot a_{r-1,s}^{(0)},
$$

$$
g^{(-1)} = g^{(-1)}_{r+1,s-1} = X_{42} \cdot a_{rs}^{(0)} + X_{32} \cdot a_{r,s-1}^{(0)},
$$

$$
g^{(-2)} = g^{(-2)}_{r-1,s+1} = X_{31} \cdot a_{rs}^{(0)} + X_{32} \cdot a_{r,s-1}^{(0)}.
$$

**Proposition 3.17.** The $K$-module generated by the vector $g^{(j)}$ is isomorphic to $\tau_{j}$ ($j=\pm 1, \pm 2$).

**Proof.** Each vector $g^{(j)}$ has weight $d_{j}$ and is annihilated by $e_{-}^{1}$ and $e_{+}^{2}$. □

Define the other weight vectors as follows:

$$
g^{(1)}_{kl} = \frac{r-k+1}{r+1} (X_{23} \cdot a_{k-1,l+1}^{(0)} + X_{24} \cdot a_{kl}^{(0)}) + \frac{k}{r+1} (X_{13} \cdot a_{k-1,l+1}^{(0)} + X_{14} \cdot a_{k-1,l}^{(0)}),
$$

$$
g^{(2)}_{kl} = \frac{s-l+1}{s+1} (X_{13} \cdot a_{kl}^{(0)} - X_{23} \cdot a_{kl}^{(0)}) + \frac{l}{s+1} (X_{24} \cdot a_{k+1,l-1}^{(0)} - X_{14} \cdot a_{k,l-1}^{(0)}),
$$

$$
g^{(-1)}_{kl} = \frac{r-k+1}{r+1} (X_{31} \cdot a_{kl}^{(0)} - X_{41} \cdot a_{kl}^{(0)}) + \frac{k}{r+1} (X_{42} \cdot a_{k+1,l+1}^{(0)} - X_{32} \cdot a_{k-1,l}^{(0)}),
$$

$$
g^{(-2)}_{kl} = \frac{s-l+1}{s+1} (X_{42} \cdot a_{kl}^{(0)} + X_{41} \cdot a_{kl}^{(0)}) + \frac{l}{s+1} (X_{31} \cdot a_{k+1,l-1}^{(0)} + X_{32} \cdot a_{k-1,l}^{(0)}).
$$

In other words, the $g^{(j)}_{kl}$'s are defined by the following recurrent relation:

$$
g^{(j)}_{kl} = \frac{1}{k+1} e_{-}^{1} \cdot g^{(j)}_{k-1,l+1} - \frac{1}{l+1} e_{+}^{2} \cdot g^{(j)}_{k+1,l-1}.
$$

These imply that an isomorphism between $\{g^{(j)}_{kl}\}$ and $V_{\tau_{j}}$ is given by $g^{(j)}_{kl} \mapsto f^{(j)}_{kl}$.

Our main result in this subsection is:

**Proposition 3.18.** Suppose that $n \geq 0$ (resp. $n < 0$). Assume $u \equiv 2s+n+1-\epsilon(-1) \pmod{4}$. Write $a_{kl}^{(j)} := a_{kl}^{(d_{j})}$ (resp. $a_{kl}^{(j)} := a_{kl}^{(d_{j})}$). Then,

$$
g^{(\pm 1)} = \frac{2\mu_{\delta(\pm 1)} + 2 + r-s \pm u}{4} a_{r+1,s-1}^{(\pm 1)},
$$

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(26) \[ g^{(\pm 2)} = \frac{2\mu_{\delta(\pm 1)} + 2 - r + s \pm u}{4} a_{r-1,s+1}, \]

where \((\delta(1), \delta(-1)) = (1, 2) (\text{resp. } (\delta(1), \delta(-1)) = (2, 1)).\)

**Proof.** It suffices to check the value of each \(g^{(j)}\) (resp. \(g_{00}^{(j)}\)) at \(l_4\) for \(j = \pm 1, \pm 2\) in the case \(n \geq 0\) (resp. \(n < 0\)). We assume that \(n \geq 0\) for clarity. Using \$2.3, we have,

\[
g_{r+1,s-1}^{(1)}(l_4) = X_{13} \cdot a_{r,s}(l_4) + X_{14} \cdot a_{r,s-1}(l_4) = (\frac{1}{4} H_1 + \frac{1}{4} (I_{2,2} + h^1 - h^2)) \cdot a_{r,s}(l_4) + (-e_2^1) \cdot a_{r,s-1}(l_4)
\]

\[= (\frac{1}{4} (\mu_1 + 3) + \frac{1}{4} (u + r - s)) a_{r,s}(l_4) - a_{r,s-1}(l_4)
\]

\[= (2\mu_1 + 2 + r - s + u)/4.\]

Here we also use the fact that \(a_{r,s}, a_{r,s-1}\) are in the image of the \(K\)-type \(\tau_{[r,s,u]}\) and that \(a_{r,s}(l_4) = 1\) by \(13).\)

Accordingly, we can check the values: \(g_{r-1,s+1}^{(1)}(l_4), g_{r+1,s-1}^{(1)}(l_4)\) and \(g_{r-2,s+1}^{(1)}(l_4),\) which give the proposition. \(\Box\)

We now discuss the case when \(\tau_0\) is 2-dimensional; thus consider the special case, \(d_0 = [0, 1; u]\) and \(d_{\pm 1} = [1, 0; u \pm 2].\)

**Corollary 3.19.** Suppose that \(n = 1\) (resp. \(n = -1\)). Assume \(u \equiv -n\epsilon(-1) \bmod 4).\)

Write \(a_{pq}^{(j)} := a_{pq,rs}^{(j)} (\text{resp. } a_{pq,00}^{(j)}).\) Then, we have,

(27) \[\begin{pmatrix} X_{24} & X_{23} \\ X_{14} & X_{13} \end{pmatrix} \begin{pmatrix} a_{00}^{(0)} \\ a_{01}^{(0)} \end{pmatrix} = \frac{(2\mu_{\delta(1)} + u + 1)}{4} \begin{pmatrix} a_{00}^{(1)} \\ a_{10}^{(1)} \end{pmatrix},\]

(28) \[\begin{pmatrix} X_{13} & -X_{23} \\ -X_{14} & X_{24} \end{pmatrix} \begin{pmatrix} a_{00}^{(-1)} \\ a_{10}^{(-1)} \end{pmatrix} = \frac{(2\mu_{\delta(2)} + u - 1)}{4} \begin{pmatrix} a_{00}^{(0)} \\ a_{01}^{(0)} \end{pmatrix},\]

(29) \[\begin{pmatrix} X_{42} & X_{41} \\ X_{32} & X_{31} \end{pmatrix} \begin{pmatrix} a_{00}^{(1)} \\ a_{10}^{(1)} \end{pmatrix} = \frac{(2\mu_{\delta(1)} - u - 1)}{4} \begin{pmatrix} a_{00}^{(0)} \\ a_{01}^{(0)} \end{pmatrix},\]

(30) \[\begin{pmatrix} X_{31} & -X_{41} \\ -X_{32} & X_{42} \end{pmatrix} \begin{pmatrix} a_{00}^{(0)} \\ a_{01}^{(0)} \end{pmatrix} = \frac{(2\mu_{\delta(2)} - u + 1)}{4} \begin{pmatrix} a_{00}^{(-1)} \\ a_{10}^{(-1)} \end{pmatrix},\]

where \((\delta(1), \delta(2)) = (1, 2) (\text{resp. } (\delta(1), \delta(2)) = (2, 1)).\)

**4. Differential Equations for Whittaker Functions**

In this section, we first recall the definition of Whittaker vectors and Whittaker functions. Then we state our main results in terms of 'radial part'.

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4.1. Whittaker functions

Choose a unitary character \( \eta \) of \( N \). We use the same notation \( \eta \) to denote its derivative. Let \( C_\eta^\infty(N \setminus G) \) be a space of the \( C^\infty \)-functions satisfying the relation:

\[
\phi(ng) = \eta(n) \phi(g) \quad (n \in N, \ g \in G).
\]

Right translation by \( G \) makes \( C_\eta^\infty(N \setminus G) \) a \( G \)-module, accordingly, it becomes a compatible (\( g, K \))-module.

Let \( (\pi, H_\pi) \) be an irreducible admissible representation of \( G \) and let \( H_\pi^K \) be the \( (g, K) \)-submodule that consists of the \( K \)-finite vectors in \( H_\pi \). We call the elements of \( \text{Hom}_{(g, K)}(H_\pi^K, C_\eta^\infty(N \setminus G)) \) the algebraic Whittaker vectors of \( \pi \).

Let \( (\tau, V_\tau) \) be an irreducible admissible representation of \( K \) and \( (\tau^*, V_\tau^*) \) be its contragredient representation. Assume that \([\pi \mid K : \tau^*] = 1\). Fix a \( K \)-injection \( i_\tau^* : V_\tau^* \to H_\pi \). By composition, there is a canonical \( K \)-homomorphism

\[
i_\tau^* : \text{Hom}_{(g, K)}(H_\pi^K, C_\eta^\infty(N \setminus G)) \to \text{Hom}_K(V_\tau^*, C_\eta^\infty(N \setminus G)),
\]

defined by \( i_\tau^* (\Phi_\pi) = \Phi_\tau \circ i_\tau^* \) for a Whittaker vector \( \Phi_\pi \). This is injective by virtue of the irreducibility of \( \pi \). Let \( C_{\eta, \tau}^\infty(N \setminus G/K) \) be the space of \( V_\tau \)-valued \( C^\infty \)-functions satisfying the relation

\[
(ngk) = \eta(n) \tau^{-1}(k) \phi(g) \quad (n \in N, \ g \in G \text{ and } k \in K).
\]

The space \( \text{Hom}_K(V_\tau^*, C_\eta^\infty(N \setminus G/K)) \) is identified with \( C_{\eta, \tau}^\infty(N \setminus G/K) \) through the identification map \( \phi \mapsto \phi_\tau \) by the rule:

\[
\phi(v^*(g)) = (v^*, \phi_\tau(g)) \quad (v^* \in V_\tau^*, \ g \in G)
\]

where \( (, ,) \) is the pairing of \( V_\tau^* \) and \( V_\tau \). In particular, we write \( \Phi_{\eta, \tau} = (i_\tau^* (\Phi_\pi))_\tau \in C_{\eta, \tau}^\infty(N \setminus G/K) \). We also call \( \Phi_{\eta, \tau} \) a Whittaker function or, more exactly, a Whittaker function of \( \pi \) with \( K \)-type \( \tau^* \).

Now any element in \( C_{\eta, \tau}^\infty(N \setminus G/K) \) is uniquely determined by its restriction to \( A \). Therefore, for \( f \in C_{\eta, \tau}^\infty(N \setminus G/K) \), we denote by \( \text{Rad}(f) = \text{Rad}_\tau(f) = f \mid_A \in C^\infty(A) \) the radial part of \( f \) and, given an operator \( D \) on \( C_{\eta, \tau}^\infty(N \setminus G/K) \), we define the radial part of \( D \), \( \text{Rad}(D) = \text{Rad}_\tau(D) \) by

\[
\text{Rad}(D) \text{Rad}(f) = \text{Rad}(D(f)).
\]

Actually the differential equations for a Whittaker function shall mean those for its radial part.

Additionally, we mention the well-known Kostant's theorem for Whittaker models here. Since the character \( \eta \) satisfies \( \eta \mid_{[n, n]} = 0 \), \( \eta \) is uniquely determined by the value of \( E_\alpha \) \( (\alpha \in \Delta_{\text{fund}}) \). We write for simplicity,

\[
\eta_2 = \sqrt{-1}\eta(E_2), \quad \eta_5 = \sqrt{-1}\eta(E_5), \quad \eta_6 = \sqrt{-1}\eta(E_6),
\]

\[
\begin{cases}
\xi = \eta(E_5) + \sqrt{-1}\eta(E_6) = -\sqrt{-1}\eta_5 + \eta_6, \\
\xi' = \eta(E_5) - \sqrt{-1}\eta(E_6) = -\sqrt{-1}\eta_5 - \eta_6, \\
\eta_0 = \xi \xi' = -(\eta_5^2 + \eta_6^2).
\end{cases}
\]

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where $\eta_j$, $(j = 2, 5, 6)$ are real numbers since $\eta$ is unitary. We say $\eta$ is non-degenerate if the $\eta_j$'s $(j = 2, 5, 6)$ are all non-zero.

The group $G = SU(2, 2)$ is quasi-split, so we have,

**Theorem 4.1** ([3, Theorem 5.5]). Let $\pi = \text{ind}_F^G(\sigma_{n, \epsilon} \otimes e^\mu + \rho)$. Assume that the unitary character $\eta$ of $N$ is non-degenerate. Then,

$$\dim_{\mathbb{C}} \text{Hom}_{(\eta, K)}(H_\pi^K, C_\infty^\omega(N \backslash G)) = |W|.$$ 

Since the order of the Weyl group $W$ is 8, we should find a holonomic system of partial differential equations of rank 8 according to Kostant's theorem.

### 4.2. Differential equations for Whittaker functions

The next theorem describes the Whittaker functions of principal series representations with the 1-dimensional $K$-types.

**Theorem 4.2.** Let $\pi = \text{ind}_F^G(\sigma_{n, \epsilon} \otimes e^\mu + \rho)$ be irreducible. Let $\tau = \tau_{[0, 0, 0]}$ be an irreducible representation of $K$ satisfying $u \equiv 1 - \epsilon(-1) \pmod{4}$. Choose a non-degenerate character $\eta$ of $N$. Let $\Phi_{\pi, \tau} \in C_\infty^\omega(N \backslash G/K)$ be a Whittaker function of $\pi$ with $K$-type $\tau^*$. Then, for $a = (a_1, a_2) \in A$, $I(a) = a_1^{-3}a_2^{-1}\Phi_{\pi, \tau}(a)$ satisfies the differential equations:

$$\begin{align*}
\left\{ \left( \partial_{\tau}^2 + \frac{\eta_1^2}{\eta_2}a^4 + \frac{\eta_2}{\eta_1}a^2 + 2\eta_0 \left( \frac{a_1}{a_2} \right)^2 \right)x = (\mu_1^2 + \mu_2^2)I, \\
\left( \left[ \left( \partial_{\tau}^2 - \left( \frac{u}{2} + 1 \right)^2 \right)x - \left( \frac{u}{2} + 1 \right)^2x \right)- \frac{\eta_1^2}{\eta_2}a^2 + \frac{\eta_2}{\eta_1}a \right)I \right. \\
\left. - 2\eta_0 \left( \frac{a_1}{a_2} \right)^2 (\partial_{\tau} + 1)(\partial_{\tau} - 1) - u \left( \frac{u}{2} + 2 \right)x \left( \frac{a_1}{a_2} \right)^2 + \eta_0 \eta_2 a \right)I, \\
+ \eta_0^2 \left( \frac{a_1}{a_2} \right)^4 \left( \mu_1^2 - \left( \frac{u}{2} + 1 \right)^2 \right)(\mu_2^2 - \left( \frac{u}{2} + 1 \right)^2)I, \\
\end{align*}$$

where we put $\partial_j = a_j(\partial / \partial a_j)$ for $j = 1, 2$.

In the case of the 2-dimensional $K$-types, we obtain,

**Theorem 4.3.** Let $|n| = 1$, $\pi = \text{ind}_F^G(\sigma_{n, \epsilon} \otimes e^\mu + \rho)$ be irreducible and $\eta$ be non-degenerate. Assume that the contragredient representation $\tau^*$ of $\tau_{[r, \epsilon, 0]}$ appears in $\pi|_K$ with multiplicity one. Let $\Phi_{\pi, \tau} \in C_\infty^\omega(N \backslash G/K)$ be the Whittaker function of $\pi$ with $K$-type $\tau^*$. For the standard basis $\{ f_{kl} \}$ of $V_r$ and $a = (a_1, a_2) \in A$, put

$$I(a) = a_1^{-3}a_2^{-1}\Phi_{\pi, \tau}(a) = \sum_{0 \leq k \leq r, 0 \leq l \leq s} b_{kl}(a) f_{kl}.$$

(i) If $r = 0, s = 1$, then $b_{00}, b_{01}$ satisfy the equations:
\[
\begin{pmatrix}
\xi'(a_1/a_2)(\partial_1 + \partial_2 + \eta_2 a_2^2 + 1) \\
(\xi(a_1/a_2)(\partial_1 + \partial_2 + \eta_2 a_2^2 + 1)
\end{pmatrix}
\begin{pmatrix}
P_2 + (u - 1)\eta_2 a_2^2 \\
(\eta_2 a_2 - 1)
\end{pmatrix}
\times \begin{pmatrix}
\mu_2^2 (b_{00}) \\
\mu_1^2 (b_{01})
\end{pmatrix}
\begin{pmatrix}
(\eta_2 a_2^2 + 1) \\
(\eta_2 a_2^2 + 1)
\end{pmatrix}
\quad (n = 1, u \equiv \epsilon(-1) \pmod{4}),
\]
\[
\begin{pmatrix}
P_2 + (u - 1)\eta_2 a_2^2 \\
(\eta_2 a_2 - 1)
\end{pmatrix}
\times \begin{pmatrix}
\mu_2^2 (b_{00}) \\
\mu_1^2 (b_{01})
\end{pmatrix}
\begin{pmatrix}
(\eta_2 a_2^2 + 1) \\
(\eta_2 a_2^2 + 1)
\end{pmatrix}
\quad (n = -1, u \equiv -\epsilon(-1) \pmod{4}).
\]
\[
(\tilde{L}_0 + (u - 1)\eta_2 a_2^2, 2\xi(a_1/a_2))\begin{pmatrix}
(b_{00}) \\
(b_{01})
\end{pmatrix}
\begin{pmatrix}
(\eta_2 a_2^2 + 1) \\
(\eta_2 a_2^2 + 1)
\end{pmatrix}
((\mu_1^2 + \mu_2^2) (b_{00}) \\
(\mu_1^2 + \mu_2^2) (b_{01})).
\]

(ii) If \( r = 1, s = 0 \), then \( b_{00}, b_{10} \) satisfy the equations:
\[
\begin{pmatrix}
P_1 - \xi'(a_1/a_2)(\partial_1 + \partial_2 - \eta_2 a_2^2 + 1) \\
-\xi(a_1/a_2)(\partial_1 + \partial_2 - \eta_2 a_2^2 + 1)
\end{pmatrix}
\times \begin{pmatrix}
\mu_2^2 (b_{00}) \\
\mu_1^2 (b_{01})
\end{pmatrix}
\begin{pmatrix}
(\eta_2 a_2^2 + 1) \\
(\eta_2 a_2^2 + 1)
\end{pmatrix}
\quad (n = 1, u \equiv -\epsilon(-1) \pmod{4}),
\]
\[
\begin{pmatrix}
P_1 - \xi'(a_1/a_2)(\partial_1 + \partial_2 - \eta_2 a_2^2 + 1) \\
-\xi(a_1/a_2)(\partial_1 + \partial_2 - \eta_2 a_2^2 + 1)
\end{pmatrix}
\times \begin{pmatrix}
\mu_2^2 (b_{00}) \\
\mu_1^2 (b_{01})
\end{pmatrix}
\begin{pmatrix}
(\eta_2 a_2^2 + 1) \\
(\eta_2 a_2^2 + 1)
\end{pmatrix}
\quad (n = -1, u \equiv \epsilon(-1) \pmod{4}).
\]
\[
\begin{pmatrix}
\tilde{L}_0 + (u - 1)\eta_2 a_2^2 \\
(2\xi(a_1/a_2))
\end{pmatrix}
\begin{pmatrix}
(b_{00}) \\
(b_{10})
\end{pmatrix}
\begin{pmatrix}
(\eta_2 a_2^2 + 1) \\
(\eta_2 a_2^2 + 1)
\end{pmatrix}
((\mu_1^2 + \mu_2^2) (b_{00}) \\
(\mu_1^2 + \mu_2^2) (b_{10})).
\]

Here we write,
\[
P_1 = \partial_1^2 + \eta_0 \left( \frac{a_1}{a_2} \right)^2,
\quad P_2 = \partial_2^2 - \eta_2 a_2^4 + \eta_0 \left( \frac{a_1}{a_2} \right)^2,
\quad \tilde{L}_0 = \partial_1^2 + \partial_2^2 - \eta_2 a_2^4 + 2\eta_0 \left( \frac{a_1}{a_2} \right)^2.
\]

**Remark 4.4.** As a matter of fact, the above differential equations are essentially the same as those in [4, Theorems (10.1) and (11.3)]. We recover those equations by putting \( \eta_2, \eta_0, u \) in place of \( -\pi \xi_2, -\pi^2 c_0^2, 2l \), respectively.

To see that the system of differential equations in Theorems 4.2 and 4.3 becomes holonomic, we must compute the corresponding characteristic variety. Put \( \mathcal{M} \) be the characteristic variety in the cotangent bundle identified with \( \mathbb{C}^4 \). Let the coordinate of it be \( (\xi_1, \xi_2, 2\xi_3, 2) \). Then, using computer software 'Kan' ([7]), which can compute a Gröbner basis of a given \( D \)-module, we find that
\[
\mathcal{M} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \text{(zero section)}
\]
where \( \mathcal{V}_1 = \{ a_2 = 0, \xi_1 = 0 \} \), \( \mathcal{V}_2 = \{ a_1 = 0, \xi_3 = 0 \} \) and \( \mathcal{V}_3 = \{ a_1 = 0, a_2 = 0 \} \). Since \( \dim \mathcal{M} = 2 \), we conclude that the system above is holonomic. We also see from the Hilbert polynomial of the zero section that the rank of this system is 8 (even if the character \( \eta \) is degenerate).

**Remark 4.5.** Our calculation is valid without the assumption that \( \pi \) is irreducible. However, the injectivity of \( i_\pi \) fails. In this case we might consider
some subquotient representation of \( \pi \) whose \( K \)-types occur in \( \pi |_K \) with multiplicity one.

5. RADIAL PART OF THE CASIMIR OPERATOR

Let \( Z(\mathfrak{g}_C) \) be the center of the universal enveloping algebra of the complexification of the Lie algebra \( \mathfrak{g} \). We mainly have an interest in the Casimir operator, a generator of \( Z(\mathfrak{g}_C) \), it will give one of the differential equations of Theorems 4.2 and 4.3. Thus we calculate the radial part of the Casimir operator and compute the infinitesimal character of the principal series representation.

5.1. Radial part of the Casimir operator

In this subsection, we calculate the radial part of the Casimir operator \( \Omega \).

By definition, the Casimir operator \( \Omega \) is of the form \( \sum_j X_j^* X_j \), with a basis \( \{ X_j \} \) of \( \mathfrak{g} \) and its dual basis \( \{ X_j^* \} \). Precisely, it is given by,

\[
\Omega = H_1^2 + H_2^2 + \frac{1}{2} I_0^2 - \frac{1}{2} \sum_{j=1,2} (E_j^* E_j + E_j E_j) + \sum_{j=3}^6 (E_j^* E_j + E_j E_j).
\]

First, we rewrite \( \Omega \) in \( Z(\mathfrak{g}_C) \) using the Poincaré–Birkhoff–Witt basis. Paying attention to \( [E_j, E_j^*] = 4H_j, \quad j = 1, 2 \), \( [E_j, E_j^*] = H_1 + H_2, \quad j = 3, 4 \) and \( [E_j, E_j^*] = H_1 - H_2, \quad j = 5, 6 \), we get

\[
\Omega = H_1^2 + H_2^2 - 6H_1 - 2H_2 + \frac{1}{2} I_0^2 + \sum_{j=1,2} E_j^* E_j + 2 \sum_{j=3}^6 E_j^* E_j
\]

\[
= H_1^2 + H_2^2 - 6H_1 - 2H_2 + \frac{1}{2} I_0^2 + \sum_{j=1,2} E_j^2 + 2 \sum_{j=3}^6 E_j^2
\]

\[
- \sum_{j=1,2} E_j(E_j - E_j^*) - 2 \sum_{j=3}^6 E_j(E_j - E_j^*).
\]

Now we determine the radial part of the Casimir operator. For notation we use the following symbols:

\[
\partial_j \phi = (H_j \cdot \phi) \bigg|_A \quad (j = 1, 2),
\]

\[
S = \frac{1}{2} e^{\lambda_1 - \lambda_2} (\eta(E_5) + \sqrt{-1} \eta(E_6)) = \frac{1}{2} \xi e^{\lambda_1 - \lambda_2},
\]

\[
S' = \frac{1}{2} e^{\lambda_1 - \lambda_2} (\eta(E_5) - \sqrt{-1} \eta(E_6)) = \frac{1}{2} \xi' e^{\lambda_1 - \lambda_2}.
\]

Lemma 5.1. Let \( \tau = \tau_{[r,s,u]} \) be an irreducible representation of \( K \). Put

\[
\phi(a) = F \big|_A (a) = \sum_{0 \leq k \leq r, 0 \leq l \leq s} c_{kl} (a) f_{kl},
\]

\[
\phi^{(\Omega)} (a) = \text{Rad}_\tau(\Omega) \phi (a) = \sum_{0 \leq k \leq r, 0 \leq l \leq s} c_{kl}^{(\Omega)} (a) f_{kl},
\]

for \( F \in C_{\eta, \tau}^{\infty}(N \backslash G / K) \). Then,
\[
(44) \quad c_{kl}^{(\Omega)} = \begin{pmatrix}
4(r - k + 1)S \\
4(s - l + 1)S \\
L_0 + (u - 2k + 2l + r - s)\eta_2 e^{2\lambda_2} + \alpha_{k,l} \\
-4(l + 1)S' \\
-4(k + 1)S'
\end{pmatrix}
\begin{pmatrix}
c_{k-1,l} \\
c_{k,l-1} \\
c_{kl} \\
c_{k,l+1} \\
c_{k+1,l}
\end{pmatrix}
\]

with,

\[L_0 = \partial_1^2 + \partial_2^2 - 6\partial_1 - 2\partial_2 - \eta_2^2 a_4^2 + 8SS',\]

\[\alpha_{k,l} = (2k - r + 2l - s)^2/2.\]

**Proof.** Using the fact that \(E_1 \cdot F = E_3 \cdot F = E_4 \cdot F = 0\) we see that

\[
\left(H_1^2 + H_2^2 - 6H_1 - 2H_2 + \frac{1}{2} I_0^2 + \sum_{j=1,2} E_j^2 + 2 \sum_{j=3}^6 \tilde{E}_j^2 \right) F(a) = (L_0 + \alpha_{k,l}) \phi(a).
\]

Furthermore, we know that

\[
E_2 + \tilde{E}_2 = \sqrt{-1}(I_{2,2} - h^1 + h^2),
\]

\[
E_5 - \tilde{E}_5 = e^1_+ - e^1_- + e^2_+ - e^2_-,
\]

\[
E_6 + \tilde{E}_6 = \sqrt{-1}(e^1_+ + e^1_- + e^2_+ + e^2_-).
\]

Therefore we get

\[
\left(- \sum_{j=1,2} E_j(E_j - \tilde{E}_j) - 2 \sum_{j=3}^6 E_j(E_j - \tilde{E}_j) \right) F(a)
\]

\[
= \sum_{k,l} \left\{ E_2 \cdot c_{kl}(a)\sqrt{-1}(I_{2,2} - h^1 + h^2)f_{kl}
\right.
\]

\[
+ 2E_5 \cdot c_{kl}(a)(e^1_+ - e^1_- + e^2_+ - e^2_-)f_{kl}
\]

\[
+ 2E_6 \cdot c_{kl}(a)\sqrt{-1}(e^1_+ + e^1_- + e^2_+ + e^2_-))f_{kl}\}
\]

\[
= \sum_{k,l} \left\{ \sqrt{-1}\eta(E_2)a^2 c_{kl}(a)(u - 2k + r + 2l - s)f_{kl}
\right.
\]

\[
+ 2\eta(E_5)(a_1/a_2)c_{kl}(a)((r - k)f_{k+1,l} - kf_{k-1,l} + (s - l)f_{k+1,l} + l f_{k,l+1})
\]

\[
+ 2\sqrt{-1}\eta(E_6)(a_1/a_2)c_{kl}(a)((r - k)f_{k+1,l} + kf_{k-1,l} + (s - l)f_{k+1,l} + l f_{k,l+1})\}
\]

\[
= \sum_{k,l} \left\{ (u - 2k + r + 2l - s)\eta_2 a^2 c_{kl}(a) + (a_1/a_2)(2(r - k + 1)\xi c_{k+1,l}(a)
\right.
\]

\[
- 2(k + 1)\xi' c_{k+1,l}(a) + 2(s - l + 1)\xi c_{k,l+1}(a) - 2(l + 1)\xi' c_{k,l+1}(a)) \right\} f_{kl}.
\]

This proves the lemma. \(\square\)

### 5.2. Infinitesimal character of a principal series representation

We again rewrite the Casimir operator \(\Omega\) in the following form:

\[
\Omega = H_1^2 + H_2^2 + 6H_1 + 2H_2 + \frac{1}{2} I_0^2 + \sum_{j=1,2} \tilde{E}_j E_j + 2 \sum_{j=3}^6 \tilde{E}_j E_j,
\]

According to [1, Proposition 8.22], the infinitesimal character \(\chi_\pi\) of \(\pi = \text{ind}_p^G \times (\sigma_{n,\nu} \otimes e^{\mu + \rho})\) is
considered as an element of \((m + a)^*_{\mathcal{C}}\) under the Harish-Chandra homomorphism. Thus, for the Casimir operator \(\Omega\), one has,

\[
\chi_\pi(\Omega) = \chi_\pi\left(H_1^2 + H_2^2 + 6H_1 + 2H_2 + \frac{1}{2}I_0^2\right)
\]
\[
= (\sigma_n + \mu)\left((H_1 - 3)^2 + (H_2 - 1)^2 + 6(H_1 - 3) + 2(H_2 - 1) + \frac{1}{2}I_0^2\right)
\]
\[
= \mu (H_1^2 + H_2^2) - 10 + \frac{n^2}{2}.
\]

Concludingly, we have,

**Lemma 5.2.** Let \(\pi = \text{ind}^G_\rho(\sigma_{\eta} \otimes e^{\mu + \rho})\) and \(\phi \in H_\pi\). Then,

\[
(46) \quad \pi(\Omega) \phi = \left(\mu_1^2 + \mu_2^2 + \frac{n^2}{2} - 10\right)\phi.
\]

### 6. RADIAL PART OF SHIFT OPERATORS

Let \((\tau, V_\tau)\) be a finite-dimensional irreducible representation of \(K\) and \(\eta\) be a unitary character of \(N\). We introduce some important operators. Let \(\{X_j\}_{j=1}^8\) be an orthonormal basis of \(\mathfrak{p}\). We define the Schmid operator \(V\) by

\[
(47) \quad V = V_{\eta, \tau} : C_{\infty}(N \backslash G/K) \ni F \mapsto \sum_{j=1}^8 X_j \cdot F(\cdot) \otimes X_j \in C_{\infty}(\eta \cdot \text{Ad}(N \backslash G/K)).
\]

It is independent of the choice of an orthonormal basis of \(\mathfrak{p}\). Besides, \(V\) is \(K\)-equivariant. Let us choose as the orthogonal basis of \(\mathfrak{p}\),

\[
\{X_{ij} + X_{ji}, \sqrt{-1}(X_{ij} - X_{ji})\}_{i=1, 2, j=3, 4}.
\]

We define the \pm-part of the Schmid operator,

\[
\nabla^\pm : C_{\infty}(N \backslash G/K) \rightarrow C_{\infty}(\eta \cdot \text{Ad}(N \backslash G/K))
\]

by

\[
(48) \quad \begin{cases} 
\nabla^+ F = \sum_{i=1, 2, j=3, 4} X_{ij} \cdot F(\cdot) \otimes X_{ij} \\
\nabla^- F = \sum_{i=1, 2, j=3, 4} X_{ij} \cdot F(\cdot) \otimes X_{ji},
\end{cases}
\]

for \(F \in C_{\infty}(N \backslash G/K)\). Then we can check that \((\text{const.})\nabla = \nabla^+ + \nabla^-\).

As explained in §3.7, \(V_{\tau} \otimes \mathfrak{p}_{\pm}\) decomposes into four irreducible components in general. Let \((\tau', V_{\tau'})\) be an irreducible component of \(\tau \otimes \text{Ad}_{\pm}\). Denoting the projector onto \(V_{\tau'}\) by \(P'\), the composition \(P' \circ \nabla^+\) maps \(C_{\infty}(N \backslash G/K)\) to \(C_{\infty}(\eta \cdot \tau'(N \backslash G/K))\). Since the space \(V_{\tau'} \otimes \mathfrak{p}_{\pm}\) has an irreducible component isomorphic to \(V_{\tau}\), we obtain an operator from \(C_{\infty}(N \backslash G/K)\) to itself by iterating this procedure. This defines a differential equation for the radial part of the
Whittaker function with $K$-type $\tau^*$. We must select an irreducible component so that the differential equation derived is appropriate.

First, we treat in §6.2 the case when $\dim \tau = 1$ and $[\pi|_K : \tau^*] = 1$ for a given principal series representation $\pi$. In this case we consider $\tau \otimes \text{Ad}_\pm \otimes \text{Ad}_\pm$. Then, in each contragredient representation of them, the unique 1-dimensional irreducible constituent occurs in $\pi|_K$ with multiplicity one. Define the operators:

\[
\mathcal{D}^{\text{up}} = P^{(-, -)}_{1, 1} \circ \nabla^+_{\tau \otimes \text{Ad}_+} \circ \nabla^+_{\tau},
\]

\[
\mathcal{D}^{\text{down}} = \tilde{P}^{(-, -)}_{1, 1} \circ \nabla^-_{\tau \otimes \text{Ad}_-} \circ \nabla^-_{\tau}.
\]

We call them the up-shift operator and the down-shift operator respectively. As a result, the composition $\mathcal{D}^{\text{down}} \circ \mathcal{D}^{\text{up}}$ will give one of the differential equations in Theorem 4.2.

Next, we treat in §6.3 the case when $\dim \tau = 2$ and $[\pi|_K : \tau^*] = 1$. In this case, $\tau \otimes \text{Ad}_+ \otimes \text{Ad}_- \otimes \text{Ad}_-$ has the unique irreducible constituent, whose contragredient is a multiplicity-one $K$-type of $\pi$. So we can define the shift operators,

\[
\mathcal{E}^{(\pm, +)} = P^{(\pm, +)} \circ \nabla^+_{\tau},
\]

\[
\mathcal{E}^{(\pm, -)} = \tilde{P}^{(\pm, -)} \circ \nabla^-_{\tau}.
\]

We call $\mathcal{E}^{(\pm, \mp)}$ the $(\pm, \mp)$-up-shift operators and $\mathcal{E}^{(\pm, \pm)}$ the $(\pm, \pm)$-down-shift operators respectively. For suitable choice of $\pm$, the composition $\mathcal{E}^{(\pm, \mp)} \circ \mathcal{E}^{(\pm, \pm)}$ will yield the differential equation.

We shall see in §7 that these shift operators characterize Whittaker functions of $\pi$ as their eigenfunctions through the calculation essentially done in §§3.8 and 3.9.

### 6.1. Explicit calculation of the radial part of the Schmid operator

In what follows, we use for notation:

\[
\mathcal{L}_1 \phi = \frac{1}{2} \partial_1 \phi,
\]

\[
\mathcal{L}_2 \pm \phi = \frac{1}{2} (\partial_2 \pm \sqrt{-1} e^{2\lambda_2 \eta(E_2)}) \phi
\]

where we put $\phi = F|_A$ for a given $F \in C^\infty_{\eta, \tau}(N \backslash G/K)$, and also keep the symbols $\partial_1, \partial_2, S$ and $S'$ already defined in (42).

The radial parts of the $\pm$-part of the Schmid operator $\nabla \pm$ are as follows.

**Proposition 6.1.** Let $F \in C^\infty_{\eta, \tau}(N \backslash G/K)$ and let $\phi = F|_A$. Then,

\[
\text{Rad}(\nabla^+) \phi = \left( \mathcal{L}_1 - \frac{\sqrt{-1}}{2} (\tau \otimes \text{Ad}_+)(H_{13}) - 3 \right) (\phi \otimes X_{13})
\]

\[
+ \left( \mathcal{L}_2^+ - \frac{\sqrt{-1}}{2} (\tau \otimes \text{Ad}_+)(H_{24}) - 1 \right) (\phi \otimes X_{24})
\]

\[
+ (S' + (\tau \otimes \text{Ad}_+)(e^t))(\phi \otimes X_{23})
\]

\[
+ (S - (\tau \otimes \text{Ad}_+)(e^t))(\phi \otimes X_{14}),
\]

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\[
\text{Rad}(\nabla^-) \phi = \left( \mathcal{L}_1 + \frac{\sqrt{-1}}{2} (\tau \otimes \text{Ad}_-)(H_{13}) - 3 \right) (\phi \otimes X_{31}) \\
+ \left( \mathcal{L}_2^+ + \frac{\sqrt{-1}}{2} (\tau \otimes \text{Ad}_-)(H_{24}) - 1 \right) (\phi \otimes X_{42}) \\
+ (S - (\tau \otimes \text{Ad}_-)(e_1^2))(\phi \otimes X_{32}) \\
+ (S' + (\tau \otimes \text{Ad}_-)(e_2^2))(\phi \otimes X_{41}).
\]

**Proof.** By definition, we can calculate the actions of \(E_j\)'s given by

\[
(E_j \cdot F) |_A(a) = \begin{cases} 
  e^{\lambda_2(\log a)}\eta(E_2) \phi(a) & (j = 2), \\
  e^{(\lambda_1 - \lambda_2)(\log a)}\eta(E_j) \phi(a) & (j = 5, 6), \\
  0 & \text{(otherwise)}. \end{cases}
\]

So, if we rewrite the Schmid operator (47) by using §2.3, we have,

\[
\nabla^+ F = \frac{1}{2} \left\{ (H_1 + \sqrt{-1}H_{13}) \cdot F \otimes X_{13} + (E_5 - \sqrt{-1}E_6 - 2e_1^1) \cdot F \otimes X_{23} \\
+ (E_5 + \sqrt{-1}E_6 + 2e_2^2) \cdot F \otimes X_{14} \\
+ (-\sqrt{-1}E_2 + H_2 + \sqrt{-1}H_{24}) \cdot F \otimes X_{24} \right\} \\
= \left( \mathcal{L}_1 + \frac{\sqrt{-1}}{2} H_{13} \right) \cdot F \otimes X_{13} + (S' - e_1^1) \cdot F \otimes X_{23} \\
+ (S \cdot e_2^2) \cdot F \otimes X_{14} + \left( \mathcal{L}_2^- + \frac{\sqrt{-1}}{2} H_{24} \right) \cdot F \otimes X_{24}.
\]

Similarly, we get

\[
\nabla^- F = \left( \mathcal{L}_1 - \frac{\sqrt{-1}}{2} H_{13} \right) \cdot F \otimes X_{31} + (S' - e_2^2) \cdot F \otimes X_{41} \\
+ (S + e_1^1) \cdot F \otimes X_{32} + \left( \mathcal{L}_2^+ - \frac{\sqrt{-1}}{2} H_{24} \right) \cdot F \otimes X_{42}.
\]

Noting that \(X \cdot F = -\tau(X)F\) for \(X \in \mathfrak{f}\), we get

\[
(\tau \otimes \text{Ad}_+)(H_{13})(F \otimes X_{13}) = (-H_{13} + 2\sqrt{-1})F \otimes X_{13}, \\
(\tau \otimes \text{Ad}_+)(e_1^1)(F \otimes X_{23}) = -e_1^1 \cdot F \otimes X_{23} + F \otimes X_{13}, \\
(\tau \otimes \text{Ad}_+)(e_2^2)(F \otimes X_{14}) = e_2^2 \cdot F \otimes X_{14} - F \otimes X_{13}, \\
(\tau \otimes \text{Ad}_-)(H_{24})(F \otimes X_{24}) = (-H_{24} + 2\sqrt{-1})F \otimes X_{24}, \\
(\tau \otimes \text{Ad}_-)(H_{13})(F \otimes X_{31}) = (-H_{13} - 2\sqrt{-1})F \otimes X_{31}, \\
(\tau \otimes \text{Ad}_-)(e_2^2)(F \otimes X_{41}) = -e_2^2 \cdot F \otimes X_{41} + F \otimes X_{31}, \\
(\tau \otimes \text{Ad}_-)(e_1^1)(F \otimes X_{32}) = -e_1^1 \cdot F \otimes X_{32} - F \otimes X_{31}, \\
(\tau \otimes \text{Ad}_-)(H_{24})(F \otimes X_{42}) = (-H_{24} - 2\sqrt{-1})F \otimes X_{42}.
\]

Our proposition follows using these relations. \(\square\)
6.2. Radial part of shift operators (1-dimensional K-type case)

In this subsection, we treat the case of $\pi = \text{ind}^F_\mathbb{Z}(\sigma_0, e \otimes e^{\mu + \rho})$ admitting a 1-dimensional $K$-type $\tau_0 = \tau_0^*|_{\mathfrak{o}(0,u)}$ with $-u \equiv 1 - e(-1) \pmod{4}$.

Define shift operators by the following:

\begin{align*}
D_{\text{up}}^+ &= P_{1,1}^{(-,-)} \circ \nabla^+_\tau \otimes \mathbb{A}_- \circ \nabla^+_\tau, \\
D_{\text{down}}^- &= \check{P}_{1,1}^{(-,-)} \circ \nabla^-\tau \otimes \mathbb{A}_- \circ \nabla^-\tau.
\end{align*}

Now we calculate the radial parts of these operators. We write $d_0 = [0, 0; u]$, $d_1 = [1, 1; u + 2]$, $d_2 = [0, 0; u + 4]$, $d_3 = [1, 1; u - 2]$, $d_4 = [0, 0; u - 4]$. For given $d_j$, we write simply $\tau_j = \tau_{d_j}, f_{pq}^{(d_j)} = f_{pq}^{(d_j)}$.

**Lemma 6.2.** Put $\psi_0(a) = F_0 |\Delta(a) = c^{(0)}(a)f_{00}^{(0)}$ for $F_0 \in C_{q,0}^\infty(N \setminus G/K)$ and put also

\begin{align*}
\phi_1(a) &= F_1 |\Delta(a) = \text{Rad}(\nabla^+)\psi_0(a) = \sum_{0 \leq k, l \leq 1} c_{kl}^{(1)}(a)f_{kl}^{(1)}, \\
\phi_2(a) &= F_2 |\Delta(a) = \text{Rad}(P_{1,1}^{(-,-)} \circ \nabla^+)\psi_1(a) = c^{(2)}(a)f_{00}^{(2)}, \\
\phi_3(a) &= F_3 |\Delta(a) = \text{Rad}(\nabla^-)\psi_0(a) = \sum_{0 \leq k, l \leq 1} c_{kl}^{(3)}(a)f_{kl}^{(3)}, \\
\phi_4(a) &= F_4 |\Delta(a) = \text{Rad}(\check{P}_{1,1}^{(-,-)} \circ \nabla^-)\psi_3(a) = c^{(4)}(a)f_{00}^{(4)}.
\end{align*}

Then the image of the shift operators can be described as follows:

\begin{align*}
\begin{cases}
c_{00}^{(1)}(a) &= S'c^{(0)}(a), \\
c_{10}^{(1)}(a) &= (\mathcal{L}_1 + u/4)c^{(0)}(a), \quad c_{11}^{(1)}(a) = -\mathcal{S}c^{(0)}(a),
\end{cases}
\end{align*}

\begin{align*}
\begin{cases}
c_{00}^{(2)}(a) &= S'c^{(0)}(a), \\
c_{10}^{(2)}(a) &= (\mathcal{L}_1 + u/4)c^{(0)}(a), \quad c_{11}^{(2)}(a) = -\mathcal{S}c^{(0)}(a),
\end{cases}
\end{align*}

\begin{align*}
\begin{cases}
c_{00}^{(3)}(a) &= S'c^{(0)}(a), \\
c_{10}^{(3)}(a) &= (\mathcal{L}_1 + u/4)c^{(0)}(a), \quad c_{11}^{(3)}(a) = -\mathcal{S}c^{(0)}(a),
\end{cases}
\end{align*}

\begin{align*}
\begin{cases}
c_{00}^{(4)}(a) &= S'c_{00}^{(1)}(a) + (\mathcal{L}_1 + u/4)c_{10}^{(1)}(a) \\
&\quad + S'c_{11}^{(1)}(a) - \mathcal{S}c_{00}^{(1)}(a),
\end{cases}
\end{align*}

\begin{align*}
\begin{cases}
c_{00}^{(5)}(a) &= S'c_{00}^{(3)}(a) + (\mathcal{L}_1 + u/4)c_{10}^{(3)}(a) \\
&\quad + S'c_{11}^{(3)}(a) - \mathcal{S}c_{00}^{(3)}(a).
\end{cases}
\end{align*}

**Proof.** First we consider $c_{ij}^{(1)}$ and $c_{ij}^{(3)}$. Using Propositions 3.10, 6.1 and Lemma 3.8,
\[ \phi_1(a) = \left( L_1 - \frac{\sqrt{-1}}{2} \tau_1(H_{13}) - 3 \right) \ell_+(F_0(a) \otimes X_{13}) \]
\[ + \left( L_2^- - \frac{\sqrt{-1}}{2} \tau_1(H_{24}) - 1 \right) \ell_+(F_0(a) \otimes X_{24}) \]
\[ + (S' + \tau_1(e_+^1)) \ell_+(F_0(a) \otimes X_{23}) \]
\[ + (S - \tau_1(e_2^2)) \ell_+(F_0(a) \otimes X_{14}) \]
\[ = (L_1 + \frac{1}{4} \tau_1(I_{2, 2} + h^1 - h^2) - 3) c^{(0)}(a) f_{10}^{(1)} \]
\[ - \left( L_2^- + \frac{1}{4} \tau_1(I_{2, 2} - h^1 + h^2) - 1 \right) c^{(0)}(a) f_{01}^{(1)} \]
\[ + (S' + \tau_1(e_+^1)) c^{(0)}(a) f_{00}^{(1)} - (S - \tau_1(e_2^2)) c^{(0)}(a) f_{11}^{(1)} \]
\[ = (L_1 + u/4) c^{(0)}(a) f_{10}^{(1)} - (L_2^- + u/4) c^{(0)}(a) f_{01}^{(1)} \]
\[ + S' c^{(0)}(a) f_{00}^{(1)} - S c^{(0)}(a) f_{11}^{(1)}. \]

We get (51) by comparing the definition of \( c_{ij}^{(1)} \) with the last equation. In a similar manner,

\[ \phi_3(a) = \left( L_1 + \frac{\sqrt{-1}}{2} \tau_3(H_{13}) - 3 \right) \ell_-(F_0(a) \otimes X_{31}) \]
\[ + \left( L_2^+ + \frac{\sqrt{-1}}{2} \tau_3(H_{24}) - 1 \right) \ell_-(F_0(a) \otimes X_{42}) \]
\[ + (S - \tau_3(e_2^2)) \ell_-(F_0(a) \otimes X_{32}) \]
\[ + (S' + \tau_3(e_+^1)) \ell_-(F_0(a) \otimes X_{41}) \]
\[ = (L_1 - \frac{1}{4} \tau_3(I_{2, 2} - h^1 + h^2) - 3) c^{(0)}(a) f_{01}^{(3)} \]
\[ - \left( L_2^+ - \frac{1}{4} \tau_3(I_{2, 2} - h^1 + h^2) - 1 \right) c^{(0)}(a) f_{10}^{(3)} \]
\[ - (S - \tau_3(e_2^1)) c^{(0)}(a) f_{11}^{(3)} + (S' + \tau_3(e_+^2)) c^{(0)}(a) f_{00}^{(3)} \]
\[ = (L_1 - u/4) c^{(0)}(a) f_{10}^{(3)} - (L_2^+ - u/4) c^{(0)}(a) f_{01}^{(3)} \]
\[ + S' c^{(0)}(a) f_{00}^{(3)} - S c^{(0)}(a) f_{11}^{(3)}. \]

The last equation implies (52).

Next we consider equations for \( c^{(2)} \) and \( c^{(4)} \). In such cases, \( K \)-equivariance of projectors \( P := P^{(\cdot, \cdot)}_{1, 1} \) and \( \bar{P} := \bar{P}^{(\cdot, \cdot)}_{1, 1} \) makes our calculation easier. By Proposition 6.1,

\[ \phi_2(a) = \text{Rad}(P \circ \nabla^+) \phi_1(a) \]
\[ = \left( L_1 - \frac{\sqrt{-1}}{2} \tau_2(H_{13}) - 3 \right) P(F_1(a) \otimes X_{13}) \]
\[ + \left( L_2^- - \frac{\sqrt{-1}}{2} \tau_2(H_{24}) - 1 \right) P(F_1(a) \otimes X_{24}) \]
\[ + (S' + \tau_2(e_+^1)) P(F_1(a) \otimes X_{23}) + (S - \tau_2(e_2^2)) P(F_1(a) \otimes X_{14}). \]

By using Lemma 3.12 and from the fact that \( \tau_2([-f, f]) = 0 \), we have,
$$\phi_2(a) = \{ - (L_1 - 3 + (u + 4)/4)c_{0i}^{(1)}(a) + (L_2^+ - 1 + (u + 4)/4)c_{i0}^{(1)}(a) \\
+ S'c_{11}^{(0)}(a) - Sc_{00}^{(0)}(a) \} f^{(0)}_{00}.$$ 

This gives (53). Correspondingly, the calculation for $\phi_4$ is as follows.

$$\phi_4(a) = \operatorname{Rad}(\tilde{P} \circ \nabla^-) \phi_3(a)$$

$$= \left( L_1 + \frac{\sqrt{-1}}{2} \tau_4(H_{13}) \right) \tilde{P}(F_3(a) \otimes X_{31})$$

$$+ \left( L_2^{+} + \frac{\sqrt{-1}}{2} \tau_4(H_{24}) - 1 \right) \tilde{P}(F_3(a) \otimes X_{42})$$

$$+ (S - \tau_4(e^1_4)) \tilde{P}(F_3(a) \otimes X_{32}) + (S' + \tau_4(e^2_4)) \tilde{P}(F_3(a) \otimes X_{41})$$

$$= (-L_1 + (u - 4)/4 + 3)c_{10}^{(3)}(a)f_{00}^{(4)}$$

$$+ (L_2^{+} - (u - 4)/4 - 1)c_{01}^{(3)}(a)f_{00}^{(4)}$$

$$- Sc_{00}^{(3)}(a)f_{00}^{(4)} + S'c_{11}^{(3)}(a)f_{00}^{(4)}.$$ 

This implies (54), hence the lemma is proved. \[ \square \]

**Proposition 6.3.** Let $F \in C^\infty_{\nu, \tau_0}(N \backslash G/K)$ and put $\phi = F \mid_A \in C^\infty(A)$. Then,

$$(55) \quad \operatorname{Rad}_{d_0}(D^{\text{up}}) \phi = 2 \{(L_1 + u/4 - 1)(L_2 + u/4) - SS'\} \phi,$$

$$(56) \quad \operatorname{Rad}_{d_0}(D^{\text{down}}) \phi = 2 \{(L_1 - u/4 - 1)(L_2^{+} - u/4) - SS'\} \phi.$$ 

This is the direct consequence of Lemma 6.2. We can also check directly that they commute with the Casimir operator, namely:

**Proposition 6.4.** Let $F \in C^\infty_{\nu, \tau_0}(N \backslash G/K)$. Then,

$$\operatorname{Rad}_{d_0}(D^{\text{up}}) \circ \operatorname{Rad}_{d_0}(\Omega) \phi = \operatorname{Rad}_{d_4}(\Omega) \circ \operatorname{Rad}_{d_0}(D^{\text{up}}) \phi,$$

$$\operatorname{Rad}_{d_0}(D^{\text{down}}) \circ \operatorname{Rad}_{d_0}(\Omega) \phi = \operatorname{Rad}_{d_4}(\Omega) \circ \operatorname{Rad}_{d_0}(D^{\text{down}}) \phi.$$

### 6.3. Radial part of shift operators (2-dimensional $K$-type case)

Now we calculate the radial part of the shift operators in the case when a given principal series representation $\tau$ has a 2-dimensional $K$-type with multiplicity one.

Put $d_0 = [r, s; u], d_{11} = [r + 1, s - 1; u \pm 2], d_{12} = [r - 1, s + 1; u \pm 2]$. We use the same convention $\tau_j, f_{pq}^{(j)}$ for given $d_j$ as in the previous subsection. Define shift operators:

$$\mathcal{E}^{(\pm, \mp)} = P^{(\pm, \mp)} \circ \nabla^+, \quad \tilde{\mathcal{E}}^{(\pm, \mp)} = \tilde{P}^{(\pm, \mp)} \circ \nabla^-.$$ 

For $\phi_0 \in C^\infty_{\nu, \tau_0}(N \backslash G/K)$, we write

$$(57) \quad \phi_0(a) = \sum_{0 \leq k \leq r, 0 \leq l \leq s} c_{kl}^{(0)}(a)f_{kl}^{(0)}.$$ 

The contragredient representations of $\tau_j$ ($j = \pm 1, \pm 2$) appear in $\pi \mid_K$ with mul-
ticiality one when $[\pi|_K : \tau_0^*] = 1$. Put $\phi_1 = \mathcal{E}^{(+,-)}\phi_0$, $\phi_2 = \mathcal{E}^{(-,+)}\phi_0$, $\phi_{-1} = \mathcal{E}^{(+,-)}\phi_0$ and $\phi_{-2} = \mathcal{E}^{(-,+)}\phi_0$. Put also,

$$\phi_j(a) = \sum_{k,l} c_{k,l}^{(j)}(a)f_{k,l}^{(j)} \quad (j = \pm 1, \pm 2).$$

Throughout this subsection we use the convention that undefined coefficients $c_{k,l}^{(j)}(a)$ are zero. Then the coefficients $c_{k,l}^{(j)}$ are described as follows:

**Lemma 6.5.** Put $\nu_1 = (-r + s + u)/4$, $\nu_2 = (r - s + u)/4$. Then,

$$c_{k,l}^{(1)} = \left( \begin{array}{c} (l-s)S \\ -(l+1)(L_1 - 3\nu_1 + u - (k-l)/2) \\ (l-s)(L_2^+ + \nu_2 - (k-l)/2) \\ -(l+1)S' \end{array} \right) \left( \begin{array}{c} c_{k-1,l}^{(0)} \\ c_{k-1,l+1}^{(0)} \\ c_{k,l}^{(0)} \\ c_{k+1,l}^{(0)} \end{array} \right),$$

$$c_{k,l}^{(2)} = \left( \begin{array}{c} (k-r)S \\ (r-k)(L_1 + \nu_1 - (k-l)/2 - 1) \\ (k+1)(L_2^+ + \nu_2 - (k-l)/2 - 1) \\ -(k+1)S' \end{array} \right) \left( \begin{array}{c} c_{k,l}^{(0)} \\ c_{k,l+1}^{(0)} \\ c_{k+1,l}^{(0)} \\ c_{k+1,l+1}^{(0)} \end{array} \right),$$

$$c_{k,l}^{(1)} = \left( \begin{array}{c} (l-s)S \\ (s-l)(L_1 - \nu_1 + (k-l)/2 - 1) \\ -(l+1)S' \end{array} \right) \left( \begin{array}{c} c_{k-1,l-1}^{(0)} \\ c_{k-1,l+1}^{(0)} \\ c_{k,l}^{(0)} \\ c_{k+1,l}^{(0)} \end{array} \right),$$

$$c_{k,l}^{(-2)} = \left( \begin{array}{c} (k-r)S \\ -(k+1)(L_1 + 3\nu_1 - u + (k-l)/2) \\ -(k+1)S' \end{array} \right) \left( \begin{array}{c} c_{k,l}^{(0)} \\ c_{k+1,l-1}^{(0)} \\ c_{k+1,l}^{(0)} \end{array} \right).$$

**Proof.** We write $c_{k,l}(a) := c_{k,l}^{(0)}(a)$. Using $K$-equivariance of $P^{(\pm, \mp)}$, $\tilde{P}^{(\pm, \mp)}$ and the fact that $[\iota, \iota]$ acts trivially on $\phi_j$'s, Proposition 6.1 says that

$$\phi_j(a) = \mathcal{E}^{(\pm, \mp)}\phi_0(a) = \sum_{k,l} P^{(\pm, \mp)} \circ \nabla^+(c_{k,l}(a)f_{k,l}^{(0)})$$

$$\left\{ \left( L_1 - \frac{\sqrt{-1}}{2} \tau_j(H_{13}) - 3 \right) c_{k,l}(a) P^{(\pm, \mp)}(f_{k,l}^{(0)} \otimes X_{13}) + \left( L_2^+ - \frac{\sqrt{-1}}{2} \tau_j(H_{24}) - 1 \right) c_{k,l}(a) P^{(\pm, \mp)}(f_{k,l}^{(0)} \otimes X_{24}) + (S' + \tau_j(e_1^+)) c_{k,l}(a) P^{(\pm, \mp)}(f_{k,l}^{(0)} \otimes X_{23}) + (S - \tau_j(e_2^+)) c_{k,l}(a) P^{(\pm, \mp)}(f_{k,l}^{(0)} \otimes X_{14}) \right\}.$$
\[
\phi_{-j}(a) = \tilde{E}^{(\pm, \mp)} \phi_0(a) = \sum_{k,l} \tilde{P}^{(\pm, \mp)} \circ \nabla^-(c_{kl}(a) f_{kl}^{(0)}) \\
= \sum_{k,l} \left\{ \left( L_1 + \frac{\sqrt{-1}}{2} \tau_{-j}(H_{13}) - 3 \right) c_{kl}(a) \tilde{P}^{(\pm, \mp)} (f_{kl}^{(0)} \otimes X_{31}) + \left( L_2 + \frac{\sqrt{-1}}{2} \tau_{-j}(H_{24}) - 1 \right) c_{kl}(a) \tilde{P}^{(\pm, \mp)} (f_{kl}^{(0)} \otimes X_{42}) + (S - \tau_{-j}(e_1^\perp)) c_{kl}(a) \tilde{P}^{(\pm, \mp)} (f_{kl}^{(0)} \otimes X_{32}) + (S' + \tau_{-j}(e_2^\perp)) c_{kl}(a) \tilde{P}^{(\pm, \mp)} (f_{kl}^{(0)} \otimes X_{41}) \right\},
\]

for \( j = 1, 2 \). We transform these equations case by case, by using Lemma 3.12.

\textbf{Case I:} \( c_{kl}^{(1)} \). We write \( f_{kl} := f_{kl}^{(1)} \). By Equation (63),

\[
\phi_1 = \sum_{k,l} \left\{ -(l(L_1 + \frac{1}{4} \tau_1(I_{2,2} + h^1 - h^2) - 3)) c_{kl} f_{k,l+1,l-1} + (l-s)(L_2^z + \frac{1}{4} \tau_1(I_{2,2} - h^1 + h^2) - 1)) c_{kl} f_{k,l+1,l-1} - l(S' + \tau_1(e_1^\perp)) c_{kl} f_{k,l+1,l-1} + (l-s)(S - \tau_1(e_2^\perp)) c_{kl} f_{k,l+1,l-1} \right\}
\]

This implies Equation (59).

\textbf{Case II:} \( c_{kl}^{(2)} \). We write \( f_{kl} := f_{kl}^{(2)} \) here. By Equation (63),

\[
\phi_2 = \sum_{k,l} \left\{ (r-k)(L_1 + \frac{1}{4} \tau_2(I_{2,2} + h^1 - h^2) - 3)) c_{kl} f_{k,l+1,l-1} + k(L_2^z + \frac{1}{4} \tau_2(I_{2,2} - h^1 + h^2) - 1)) c_{kl} f_{k,l-1,l+1} - k(S' + \tau_2(e_1^\perp)) c_{kl} f_{k,l-1,l+1} + (k-r)(S - \tau_2(e_2^\perp)) c_{kl} f_{k,l-1,l+1} \right\}
\]

This implies Equation (59).
This shows Equation (60).

**Case III: $c_{kl}{^{(-1)}}$.** In this case, we set $f_{kl} := f_{kl}{^{(-1)}}$. Then Equation (64) implies,

$$
\phi_{-1} = \sum_{k,l} \{(s - l)(L_1 - \frac{1}{4} \tau_{-1}(J_{2,2} + h^1 - h^2) - 3)c_{kl}f_{kl}
+ l(L_2 - \frac{1}{4} \tau_{-1}(J_{2,2} - h^1 + h^2) - 1)c_{kl}f_{k,l-1}
+ (l - s)(S - \tau_{-1}(e_+^1))c_{kl}f_{k+1,l-1} - l(S' + \tau_{-1}(e_+^2))c_{kl}f_{k,l-1}\}
= \sum_{k,l} \{(s - l)(L_1 - \nu_1 + (k - l)/2 - 1)c_{kl}f_{kl} + (l - s)S'c_{kl}f_{k+1,l-1}\}
+ (l + 1)(L_2 - \nu_2 + (k - l)/2 - 1)c_{k-1,l+1}(l + 1)S'c_{k,l+1}f_{kl}.
$$

Therefore Equation (61) follows.

**Case IV: $c_{kl}{^{(-2)}}$.** Set $f_{kl} := f_{kl}{^{(-2)}}$, then Equation (64) shows,

$$
\phi_{-2} = \sum_{k,l} \{(-k)(L_1 - \frac{1}{4} \tau_{-2}(J_{2,2} + h^1 - h^2) - 3)c_{kl}f_{k-1,l+1}
+ (k - r)(L_2 - \frac{1}{4} \tau_{-2}(J_{2,2} - h^1 + h^2) - 1)c_{kl}f_{k,l}
+ (k - r)(S - \tau_{-2}(e_1^2))c_{kl}f_{k,l+1} - k(S' + \tau_{-2}(e_1^2))c_{kl}f_{k-1,l-1}\}
= \sum_{k,l} \{(-k)(L_1 - \nu_1 + (k - l)/2 - r + s - 1)c_{kl}f_{k-1,l+1} - kS'c_{kl}f_{k,l-1}\}
+ (k - r)(L_2 - \nu_2 + (k - l)/2)c_{kl}f_{k,l} + (k - r)S'c_{kl}f_{k,l+1}\}
= \sum_{k,l} \{(-k + 1)(L_1 + 3\nu_1 - u + (k - l)/2)c_{k+1,l-1} + (k - r)S'c_{k,l-1}\}
+ (k - r)(L_2 - \nu_2 + (k - l)/2)c_{kl} - (k + 1)S'c_{k+1,l}f_{kl}.
$$

This implies Equation (62) and the lemma is proved.

In particular, $\tau_{[r,s,u]}$ is 2-dimensional if and only if $(r,s) = (0,1)$ or $(1,0)$. So we rewrite Lemma 6.5 as follows:

**Corollary 6.6.**

(i) Let $r = 0$ and $s = 1$. Then,

$$
\begin{pmatrix}
(0)_{c_{00}}^{(1)} \\
(1)_{c_{10}}^{(1)}
\end{pmatrix}
= \begin{pmatrix}
-(L_2 - (u - 1)/4) & -S' \\
-S & -(L_1 + (u - 5)/4)
\end{pmatrix}
\begin{pmatrix}
(0)_{c_{00}}^{(0)} \\
(0)_{c_{01}}^{(0)}
\end{pmatrix},
$$

(ii) Let $r = 1$ and $s = 0$. Then,

$$
\begin{pmatrix}
(0)_{c_{00}}^{(-1)} \\
(-1)_{c_{10}}^{(-1)}
\end{pmatrix}
= \begin{pmatrix}
L_1 - (u + 5)/4 & -S' \\
-S & L_2' - (u + 1)/4
\end{pmatrix}
\begin{pmatrix}
(0)_{c_{00}}^{(0)} \\
(0)_{c_{01}}^{(0)}
\end{pmatrix}.$$

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We see from this corollary, that Equation (66) (resp. (68)) can be recovered from (65) (resp. (67)) with the following rule:

\[
\begin{pmatrix}
    c_{00}^{(0)} \\
    c_{01}^{(0)} \\
    c_{10}^{(1)} \\
    c_{11}^{(1)}
\end{pmatrix} 
\rightarrow 
\begin{pmatrix}
    -c_{01}^{(0)} \\
    -c_{00}^{(0)} \\
    c_{10}^{(-1)} \\
    c_{11}^{(-1)}
\end{pmatrix},
\]

In addition, one can check the following directly:

**Proposition 6.7.** Let \( \text{Rad}_d = \text{Rad}_{\tau_d} \) be the radial part. Then we have,

\[
\begin{align*}
\text{Rad}_{d_0}(\mathcal{E}^{(+,-)}) &\circ \text{Rad}_{d_0}(\Omega) = \text{Rad}_{d_1}(\Omega) \circ \text{Rad}_{d_0}(\mathcal{E}^{(+,-)}), \\
\text{Rad}_{d_0}(\mathcal{E}^{(-,+)} ) &\circ \text{Rad}_{d_0}(\Omega) = \text{Rad}_{d_2}(\Omega) \circ \text{Rad}_{d_0}(\mathcal{E}^{(-,+)}), \\
\text{Rad}_{d_0}(\mathcal{E}^{(+,-)} ) &\circ \text{Rad}_{d_0}(\Omega) = \text{Rad}_{d_{-1}}(\Omega) \circ \text{Rad}_{d_0}(\mathcal{E}^{(+,-)}), \\
\text{Rad}_{d_0}(\mathcal{E}^{(-,+)} ) &\circ \text{Rad}_{d_0}(\Omega) = \text{Rad}_{d_{-2}}(\Omega) \circ \text{Rad}_{d_0}(\mathcal{E}^{(-,+)}).
\end{align*}
\]

7. **Proofs of the Main Results**

Throughout this section, we use for notation \( \tau_j, f_{kl}^{(j)}, c_{kl}^{(j)} \) for a given \( d_j \), as in the previous section.

7.1. **Proof of Theorem 4.2**

For \( d_0 = [0,0;u] \), put \( \Phi = \Phi_{\tau_0} = c_{00}^{(0)} f_{00}^{(d_0)} \). We show Equation (33) firstly. Lemma 5.2 says that the Casimir operator acts on Whittaker vectors by a constant multiple. Putting \( r = 0, s = 0 \) in Lemma 5.1, we have

\[
(L_0 + u \eta_2 a_2^2) \Phi = (\mu_1 \mu_2 - 10) \Phi.
\]

The translation from \( \Phi(a) \) to \( I(a) = a_1^{-2} a_2^{-1} c_{00}^{(0)}(a) \) yields (33).

Next, we verify Equation (34). Put \( d_1 = [1,1;u+2] \) and \( d_2 = [0,0;u+4] \). Write,

\[
\nabla^+ \Phi = \sum_{0 \leq k \leq 1, 0 \leq l \leq 1} c_{kl}^l f_{kl}^{(1)}, \quad \mathcal{D}^{++} \Phi = c_{00}^{(0)} f_{00}^{(2)},
\]

\[
\nabla^- \circ \mathcal{D}^{++} \Phi = \sum_{0 \leq k \leq 1, 0 \leq l \leq 1} c_{kl}^m f_{kl}^{(1)}.
\]

Considering (48), (17) and Lemma 3.12, we have the following equations:

\[
\nabla^+ \Phi = (X_{32} \cdot c_{00}^{(0)}) f_{00}^{(1)} - (X_{42} \cdot c_{00}^{(0)}) f_{01}^{(1)} + (X_{31} \cdot c_{00}^{(0)}) f_{10}^{(1)}
\]

\[
- (X_{41} \cdot c_{00}^{(0)}) f_{11}^{(1)}.
\]
Concluding, we obtain,

\[
\begin{pmatrix}
-X_{23} \\
X_{24} \\
-X_{13} \\
X_{14}
\end{pmatrix}
\begin{pmatrix}
X_{14} \\
X_{13} \\
-X_{24} \\
X_{32}
\end{pmatrix}
\begin{pmatrix}
-X_{41} \\
-X_{31} \\
X_{42} \\
X_{31}
\end{pmatrix}
\begin{pmatrix}
X_{32} \\
X_{10} \\
-X_{41}
\end{pmatrix}
\Phi = \mathcal{D}^{\text{down}} \circ \mathcal{D}^{\text{up}} \Phi.
\]

(70)

We know from Proposition 3.15 that the left-hand side of (70) has \( \Phi \) as an eigenfunction with eigenvalue \((\pi - (u/2) - 1)(\pi^2 - (u/2) - 1)(\pi + (u/2) + 1)\times(\pi^2 + (u/2) + 1)\). On the other hand, using Proposition 6.3, we have,

\[
I' = a_1^{-3}a_2^{-1} \text{Rad}_{\mathcal{D}^{\text{up}}} \mathcal{D}^{(0)}
\]

\[
= 2\left\{ \left( \mathcal{L}_1 + \frac{u}{4} + \frac{1}{2} \right) \left( \mathcal{L}_2 - \frac{u}{4} + \frac{1}{2} \right) - SS' \right\} I,
\]

\[
a_1^{-3}a_2^{-1} \text{Rad}_{\mathcal{D}^{\text{down}} \circ \mathcal{D}^{\text{up}}} \mathcal{D}^{(0)}
\]

\[
= 2\left\{ \left( \mathcal{L}_1 - \frac{u}{4} - \frac{1}{2} \right) \left( \mathcal{L}_2 - \frac{u}{4} - \frac{1}{2} \right) - SS' \right\} I'.
\]

Therefore the radial part of \( \mathcal{D}^{\text{down}} \circ \mathcal{D}^{\text{up}} \) is,

\[
a_1^{-3}a_2^{-1} \text{Rad}_{\mathcal{D}^{\text{down}} \circ \mathcal{D}^{\text{up}}} \mathcal{D}^{(0)}
\]

\[
= \frac{1}{4} \left\{ \left( \partial_1 - \frac{u}{2} + 1 \right) \left( \partial_2 + \frac{u}{2} + 1 \right) + \eta_2 a_2^2 \right\} - \eta_0 \frac{1}{a_2^2} \left\{ \left( \partial_1 + \frac{u}{2} + 1 \right) \left( \partial_2 + \frac{u}{2} + 1 \right) - \eta_0 \frac{1}{a_2^2} \right\} I
\]

\[
= \frac{1}{4} \left\{ \left( \partial_1^2 - \frac{u^2}{2} + 1 \right) \left( \partial_2^2 - \frac{u^2}{2} + 1 \right) + \eta_2 a_2^2 \right\} \left( 2(\partial_1 + 1)(\partial_2 - 1) + \frac{u^2}{2} + 2u - u\eta_2 a_2^2 + \eta_0 \frac{a_1}{a_2} \right) I.
\]

Remark 7.1. We also obtain these equations by the method used in [4, §10]. See also [2]. (cf. [4, Remark (12.1)].)

7.2. Proof of Theorem 4.3

Assume \( |n| = 1 \). Put \( d_{\pm 1} = [0, 1; \pm u] \), \( d_{\pm 2} = [0, 1; \pm (u + 2)] \), \( d_{\pm 3} = [1, 0; \pm u] \) and \( d_{\pm 4} = [0, 1; \pm (u + 2)] \). Then, \( \pi \mid_{x : \tau_1} = 1 \) if \( u \equiv ne(-1) \) and \( \pi \mid_{x : \tau_3} = 1 \) if \( u \equiv -ne(-1) \). We identify \( \tau_{-j} \) with \( \tau_{j} \) through the map in Proposition 3.9.
Let $\Phi_\pi$ be a Whittaker vector and let $\Phi_{\pi, \tau_j}$ be a Whittaker function of $\pi$ with $K$-type $\tau_j^*$, $(j = 1, \ldots, 4)$. Namely, for any $\nu^* \in V_{\tau_j}^*$,

\begin{equation}
\langle \nu^*, \Phi_{\pi, \tau_j}(g) \rangle = \Phi_{\pi}(\nu^*(v^*))(g).
\end{equation}

We write $\Phi_{\pi, \tau_j}(g) = \sum_{k, l} c_{kl}^{(j)}(g) f_{kl}^{(j)}$. By Corollary 3.7, we can take $a_{kl}^{(-j)}$ in place of $\nu^*$ in (71); then we have,

\begin{equation}
c^{(1)}_{00}(g) = \langle f^{(1)}_{00}, \Phi_{\pi, \tau_j}(g) \rangle = \Phi_{\pi}(\nu_{\tau_j}(f^{(-1)}_{00}))(g) = \Phi_{\pi}(a^{(-1)}_{00})(g).
\end{equation}

Similarly,

\begin{equation}
c^{(j)}_{00} = \Phi_{\pi}(a^{(-j)}_{00}), \quad c^{(j)}_{01} = -\Phi_{\pi}(a^{(-j)}_{00}) \quad \text{for } j = 1, 4,
\end{equation}

\begin{equation}
c^{(j)}_{00} = \Phi_{\pi}(a^{(-j)}_{10}), \quad c^{(j)}_{01} = -\Phi_{\pi}(a^{(-j)}_{10}) \quad \text{for } j = 2, 3.
\end{equation}

Since $\Phi_{\pi}$ is in particular a $\mathfrak{g}$-homomorphism, one obtains the following equations from Corollary 3.19:

\begin{equation}
\begin{pmatrix}
X_{42} \\
X_{41}
\end{pmatrix}
\begin{pmatrix}
X_{32} \\
X_{31}
\end{pmatrix}
= \frac{2(\mu_{\delta(2)} + u + 1)}{4}
\begin{pmatrix}
c^{(1)}_{00} \\
c^{(1)}_{01}
\end{pmatrix},
\end{equation}

\begin{equation}
\begin{pmatrix}
X_{24} \\
X_{23}
\end{pmatrix}
\begin{pmatrix}
X_{14} \\
X_{13}
\end{pmatrix}
= \frac{2(\mu_{\delta(2)} - u - 1)}{4}
\begin{pmatrix}
c^{(2)}_{00} \\
c^{(2)}_{01}
\end{pmatrix},
\end{equation}

\begin{equation}
\begin{pmatrix}
X_{31} \\
-X_{32}
\end{pmatrix}
\begin{pmatrix}
X_{32} \\
X_{42}
\end{pmatrix}
= \frac{2(\mu_{\delta(1)} + u + 1)}{4}
\begin{pmatrix}
c^{(3)}_{00} \\
c^{(3)}_{01}
\end{pmatrix},
\end{equation}

\begin{equation}
\begin{pmatrix}
-X_{41} \\
-X_{42}
\end{pmatrix}
\begin{pmatrix}
X_{42} \\
X_{24}
\end{pmatrix}
= \frac{2(\mu_{\delta(1)} - u - 1)}{4}
\begin{pmatrix}
c^{(4)}_{00} \\
c^{(4)}_{01}
\end{pmatrix},
\end{equation}

where $(\delta(1), \delta(2)) = (1, 2)$, (resp. $(\delta(1), \delta(2)) = (2, 1)$) if $n = 1$, (resp. $n = -1$). We actually have obtained shift operators, i.e., by Definition (48) and Lemma 3.12, Equations (72) and (73) (resp. (74) and (75)) turn out to be,

\begin{equation}
\mathcal{E}^{(-, +)} \circ \mathcal{E}^{(+, -)} \Phi_{\pi, \tau_1} = \left( \left( \frac{\mu_{\delta(2)}}{2} \right)^2 - \left( \frac{u + 1}{4} \right)^2 \right) \Phi_{\pi, \tau_1},
\end{equation}

\begin{equation}
\text{resp. } \mathcal{E}^{(+, +)} \circ \mathcal{E}^{(-, -)} \Phi_{\pi, \tau_1} = \left( \left( \frac{\mu_{\delta(1)}}{2} \right)^2 - \left( \frac{u + 1}{4} \right)^2 \right) \Phi_{\pi, \tau_1}.
\end{equation}

Now, we take the radial part of the both sides of the above equations, which will finish this proof. Corollary 6.6 says that, from (76),

\begin{equation}
\begin{pmatrix}
L_2^+ - (u + 3)/4 \\
S
\end{pmatrix}
\begin{pmatrix}
S' \\
L_1 - (u + 7)/4
\end{pmatrix}
= \frac{1}{4}
\begin{pmatrix}
c^{(1)}_{00} \\
c^{(1)}_{01}
\end{pmatrix}
\begin{pmatrix}
\mu_{\delta(2)}^2 - \left( \frac{u + 1}{2} \right)^2 \\
\mu_{\delta(2)}^2 - \left( \frac{u + 1}{2} \right)^2
\end{pmatrix}
\begin{pmatrix}
c^{(1)}_{00} \\
c^{(1)}_{01}
\end{pmatrix}.
\end{equation}

By multiplying matrices in the left-hand side, we have, by Equation (78),
Keeping in sight of (69), we obtain the differential equations of $\Phi_{\tau_1, \tau_2}$ immediately from (79). Therefore, translating Equations (79) to those of $b_{kl}^{(j)} := a_1^{-3} a_2^{-1} c_{kl}^{(j)}$, we obtain Equations (35) and (37).

Equations (36) and (38) are a direct consequence of the Casimir operator, Lemmas 5.1 and 5.2. □

Remark 7.2. Put $\tau = \tau_{r,s,u}$ and $\tau' = \tau_{r,s,u+4}$. If $r = 0$ and $s = 0$, $D^{up} \Phi_{\tau, \tau}$ becomes a Whittaker function of $\pi$ with $K$-type $\tau'$. This can be shown by seeing that $D^{up} \Phi_{\tau, \tau}$ satisfies the differential equations (33) and (34) with $u + 4$ in place of $u$.

Similarly if $r = 0$ and $s = 1$ (resp. $r = 1$ and $s = 0$), $\mathcal{E}^{(+,-)} \Phi_{\tau, \tau}$ (resp. $\mathcal{E}^{(-,+)} \Phi_{\tau, \tau}$) satisfies the differential equations (37) and (38) with a suitable change of parameter.

Remark 7.3. We also obtain another differential equation using the radial part of the operator $\mathcal{E}^{(-,+)} \circ \mathcal{E}^{(+,-)}$ instead. Apparently this equation is dependent; in fact the operator $\mathcal{E}^{(-,+)} \circ \mathcal{E}^{(+,-)} + \mathcal{E}^{(-,+)} \circ \mathcal{E}^{(+,-)}$ differs from the Casimir operator only on constant terms.

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