Seminorms on ordered vector spaces that extend to Riesz seminorms on larger Riesz spaces

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Communicated by Prof. M.S. Keane at the meeting of November 25, 2002

ABSTRACT

As a generalization of the notion of Riesz seminorm, a class of seminorms on directed partially ordered vector spaces is introduced, such that (1) every seminorm in the class can be extended to a Riesz seminorm on every larger Riesz space that is majorized and (2) a seminorm on an order dense linear subspace of a Riesz space is in the class if and only if it can be extended to a Riesz seminorm on the Riesz space. The latter property yields that if a directed partially ordered vector space has an appropriate Riesz completion, then a seminorm on the space is in the class if and only if it is the restriction of a Riesz seminorm on the Riesz completion. An explicit formula for the extension is given. The class of seminorms is described by means of a notion of solid unit ball in partially ordered vector spaces. Some more properties concerning restriction and extension are studied, including extension to L- and M-seminorms.

1. INTRODUCTION

A major part of the theory of norms on Riesz spaces (vector lattices) is devoted to Riesz norms. Recall that a (semi)norm \( \rho \) on a Riesz space \( E \) is called a Riesz (semi)norm if for every \( x, y \in E \):

\[
(1) \quad 0 \leq x \leq y \implies \rho(x) \leq \rho(y) \quad \text{(monotonicity), and}
\]

\[
(2) \quad \rho(\|x\|) = \rho(x).
\]

As the notion of Riesz norm involves absolute values, it has no direct generalization to partially ordered vector spaces. Monotonicity does make sense on partially ordered vector spaces but is in general much weaker. Especially results concerning dual spaces and operators rely heavily on condition (2).
There are several ways to extend the notion of Riesz norm or Riesz seminorm to larger classes of partially ordered vector spaces than the Riesz spaces. This paper considers seminorms and presents three such ways. It will come out that they all lead to the same class of norms. The first approach is based on a formulation of the notion of 'solid unit ball' without using lattice operations, the second one on extendability to Riesz seminorms on larger Riesz spaces, and the third one on restriction of Riesz seminorms on Riesz completions.

Let us list some terminology before we discuss the results. Let $E$ and $F$ be partially ordered vector spaces. A linear map $A: E \to F$ is called bipositive if for every $x \in E$ one has that $x \geq 0 \iff Ax \geq 0$. A linear subspace $E_0$ of $E$ is called majorizing if for every $x \in E$ there is a $y \in E_0$ such that $y \geq x$ and it is called order dense if $x = \inf\{y \in E_0; y \geq x\}$ for every $x \in E$. The space $E$ is called directed if for every $x, y \in E$ there is a $z \in E$ with $z$ and $z \geq y$, and integrally closed if $nx \leq y$ for all $n \in \mathbb{N}$ implies that $x \leq 0$, for every $x, y \in E$. For other standard terminology, we refer to [1, 3, 9, 11]. By terms as 'isomorphism' and 'embedding' without further specifications we will mean maps that are linear and bipositive.

For our third approach we need to restrict to spaces that have proper Riesz completions, that is, for which there exist 'smallest' Riesz spaces in which they can be embedded. Existence of such spaces is rather delicate and depends on the kind of isomorphisms and embedding maps that are considered. We will follow the definition of Van Haandel in [7] with the additional condition that the spaces can linearly, bipositively be embedded in their Riesz completions. Then a directed partially ordered vector space has a Riesz completion if and only if it can be embedded as an order dense linear subspace in a Riesz space. We will mainly work with the latter formulation and do not come back to details of Riesz completions before Section 5. It can be shown that every directed partially ordered vector space that is integrally closed has an essentially unique Riesz completion.

Our three approaches lead to a class of seminorms on partially ordered vector spaces that we will call pre-Riesz seminorms. They are such that:

(a) each pre-Riesz seminorm on a directed partially ordered vector space $E$ can be extended to a Riesz seminorm on every Riesz space in which $E$ can be embedded as a majorizing linear subspace (Theorem 3.4),

(b) a seminorm on an order dense linear subspace of a Riesz space is pre-Riesz if and only if it can be extended to a Riesz seminorm on the Riesz space (Theorem 3.6), and

(c) a seminorm on a partially ordered vector space that has a Riesz completion is pre-Riesz if and only if it is the restriction of a Riesz seminorm on the Riesz completion (Theorem 5.5).

In Section 2 we discuss a suitable notion of 'solid unit ball' in partially ordered vector spaces. We consider a joint generalization of solidness and convexity, called solvexitv. The pre-Riesz seminorms are defined as the seminorms with solvex unit balls. In Section 3 we consider extensions and infer properties (a) and (b). Section 4 studies uniqueness of extensions and extension to L- and
M-semnorms. Section 5, finally, deals with Riesz completions and establishes (c).

The philosophy behind our methods is given by a result in [8]: every partially
ordered vector space can be embedded in a Riesz space and, consequently, ev-
every directed partially ordered vector space is isomorphic to a majorizing sub-
space of a Riesz space. We need this theorem only in Proposition 4.2.

2. SOLID SETS AND SOLVEX SETS

A seminorm on a Riesz space is a Riesz seminorm if and only if its unit ball is a
solid set. Therefore extending the notion of Riesz seminorm to all partially or-
dered vector spaces comes down to extending the notion of solid unit ball.

Recall that a subset $S$ of a Riesz space $E$ is called solid if $|x| \leq |y|$ with $x \in E$
and $y \in S$ implies that $x \in S$. The assertion $|x| \leq |y|$ means the same as 'every
upper bound of $\{y, -y\}$ is an upper bound of $\{x, -x\}$. The latter formulation
also makes sense if $E$ is a partially ordered vector space that is not a lattice. We
denote it shortly by $x \.Cascade$ $y$.

For subsets $A$ and $B$ of a partially ordered vector space $E$ we will denote the
assertion 'every upper bound of $A \cup -A$ is an upper bound of $B \cup -B$' by $A \.Cascade B$
or $B \.Cascade A$ (in words: 'A is wider than B' or 'B is narrower than A', respectively).
We will also write $a \.Cascade B$ if $A = \{a\}$ and use similar notations with their obvious
meanings. Remark that the relation $\prec$ strongly depends on the space in which
the upper bounds are taken. For instance, the assertion $(x \prec 1) \prec (x \prec x)$ is true
in the space of affine functions on $[-1, 1]$ but fails in $C[-1, 1]$. If there is risk of
confusion, the space will be explicitly specified.

Definition 2.1. Let $E$ be a partially ordered vector space. A set $S \subset E$ is called
solid if for every $x \in E$ and $y \in S$ such that $x \prec y$ one has that $x \in S$. For $S \subset E$
the set $\text{sol} S := \{x \in E: \text{there exists } y \in S \text{ such that } x \prec y\}$ is the smallest solid
set in $E$ containing $S$ and it is called the solid hull of $S$.

Note that the intersection of a solid set and a linear subspace need not be solid
in the subspace. For instance, the closed unit ball of the 1-norm $\|f\|_1 = \int_0^1 |f(t)| dt$
on $C[0, 1]$ is solid but its intersection with the subspace of affine functions is not solid in that subspace. In case that $E_0$ is an order dense
linear subspace of a Riesz space $E$, one has for each solid subset $S$ of $E$
that $E_0 \cap S$ is solid in $E_0$. Indeed, for every $x \in E_0$ and $y \in E_0 \cap S$ with $x \prec y$ in $E_0$ one has that $|x| = \inf \{z \in F_0^+: z \geq |x|\} \leq \inf \{z \in F_0^+: z \geq |y|\} = |y|$, so that
$x \in E_0 \cap S$.

By the unit ball of a seminorm $p$ on a vector space $E$ we mean the closed ball
$\{x \in E: p(x) \leq 1\}$. Notice that for any convex, circled, absorbing set $B$ the set
$\bar{B} := \cap_{\alpha \geq 1} \alpha B$ is a unit ball.

The unit ball of a seminorm $p$ on a partially ordered vector space $E$ is solid if
and only if $p(x) \leq p(y)$ for all $x, y \in E$ with $x \prec y$. In order to extend a seminorm
with a solid unit ball $B$ we want to take the convex hull of the solid hull (denoted
by $\text{co sol}$) in the larger space. The convex hull of a solid set in a Riesz space is
solid (see [1, Theorem 1.3], [3, Lemma 14J(d)], or [11, Proposition 4.1]). However, co sol $B$ may become too large in the sense that co sol $B$ intersected with the subspace may be strictly larger than $B$. Such a ball is presented in Example 2.8.

The situation improves if we consider sets that are both solid and convex, instead of generalizing the notion of solid set on its own. For sake of readability we write in the sequel $\varepsilon = \pm 1$ instead of $\varepsilon \in \{-1,1\}$.

**Definition 2.2.** Let $E$ be a partially ordered vector space. A subset $S$ of $E$ is called *solvex* if for every $x \in E$, $x_1, \ldots, x_n \in S$, and $\lambda_1, \ldots, \lambda_n \in (0, 1]$ with $\sum_k \lambda_k = 1$ such that $x \in \{\sum_k \varepsilon_k \lambda_k x_k : \varepsilon_k = \pm 1\}$ one has that $x \in S$.

**Lemma 2.3.** Let $E$ be a partially ordered vector space. Every solvex set in $E$ is solid and convex. If $E$ is a Riesz space, then a set is solvex in $E$ if and only if it is solid and convex.

**Proof.** Let $S \subset E$ be solvex. It is clear that $S$ is solid, by taking $n = 1$ in the definition of solvex sets. To prove that $S$ is convex, let $x_1, \ldots, x_n \in S$ and let $\lambda_1, \ldots, \lambda_n \in (0, 1]$ with $\sum_k \lambda_k = 1$. Then $\sum_k \lambda_k x_k \in \{\sum_k \varepsilon_k \lambda_k x_k : \varepsilon_k = \pm 1\}$, so $\sum_k \lambda_k x_k \in S$, by solvexity of $S$. Hence $S$ is convex.

Assume that $E$ is a Riesz space and let $S$ be a solid and convex set in $E$. Let $x \in E$, $x_1, \ldots, x_n \in S$, and $\lambda_1, \ldots, \lambda_n \in (0, 1]$ with $\sum_k \lambda_k = 1$ be such that $x \in \{\sum_k \varepsilon_k \lambda_k x_k : \varepsilon_k = \pm 1\}$. As $S$ is solid, $|x_k| \in S$ for all $k$ and then $\sum_k \lambda_k |x_k| \in S$ by convexity of $S$. Because $|\sum_k \varepsilon_k \lambda_k x_k| \leq \sum_k \lambda_k |x_k|$ for any $\varepsilon_k \in \{-1,1\}$, $k = 1, \ldots, n$, every upper bound of $\{\sum_k \lambda_k |x_k|, -\sum_k \lambda_k |x_k|\}$ is an upper bound of $\{\sum_k \varepsilon_k \lambda_k x_k : \varepsilon_k = \pm 1\}$ and therefore one of $\{x, -x\}$. As $S$ is solid, it follows that $x \in S$. Thus, $S$ is solvex. \(\square\)

The following observation will be used many times.

**Lemma 2.4.** Let $E$ be a Riesz space and let $x_1, \ldots, x_n \in E$. Then $\sum_k |x_k| \in \{\sum_k \varepsilon_k x_k : \varepsilon_k = \pm 1\}$.

**Proof.** By induction with respect to $n$. First, observe that $|x_1| \in \{x_1, -x_1\}$ holds trivially. Then, assume that $\sum_{k=1}^{n} |x_k| \in \{\sum_k \varepsilon_k x_k : \varepsilon_k = \pm 1\}$. Let $z$ be an upper bound of $\{\sum_{k=1}^{n+1} \varepsilon_k x_k : \varepsilon_k = \pm 1\}$. Then $z - x_{n+1}$ and $z + x_{n+1}$ are upper bounds of $\{\sum_{k=1}^{n+1} \varepsilon_k x_k : \varepsilon_k = \pm 1\}$, so they are greater than $\sum_{k=1}^{n} |x_k|$. Thus, $z \geq \sum_{k=1}^{n+1} |x_k|$. In other words: $\sum_{k=1}^{n+1} |x_k| \in \{\sum_{k=1}^{n+1} \varepsilon_k x_k : \varepsilon_k = \pm 1\}$. \(\square\)

We have the following result on restrictions of solvex sets. It includes the cases that $E_0$ is order dense in $E$ and that $E_0$ is a Riesz subspace of $E$.

**Lemma 2.5.** Let $E$ be a Riesz space with a linear subspace $E_0$ such that for every $x_1, \ldots, x_n \in E_0$ one has $|x_1| + \cdots + |x_n| = \inf\{y \in E_0 : y \geq |x_1| + \cdots + |x_n|\}$. If $S$ is a solvex subset of $E$, then $E_0 \cap S$ is solvex in $E_0$. 

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Proof. Let $x \in E_0$, $x_1, \ldots, x_n \in E_0 \cap S$, and $\lambda_1, \ldots, \lambda_n \in (0, 1]$ with $\sum_k \lambda_k = 1$ be such that $x \notin \{\sum_k \varepsilon_k \lambda_k x_k : \varepsilon_k = \pm 1\}$. Then, with aid of the previous lemma,

$$|x| \leq \inf \{y \in E_0 : y \geq \sup(\sum_k \varepsilon_k \lambda_k x_k : \varepsilon_k = \pm 1)\} = \inf \{y \in E_0 : y \geq |\lambda_1 x_1| + \cdots + |\lambda_n x_n|\} = |\lambda_1 x_1| + \cdots + |\lambda_n x_n| \leq u,$$

for every $u \in E$ that is an upper bound of $\{\sum_k \varepsilon_k \lambda_k x_k : \varepsilon_k = \pm 1\}$. As $S$ is solvex, it follows that $x \in S$ and therefore $x \in E_0 \cap S$. Thus $E_0 \cap S$ is solvex in $E_0$.

The next lemma serves to extend solvex sets to larger Riesz spaces.

Lemma 2.6. Let $E$ be a Riesz space with a linear subspace $E_0$. If $S \subset E_0$ is solvex in $E_0$, then for the convex hull of the solid hull in $E$ one has $E_0 \cap \co \sol S = S$.

Proof. Let $x \in E_0 \cap \co \sol S$. Then there are $x_1, \ldots, x_n \in \sol S$ and $\lambda_1, \ldots, \lambda_n \in (0, 1]$ with $\sum_k \lambda_k = 1$ such that $x = \sum_k \lambda_k x_k$. There are $u_1, \ldots, u_n \in S$ such that $|x_k| \leq |u_k|$ for all $k$. If $z \in E_0$ is an upper bound of $\{\sum_k \varepsilon_k \lambda_k u_k : \varepsilon_k = \pm 1\}$, then Lemma 2.4 yields that $z \geq \sum_k \lambda_k |u_k| \geq \sum_k \lambda_k |x_k| \geq -z$. Because $S$ is solvex in $E_0$, it follows that $x \in S$ and therefore $E_0 \cap \co \sol S \subset S$. Conversely, $\co \sol S \supset S$, which completes the proof.

The seminorms with a solvex unit ball will be the pre-Riesz seminorms as in the introduction and the desired properties will be shown in the next sections. Let us conclude this section by an example showing that merely a solid unit ball is too weak for the desired extendability to Riesz seminorms. First we show that if there is an extending Riesz seminorm, then there is also a greatest one.

Lemma 2.7. Let $E$ be a Riesz space with a majorizing linear subspace $E_0$ and let $p$ be a seminorm on $E_0$ with unit ball $B$. Then $\co \sol B$ is the unit ball of the greatest seminorm $q$ with a solid unit ball on $E$ such that $q < p$ on $E_0$. If there exists a seminorm with a solid unit ball on $E$ extending $p$, then $q$ extends $p$ as well.

Proof. The set $B$ is absorbing in $E_0$ and $E_0$ is majorizing in $E$, so $\sol B$ is absorbing in $E$. Hence, $\co \sol B$ is the unit ball of a seminorm $q$ on $E$ and it is solvex. Moreover, $\co \sol B$ is the smallest convex, solid set in $E$ containing $B$, so $q$ is the greatest seminorm on $E$ with a solid unit ball that is less than or equal to $p$ on $E_0$. If $B'$ is the solid unit ball of a seminorm on $E$ extending $p$, then $E_0 \cap B' = B$, and $B' \supset \co \sol B$, so $B \subset E_0 \cap \co \sol B \subset E_0 \cap B' = B$. This yields that $q$ extends $p$.

If in the situation of the above lemma $E_0 \cap \co \sol B \neq B$, then there is no Riesz seminorm on $E$ extending $p$. The following example presents such a ball $B$. It also shows that the solid hull of a convex set that is solid in a subspace need not be convex.
Example 2.8. A Riesz space $E$ with an order dense linear subspace $E_0$ that generates $E$ as a Riesz space and a seminorm with a solid unit ball on $E_0$ that cannot be extended to a Riesz seminorm on $E$. Let $D$ be the unit disk in $\mathbb{R}^2$ and let $S$ be the unit circle. Take for $E_0$ the space of all restrictions to $D$ of affine functions from $\mathbb{R}^2$ to $\mathbb{R}$, with pointwise ordering on $D$. Then $E_0$ is a directed and integrally closed partially ordered vector space. Note that the ordering is the same as the pointwise ordering on $S$, so $E_0$ can be interpreted as a subspace of $C(S)$. Let $E$ be the Riesz subspace of $C(S)$ generated by $E_0$.

Take $p(x) := (\int_D x(t)^+ dt) \vee (\int_D x(t)^- dt), x \in E_0$. Then $p$ is a norm on $E_0$ such that $-y \leq x \leq y$ implies that $p(x) \leq p(y)$ for all $x, y \in E_0$. Denote for an angle $\phi \in (-\pi, \pi]$ the corresponding point in $S$ by $s_\phi$, that is, $s_\phi = (\cos \phi, \sin \phi)$. Let $R_\phi: E_0 \to E_0$ be the operator of rotation over the angle $\phi$:

$$R_\phi x(t_1, t_2) := x(t_1 \cos \phi + t_2 \sin \phi, t_2 \cos \phi - t_1 \sin \phi),$$

$(t_1, t_2) \in \mathbb{R}^2, x \in E_0, \phi \in \mathbb{R}$. The example is completed by the six claims below.

(a) For every $\alpha \in (0, \pi/2)$ and $M \in [0, \infty)$ there is a $z \in E_0$ such that $z(s_0) = -1, z \geq -1$ on $S$, and $z(s_\phi) \geq M$ for all $\phi \in (-\pi, \pi] \setminus (-\alpha, \alpha)$: Take

$$z(t_1, t_2) := \frac{M + 1}{1 - \cos \alpha} (1 - t_1) - 1, \quad (t_1, t_2) \in \mathbb{R}^2,$$

and verify that $z \in E_0, z(1, 0) = -1, z \geq -1$ on $S$, and $z(s_\phi) > M$ for all $\phi$ such that $\cos \phi \leq \cos \alpha$, that is, for all $\phi \in (-\pi, \pi] \setminus (-\alpha, \alpha)$.

(b) For every $x \in C(S)$ one has $x(t) = \inf\{u(t) : u \in E_0, u \geq x \text{ on } S\}$ for all $t \in S$: Let $x \in C(S)^+$. Because of rotation symmetry, it suffices to prove the equation for $t = s_0 = (1, 0)$. Let $\epsilon \in (0, \infty)$. Because $t \mapsto x(t)$ is continuous on $S$, there exists an $\alpha \in (0, \pi/2)$ such that $x(s_\phi) \leq x(s_0) + \epsilon$ for all $\phi \in (-\alpha, \alpha)$. Observe that $x$ is bounded on $S$ and take $M \in [0, \infty)$ such that $\epsilon M \geq x \text{ on } S$. For these $M$ and $\alpha$, take a $z \in E_0$ as in (a). Let $y := (x(s_0) + 2\epsilon) \mathbb{1} + \epsilon z$. Then $y \in E_0$, $y(s_0) = x(s_0) + \epsilon, y(s_\phi) \geq x(s_0) + \epsilon \geq x(s_\phi)$ for all $\phi \in (-\alpha, \alpha)$, and $y(s_\phi) \geq \epsilon z(s_\phi) \geq \epsilon M \geq x(s_\phi)$ for all $\phi \in (-\pi, \pi] \setminus (-\alpha, \alpha), \text{ so } y \geq x \text{ on } S$. Thus, $\inf\{u(s_0) : u \in E_0, u \geq x \text{ on } S\} \leq y(s_0) = x(s_0) + \epsilon$. From this, the assertion follows for $x \in C(S)^+$. To complete the proof, note that $E_0$ is majorizing in $C(S)$, so that to every $x \in C(S)$ an element of $E_0$ can be added such that the sum is positive.

(c) $E_0$ is order dense in $E$: By (b), $E_0$ is order dense in $C(S)$, and therefore an order dense subspace of $E$.

(d) The unit ball $B$ of $p$ is solid in $E_0$: Let $x, y \in E_0$ be such that $x \prec y$. If $y \geq 0$, then $-y \leq x \leq y$ on $D$, hence $p(x) \leq p(y)$. Similarly, if $y \leq 0$, then $y \leq x \leq -y$, so $p(x) \leq p(y)$. If $y$ is neither nonpositive nor nonnegative, then there are two points of $S$ where $y$ is zero. Then $x$ is zero in those points, too, because, by (b), $x \prec y$ implies that $|x(t)| \leq |y(t)|$ for all $t \in S$. Thus, $x$ must be a scalar multiple of $y$. Because $x \prec y$ this yields that $p(x) \leq p(y)$. Hence, $B$ is solid.
(e) \( \text{sol} B = \{ x \in E : \text{there is a } y \in E \text{ such that } |x| \leq |y| \text{ on } D \} \) is not convex. Let \( y(t_1, t_2) := (3/2)t_1, (t_1, t_2) \in \mathbb{R}^2 \). With the rotations \( R_\phi \) defined above, we find that \( p(R_\phi y) = p(y) = \int_D (3/2)t_1^2 d(t_1, t_2) = \int_{\pi/2}^{\pi/2} \int_0^1 r \cos \phi r dr d\phi = 1 \), so \( R_\phi y \in B \) for all \( \phi \in \mathbb{R} \), hence \( (R_\phi y)^+ \in \text{sol} B \) for all \( \phi \). Choose an \( \alpha \in (0, 1/5) \) and take \( x_1 := y^+ \) and \( x_2 := (R_{\pi - \alpha} y)^+ \) (see fig. 1). To show that \( 2^{-1}(x_1 + x_2) \notin \text{sol} B \), let \( z \in E_0 \) be such that \( |z| \geq x_1 + x_2 \) in \( E \). Then \( |z| \) vanishes nowhere on the set \( \{ s_\phi : \phi \in [0, \pi - \alpha] \} \), so \( z \) has the same sign in the end points \( s_0 \) and \( s_{\pi - \alpha} \), and there \( |z| \geq 3/2 \), because \( x_1(s_0) = x_2(s_{\pi - \alpha}) = 3/2 \). The function \( z \) is affine, so \( |z| \) cannot be lower than \( 3/2 \) on both sides of the line through \( s_0 \) and \( s_{\pi - \alpha} \), hence \( p(z) \geq (3/2)(\pi/2 - \sin \alpha) > 2 \). It follows that \( 2^{-1}(x_1 + x_2) \notin \text{sol} B \). This shows that \( \text{sol} B \) is not convex.

(f) \( E_0 \cap \text{co} \text{sol} B \supseteq B \): Let \( y \) be as in (e) and let \( x_1 := |y| \) and \( x_2 := |R_{\pi/2} y| \). Then \( x_1, x_2 \in \text{sol} B \), so \( x := 2^{-1}(x_1 + x_2) \in \text{co} \text{sol} B \). On \( S \), \( x \) is of the form \( s_\phi \mapsto 3(|\sin \phi| + |\cos \phi|)/4 \), so \( x \geq 3/4 \) on \( S \). Thus, \( 0 \leq (3/4) 1 \leq x \), so \( (3/4) 1 \in \text{co} \text{sol} B \), because in a Riesz space the convex hull of a solid set is solid. Furthermore, \( (3/4) 1 \in E_0 \), whereas \( p((3/4) 1) = 3\pi/4 > 1 \), so \( (3/4) 1 \notin B \).

**Example 2.9.** A solid, convex set that is not solvex. Consider the situation of Example 2.8. It has been proved that \( B = \{ x \in E_0 : p(x) \leq 1 \} \) is solid and convex in \( E_0 \). To show that \( B \) is not solvex, let \( y_1(t_1, t_2) := (3/2)t_1 \) and \( y_2(t_1, t_2) := (3/2)t_2, (t_1, t_2) \in \mathbb{R}^2 \). Then \( y_1, y_2 \in B \) and \( (3/4) 1 \notin \{ 2^{-1}(y_1 + y_2), 2^{-1}(y_1 - y_2) \} \). Indeed, \( y_1 + y_2 \geq 3/2 \) on the first quadrant, and \( y_2 - y_1, y_1 - y_2, -y_1 - y_2 \geq 3/2 \) on the second, third, fourth quadrant, respectively. However, \( (3/4) 1 \notin B \), so \( B \) is not solvex.

### 3. PRE-RIESZ SEMINORMS

We will define the pre-Riesz seminorms by means of solvexity and show that they have the extension and restriction properties as announced in Section 1.

**Definition 3.1.** Let \( E \) be a partially ordered vector space. A seminorm on \( E \) is called **pre-Riesz** if its unit ball is solvex.

Since a subset of a Riesz space is solvex if and only if it is solid and convex, a seminorm on a Riesz space is pre-Riesz if and only if it is a Riesz seminorm. We have the following explicit reformulation.

**Proposition 3.2.** Let \( E \) be a partially ordered vector space with a seminorm \( p \). Then \( p \) is pre-Riesz if and only if
\[ p(x) = \inf \left\{ p(x_1) \lor \cdots \lor p(x_n) : x_1, \ldots, x_n \in E \text{ such that} \right. \\
\left. \exists \lambda_1, \ldots, \lambda_n \in (0, 1] \text{ with } \sum_k \lambda_k = 1 \text{ and} \right. \\
\left. x < \left\{ \sum_k \epsilon_k \lambda_k x_k : \epsilon_k = \pm 1 \right\} \right\} \\
\text{for all } x \in E. \\
\]

**Proof.** Let \( B \) be the unit ball of \( p \).

\( \Rightarrow \) Let \( x \in E \), \( x_1, \ldots, x_n \in E \), and \( \lambda_1, \ldots, \lambda_n \in (0, 1] \) with \( \sum_k \lambda_k = 1 \) be such that \( x < \left\{ \sum_k \epsilon_k \lambda_k x_k : \epsilon_k = \pm 1 \right\} \). Clearly, \( x_1, \ldots, x_n \in (p(x_1) \lor \cdots \lor p(x_n))B \) and this set is solvex, because \( B \) is solvex. It follows that \( x \in (p(x_1) \lor \cdots \lor p(x_n))B \), or, in other words, \( p(x) \leq p(x_1) \lor \cdots \lor p(x_n) \), which proves that \( p(x) \) is less than or equal to the infimum at the right hand side. By taking \( n = 1 \) and \( x_1 = x \), it is clear that the infimum is not strictly greater than \( p(x) \). Thus, the equality is established.

\( \Leftarrow \) Let \( x \in E \), \( \lambda_1, \ldots, \lambda_n \in (0, 1] \) with \( \sum_k \lambda_k = 1 \), and \( x_1, \ldots, x_n \in B \) be such that \( x < \left\{ \sum_k \epsilon_k \lambda_k x_k : \epsilon_k = \pm 1 \right\} \). Then, by assumption, \( p(x) \leq p(\lambda_1 x_1) \lor \cdots \lor p(\lambda_n x_n) \leq 1 \), so \( x \in B \). Hence \( B \) is solvex and \( p \) is pre-Riesz. \( \square \)

Let us now consider extension of pre-Riesz seminorms. A Riesz seminorm \( \rho \) on a Riesz space \( E \) can be extended to a Riesz seminorm \( \tilde{\rho} \) on any Riesz space \( F \) that contains \( E \) as a majorizing subspace by the formula

\[ \tilde{\rho}(x) := \inf \{ \rho(y) : y \in E, y \geq |x| \}, \quad x \in F. \]

Hence a seminorm on \( E \) is Riesz if and only if it can be extended to a Riesz seminorm on every Riesz space that contains \( E \) as a majorizing subspace. The latter formulation also makes sense if \( E \) is only a partially ordered vector space. Pre-Riesz seminorms generalize this property of Riesz seminorms. Indeed, if \( E_0 \) is a majorizing subspace of a Riesz space \( E \) and \( B \) is a solid unit ball in \( E_0 \), then \( B' := \text{co sol } B \) is the unit ball of a Riesz seminorm \( \rho \) on \( E \), since \( B' \) is solid, convex, absorbing, contains 0, and \( B' = \cap_{\alpha \geq 1} \alpha B' \). If \( B \) is solvex, then \( \rho \) extends the seminorm corresponding to \( B \), because of Lemma 2.6. Thus we obtain property (a) of Section 1. Moreover, \( B' \) is the smallest unit ball extending \( B \), hence \( \rho \) the greatest extending Riesz seminorm.

With some more effort an explicit extension formula can be given that yields extension to arbitrary majorized directed partially ordered vector spaces. Thus we obtain an extension theorem entirely similar to those in [4]. We need the following lemma.

**Lemma 3.3.** Let \( E \) be a partially ordered vector space. If \( x, x_1, \ldots, x_n \in E \) and \( y, y_{k,i} \in E \), \( i = 1, \ldots, m_k \), \( k = 1, \ldots, n \), are such that \( x_k \in \left\{ \sum_{i=1}^{m_k} \epsilon_i y_{k,i} : \epsilon_i = \pm 1 \right\} \) for \( k = 1, \ldots, n \), and \( x < \left\{ \sum_{k=1}^{n} \sum_{i=1}^{m_k} \epsilon_{k,i} y_{k,i} : \epsilon_{k,i} = \pm 1 \right\} \), then

\[ x < \left\{ \sum_{k=1}^{n} \sum_{i=1}^{m_k} \epsilon_{k,i} y_{k,i} : \epsilon_{k,i} = \pm 1 \right\}. \]
Proof. Let $u$ be an upper bound of $\{\sum_{k=1}^n \sum_{i=1}^{m_k} \epsilon_{k,i} y_{k,i}: \sum_{i=1}^{m_k} \epsilon_{k,i} = \pm 1\}$. Then $u - \sum_{k=1}^{n-1} \sum_{i=1}^{m_k} \epsilon_{k,i} y_{k,i}$ is an upper bound of $\{(\sum_{i=1}^{m_k} \epsilon_{i} y_{n,i}, \sum_{i=1}^{m_k} \epsilon_{i} y_{n,i})| \epsilon_{i} = \pm 1\}$ and therefore of $\{x_n, -x_n\}$, for any $\epsilon_{k,i} \in \{-1, 1\}$, $i = 1, \ldots, m_k$, $k = 1, \ldots, n-1$. This yields that $u - x_n, u + x_n$ are upper bounds of $\{\sum_{k=1}^{n} \sum_{i=1}^{m_k} \epsilon_{k,i} y_{k,i}: \sum_{i=1}^{m_k} \epsilon_{k,i} = \pm 1\}$. By induction, it follows that $u$ is an upper bound of $\{\sum_{k=1}^{n} \epsilon_{k} x_{k}: \sum_{i=1}^{m_k} \epsilon_{k,i} = \pm 1\}$ and therefore of $\{x, -x\}$. □

Theorem 3.4. Let $E$ be a directed partially ordered vector space with a majorizing linear subspace $E_0$. Let $p$ be a seminorm on $E_0$. Define for $x \in E$:

$$\tilde{p}(x) := \inf \left\{ p(y_1) + \cdots + p(y_n) : y_1, \ldots, y_n \in E_0 \text{ such that } x \in \left\{ \sum_{k=1}^{n} \epsilon_{k} y_{k} : \epsilon_{k} = \pm 1 \right\} \right\}.$$ 

Then $\tilde{p}$ is the greatest pre-Riesz seminorm on $E$ that is on $E_0$ less than or equal to $p$. Moreover, $\tilde{p}$ extends $p$ if and only if $p$ is pre-Riesz.

Proof. It follows directly from the definition that $\tilde{p}$ is a seminorm on $E$. For $x \in E_0$ it is clear that $\tilde{p}(x) \leq p(x)$.

To prove that $\tilde{p}$ is pre-Riesz, it suffices to establish for every $x \in E$ that $\tilde{p}(x) = \inf \{p(x_1) \vee \cdots \vee p(x_n) : x_1, \ldots, x_n \in E \text{ such that there are } \lambda_1, \ldots, \lambda_n \in (0, 1) \text{ with } \sum_{k} \lambda_k = 1 \text{ and } x \in \left\{ \sum_{k} \lambda_k x_{k} : \sum_{i=1}^{m_k} \epsilon_{i} y_{k,i} = \pm 1 \right\} \}$ and to use Proposition 3.2. Let $x \in E$. It is clear that $\tilde{p}(x)$ is greater than the infimum at the right hand side.

To prove that it is not strictly greater, let $x_1, \ldots, x_n \in E$ and $\lambda_1, \ldots, \lambda_n \in (0, 1)$ with $\sum_{k} \lambda_k = 1$ such that $x \in \left\{ \sum_{k} \lambda_k x_{k} : \sum_{i=1}^{m_k} \epsilon_{i} y_{k,i} = \pm 1 \right\}$, and let $\epsilon > 0$. By the definition of $\tilde{p}$, for each $k$ there are $y_{k,1}, \ldots, y_{k,m_k} \in E_0$ such that $x_k \in \left\{ \sum_{i=1}^{m_k} \epsilon_{k,i} y_{k,i} : \epsilon_{k,i} = \pm 1 \right\}$ and $\tilde{p}(x_k) \geq p(y_{k,1}) + \cdots + p(y_{k,m_k}) - \epsilon$. Then, by the previous lemma, $x \in \left\{ \sum_{k=1}^{n} \sum_{i=1}^{m_k} \epsilon_{k,i} y_{k,i} : \sum_{i=1}^{m_k} \epsilon_{k,i} y_{k,i} = \pm 1 \right\}$, so $\tilde{p}(x) \leq \sum_{k=1}^{n} \sum_{i=1}^{m_k} \epsilon_{k,i} y_{k,i} = \sum_{k=1}^{n} \lambda_k \sum_{i=1}^{m_k} p(y_{k,i}) = \sum_{k=1}^{n} \lambda_k \sum_{i=1}^{m_k} \tilde{p}(y_{k,i}) + \epsilon \leq \tilde{p}(x_1) \vee \cdots \vee \tilde{p}(x_n) + \epsilon$. It follows that $\tilde{p}$ is pre-Riesz.

Let $q$ be a pre-Riesz seminorm on $E$ with $q \leq p$ on $E_0$. Let $x \in E$ with $\tilde{p}(x) < 1$. Then there are $y_1, \ldots, y_n \in E_0$ such that $x \in \left\{ \sum_{k=1}^{n} \lambda_k x_{k} : \sum_{i=1}^{m_k} \epsilon_{i} y_{k,i} = \pm 1 \right\}$ and $p(y_1) + \cdots + p(y_n) < 1$. Choose $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\lambda_k > p(y_{k}) \geq q(y_{k})$ for all $k$ and $\sum_{k} \lambda_k = 1$. Then $x \in \left\{ \sum_{k=1}^{n} \lambda_k x_{k} : \sum_{i=1}^{m_k} \epsilon_{i} y_{k,i} = \pm 1 \right\}$, so that $q(x) < 1$, because the unit ball of $q$ is solvex. It follows that $q(x) < \tilde{p}(x)$ for all $x \in E$, so $\tilde{p}$ is the greatest pre-Riesz seminorm on $E$ that is less than or equal to $p$ on $E_0$.

If $p$ is pre-Riesz, the same argument shows for an $x \in E_0$ that $p(x) \leq \tilde{p}(x)$, so that then $\tilde{p} = p$ on $E_0$. Conversely, if $\tilde{p}$ extends $p$, then for every $x \in E_0$ and $x_1, \ldots, x_n \in E_0$ with $p(x_k) < 1$ for all $k$ and $\lambda_1, \ldots, \lambda_n \in (0, 1)$ with $\sum_{k} \lambda_k = 1$ such that $x \in \left\{ \sum_{k=1}^{n} \lambda_k x_{k} : \sum_{i=1}^{m_k} \epsilon_{i} y_{k,i} = \pm 1 \right\}$, the definition of $\tilde{p}$ implies that $p(x) = \tilde{p}(x) \leq p(\lambda_1 x_1) + \cdots + p(\lambda_n x_n) \leq 1$. Hence the unit ball of $p$ is solvex, or, in other words, $p$ is pre-Riesz. □

Lemma 2.5 yields the following result on restriction. More refined results are in Section 5.
Proposition 3.5. Let $E$ be a Riesz space with a Riesz seminorm $p$. If $E_0$ is a linear subspace such that for every $x_1, \ldots, x_n \in E_0$ one has $|x_1| + \cdots + |x_n| = \inf \{ y \in E_0 : y \geq |x_1| + \cdots + |x_n| \}$, then the restriction $p|_{E_0}$ of $p$ to $E_0$ is pre-Riesz. In particular, if $E_0$ is an order dense subspace, then $p|_{E_0}$ is pre-Riesz.

Combination of Theorem 3.4 and Proposition 3.5 yields property (b) of the introduction:

Theorem 3.6. A seminorm on an order dense linear subspace of a Riesz space is pre-Riesz if and only if it can be extended to a Riesz seminorm on the Riesz space.

We conclude this section by an example that shows that pre-Riesz seminorms are not determined by their restrictions to the positive cone. It turns out to be a major difference with Riesz seminorms. A notion of seminorms on partially ordered vector spaces based on this property of Riesz seminorms is that of regular seminorm, similar to the r-norm on spaces of operators. Some properties may be found in [5, 6].

Example 3.7. A directed partially ordered vector space $E$ with pre-Riesz seminorms $p$ and $q$ such that $p = q$ on $E^+$ and such that $p$ and $q$ are not equivalent. Let $S$ be the unit circle in $\mathbb{R}^2$, $D$ the unit disk, and take for $E$ the space of restrictions to $D$ of affine functions from $\mathbb{R}^2$ to $\mathbb{R}$. In Example 2.8 it is proved that $E$ can be seen as an order dense subspace of $C(S)$. Take $p(x) := |x(1,0)| + |x(-1,0)|$ and $q(x) := |x(0,1)| + |x(0,-1)|$, $x \in E$. Clearly, $p$ and $q$ are restrictions of Riesz seminorms on $C(S)$ and therefore they are pre-Riesz seminorms, by Proposition 3.5. Any $x \in E^+$ is affine, so $p(x) = x(1,0) + x(-1,0) = x(0,0)$ and, similarly, $q(x) = x(0,0)$, hence $p = q$ on $E^+$. To show that $p$ and $q$ are not equivalent on $E$, take $x(s,t) := s$, $(s,t) \in D$. Then $x \in E$, $p(x) = 2$, and $q(x) = 0$, and therefore $p$ and $q$ are not equivalent.

4. MORE ON EXTENSIONS

Theorem 3.4 describes how to extend pre-Riesz seminorms. In this section we will investigate when the greatest extension is an L- or M-seminorm and we start with a result about uniqueness of extensions similar to those in [4]. Given a seminorm $p$ on a vector space $E$, expressions as ‘norm dense’ and ‘norm complete’ refer to the (non-Hausdorff) topology and Cauchy-sequences induced by $p$.

Proposition 4.1. Let $E$ be a Riesz space with a majorizing linear subspace $E_0$ and let $p$ be a pre-Riesz seminorm on $E_0$.

(i) If $E_0$ is norm dense with respect to the greatest Riesz seminorm $\rho$ on $E$ extending $p$, then $\rho$ is the only Riesz seminorm on $E$ extending $p$.

(ii) If $E_0$ is a Riesz subspace and not norm dense in $E$ with respect to the greatest Riesz seminorm on $E$ extending $p$, then there is a Riesz seminorm on $E_0$ which is equivalent to $p$ and which has distinct extensions.
Proof.

(i) Every Riesz extension of $p$ is $\rho$-continuous, so if $E_0$ is $\rho$-dense in $E$, then the extensions coincide.

(ii) Let $\rho$ be the greatest Riesz extension of $p$. As a consequence of Mazur (see [4, Lemma 7]), there exist a continuous linear functional $\varphi: E \to \mathbb{R}$ and an $x_0 \in E^+$ such that $\varphi(x_0) \neq 0$ and $\varphi = 0$ on $E_0$. Take $\rho_1(x) = \rho(x) + \varphi^+ (|x|)$, $\rho_2(x) = \rho(x) + \varphi^- (|x|)$, $x \in E$. Then $\rho_1$ and $\rho_2$ are Riesz seminorms on $E$ and they are equivalent to $\rho$, because $\varphi^+$ and $\varphi^-$ are continuous. As $\varphi = \varphi^-$ on $E_0$, $\rho_1 = \rho_2$ on $E_0$, and $\varphi^+(x_0) \neq \varphi^-(x_0)$, so $\rho_1(x_0) \neq \rho_2(x_0)$. \qed

A Riesz seminorm $\rho$ on a Riesz space $E$ is called an $M$-seminorm if $\rho(x \vee y) = \rho(x) \vee \rho(y)$ for all $x, y \in E^+$ and an $L$-seminorm if $\rho(x + y) = \rho(x) + \rho(y)$ for all $x, y \in E^+$.

**Proposition 4.2.** Let $E_0$ be a directed partially ordered vector space with a pre-Riesz seminorm $p$. The following statements are equivalent:

(i) For every Riesz space $E$ that contains $E_0$ as a majorizing linear subspace, the greatest Riesz seminorm on $E$ extending $p$ is an $M$-seminorm.

(ii) There is a Riesz space $E$ that contains $E_0$ as a majorizing linear subspace such that the greatest Riesz seminorm on $E$ extending $p$ is an $M$-seminorm.

(iii) For every $x_1, \ldots, x_m \in E_0$ with $p(x_1), \ldots, p(x_m) < 1$ there are $y_1, \ldots, y_n \in E_0$ such that $\{x_1, \ldots, x_m\} \subset \{x \in \mathbb{R}^n : \sum_k \varepsilon_k y_k : \varepsilon_k = \pm 1\}$ and $p(y_1) + \cdots + p(y_n) < 1$.

Proof.

(i) $\Rightarrow$ (ii) Since $E_0$ is directed, it follows from [8, Theorem 4.1] that there is a Riesz space $E$ that contains $E_0$ as a majorizing linear subspace. The greatest Riesz extension of $p$ on this space is an $M$-seminorm, by assumption (i).

(ii) $\Rightarrow$ (iii) Let $E$ be a Riesz space that contains $E_0$ as a majorizing linear subspace and let $\rho$ be the greatest Riesz extension of $p$. Let $x_1, \ldots, x_m \in E_0$ be such that $p(x_1), \ldots, p(x_m) < 1$. Then $\rho(|x_1| \vee \cdots \vee |x_m|) = p(x_1) \vee \cdots \vee p(x_m) = p(\rho_1(x_1) \vee \cdots \vee \rho_1(x_m)) < 1$. By the formula for $\rho$ given by Theorem 3.4, it follows that there exist $y_1, \ldots, y_n \in E_0$ such that $p(y_1) + \cdots + p(y_n) < 1$ and $|x_1| \vee \cdots \vee |x_m| \leq |y_1| + \cdots + |y_n|$ and therefore $\{x_1, \ldots, x_m\} \subset \{y \in \mathbb{R}^n : \sum_k \varepsilon_k y_k : \varepsilon_k = \pm 1\}$.

(iii) $\Rightarrow$ (i) Let $E$ be a Riesz space that contains $E_0$ as a majorizing linear subspace and let $\rho$ be the greatest Riesz extension of $p$. For any $x, y \in E$, $\rho(|x| \vee |y|) \geq \rho(|x|) \vee \rho(|y|)$, because $\rho$ is Riesz. To prove that $\rho(|x| \vee |y|) \leq \rho(|x|) \vee \rho(|y|)$, let $x, y \in E$ be such that $\rho(x), \rho(y) < 1$. It suffices to show that $\rho(|x| \vee |y|) < 1$. There exist $x_1, \ldots, x_l, x_{l+1}, \ldots, x_m \in E_0$ such that $|x| \leq |x_1| + \cdots + |x_l|$, $|y| \leq |x_{l+1}| + \cdots + |x_m|$, $p(x_1) + \cdots + p(x_l) < 1$, and $p(x_{l+1}) + \cdots + p(x_m) < 1$. Then there exist $\lambda_1, \ldots, \lambda_l, \lambda_{l+1}, \ldots, \lambda_m \in \mathbb{R}$ such that $\lambda_1 + \cdots + \lambda_l < 1$, $\lambda_{l+1} + \cdots + \lambda_m < 1$, and $\lambda_i > p(x_i)$ for all $i$. By assumption, there are $y_1, \ldots, y_n \in E_0$ such that $p(y_1) + \cdots + p(y_n) < 1$ and $\{\lambda_1^{-1} x_1, \ldots, \lambda_m^{-1} x_m\} \subset \{y \in \mathbb{R}^n : \sum_k \varepsilon_k y_k : \varepsilon_k = \pm 1\}$. Then $|x_i| \leq \lambda_i (|y_1| + \cdots + |y_n|) \leq |y_1| + \cdots + |y_n|$, and
\[ |y| \leq |y_1| + \cdots + |y_n|, \text{ hence } |x| \vee |y| \leq |y_1| + \cdots + |y_n|. \] This yields that 
\[ \rho(|x| \vee |y|) \leq \rho(y_1) + \cdots + \rho(y_n) < 1, \] which completes the proof. \( \square \)

**Corollary 4.3.** Let \( E \) be a Riesz space with a majorizing linear subspace \( E_0 \) and let \( p \) be a pre-Riesz seminorm on \( E \). Then the greatest Riesz seminorm on \( E \) extending \( p \) is an \( M \)-seminorm if \( p \) has at least one of the following properties:

(i) The unit ball of \( p \) is directed.

(ii) The unit ball \( B \) of \( p \) is directed in the sense that for every \( x_1, x_2 \in B \) there is a \( y \in B \) such that \( y \notin \{x_1, x_2\} \).

**Proof.** Observe that (i) implies (ii). If (ii) is satisfied, it follows by induction that for every \( x_1, \ldots, x_m \in E_0 \) with \( p(x_1) + \cdots + p(x_m) < 1 \) there is a \( y \in B \) such that \( y \notin \{x_1, \ldots, x_m\} \) and \( p(y) < 1 \). Now apply Proposition 4.2. \( \square \)

Since the unit ball of an \( M \)-seminorm on a Riesz space is directed, it follows that the greatest Riesz extension of an \( M \)-seminorm is an \( M \)-seminorm.

The greatest Riesz extension of an \( L \)-seminorm is only an \( L \)-seminorm in case of norm denseness. This will be proved via dual spaces. Let \( E' \) denote the norm dual of a directed partially ordered vector space \( E \) with a seminorm, equipped with the usual ordering. The following result is well-known for norms (see e.g. \([10, \text{Proposition 1.4.7}]\)). To deal with a seminorm, one can factor out its kernel and apply the result for the norm case to the quotient space.

**Lemma 4.4.** Let \( E \) be a Riesz space with a Riesz seminorm \( \rho \). Then \( \rho \) is an \( L \)-seminorm if and only if the norm of \( E' \) is an \( M \)-norm.

**Lemma 4.5.** Let \( E \) be a Riesz space with a majorizing Riesz subspace \( E_0 \) and let \( p \) be a Riesz seminorm on \( E_0 \). Let \( \rho \) be the greatest Riesz seminorm on \( E \) extending \( p \). For every \( \varphi \in (E, \rho)^{++} \) one has that \( \varphi|_{E_0} \in (E_0, p)^{++} \) and \( \|\varphi|_{E_0}\|_{E_0} = \|\varphi\|_{E'} \).

**Proof.** Because \( E_0 \) is a Riesz subspace of \( E \), the formula for \( \rho \) reduces to \( \rho(x) = \inf\{p(y): y \in E_0, y \geq |x|\}, \ x \in E \). Let \( \varphi \in (E, \rho)^{++} \). Clearly, \( \varphi|_{E_0} \in (E_0, p)^{++} \) and \( \|\varphi|_{E_0}\|_{E_0} \leq \|\varphi\|_{E'} \). For \( x \in E \) with \( \rho(x) < 1 \), there is \( y \in E_0 \) with \( y \geq |x| \) and \( p(y) < 1 \). Then \( |\varphi(x)| \leq \varphi(|x|) \leq \varphi(y) = \varphi|_{E_0}(y) \). Hence \( \|\varphi\|_{E'} = \sup\{|\varphi(x)|: x \in E, \rho(x) < 1 \} \leq \sup\{\varphi|_{E_0}(y): y \in E_0, p(y) < 1 \} = \|\varphi|_{E_0}\|_{E_0} \). \( \square \)

**Proposition 4.6.** Let \( E \) be a Riesz space with a majorizing Riesz subspace \( E_0 \) and let \( p \) be an \( L \)-seminorm on \( E_0 \). Let \( \rho \) be the greatest Riesz seminorm on \( E \) extending \( p \). Then \( \rho \) is an \( L \)-seminorm if and only if \( E_0 \) is \( \rho \)-dense in \( E \).

**Proof.**

\( \Leftarrow \) Let \( x, y \in E^+ \). Then there are sequences \( (x_n)_n \) and \( (y_n)_n \) in \( E_0^+ \) with \( \rho(x_n - x) \to 0, \rho(y_n - y) \to 0 \). Then \( \rho(x + y) = \lim_n \rho(x_n + y_n) = \lim_n (p(x_n) + p(y_n)) = \rho(x) + \rho(y) \).

\( \Rightarrow \) Suppose that \( E_0 \) is not \( \rho \)-dense in \( E \). Because of Lemma 4.4 it suffices
to prove that the norm of $E'$ is not an M-norm. There is a $\varphi \in E'$, $\varphi \neq 0$, with $\varphi = 0$ on $E_0$. Then $\varphi^+, \varphi^- \in E'$ and $\varphi^+ \vee \varphi^- = \varphi^+ + \varphi^-$. Denote $\varphi_0 := \varphi^+|_{E_0} = \varphi^-|_{E_0}$. Because of the previous lemma and Lemma 4.4 again, it follows that

$$\|\varphi^+ \vee \varphi^-\|_{E'} = \|\varphi^+ + \varphi^-\|_{E'} = \|\varphi_0 + \varphi_0\|_{E'} = 2(\|\varphi_0\|_{E'} \vee \|\varphi_0\|_{E'}) = 2(\|\varphi^+\|_{E'} \vee \|\varphi^-\|_{E'}) \geq 2(\|\varphi^+\|_{E'} \vee \|\varphi^-\|_{E'}),$$

because $\varphi^+$ or $\varphi^- \neq 0$ on $E$. So $\|\cdot\|_{E'}$ is not an M-norm, hence $\rho$ is not an L-seminorm.

5. MORE ON RESTRICTIONS

There are more general cases than the one in Proposition 3.5 in which the restriction of a Riesz seminorm is pre-Riesz. We will present one below. It involves the same terminology as the notion of Riesz completion that was mentioned in Section 1. We first give a summary of notions and results presented by Van Haandel in [7] and their connection with pre-Riesz seminorms.

**Definition 5.1.** Let $E$ and $F$ be directed partially ordered vector spaces and let $h : E \to F$ be a linear map. The map $h$ is called a Riesz* homomorphism if for every finite subset $S$ of $E$ (or, equivalently, every $S$ consisting of two elements of $E$) and every $x \in E$ one has: if $x \leq u$ for every upper bound $u$ of $S$, then $h(x) \leq v$ for every upper bound $v$ of $h(S)$. ([7, Definition 5.1, Theorem 5.3]).

**Lemma 5.2.** Let $E$ be a Riesz space, let $E_0$ be a partially ordered vector space, and let $i : E_0 \to E$ be a bipositive linear map. Then $i$ is a Riesz* homomorphism in both of the following cases:

(i) $i(E_0)$ is order dense in $E$.

(ii) $i(E_0)$ is a Riesz subspace of $E$.

**Proof.**

(i) Let $x, y \in E_0$. Let $z \in E_0$ be such that $z \leq u$ for every $u \in E_0$ with $u \geq x, y$. Then $i(z) \leq i(u)$ for all such $u$. Hence, because $i(E_0)$ is order dense and $i$ is bipositive, $i(x) \vee i(y) = \inf\{i(u) : u \in E_0 \text{ such that } i(u) \geq i(x), i(y)\} = \inf\{i(u) : u \in E_0 \text{ such that } u \geq x, y\} \geq i(z)$, so that $i(z) \leq v$ for every upper bound $v$ of $i(x), i(y)$.

(ii) In this situation $E_0$ is a Riesz space and the map $i$ is a Riesz homomorphism, hence a Riesz* homomorphism ([7, Remark 5.2(i)]).

A special role is played by the directed partially ordered vector spaces that can be embedded in Riesz spaces by bipositive Riesz* homomorphisms. Those are also the spaces that have proper Riesz completions.

**Definition 5.3.** A partially ordered vector space $E$ is called pre-Riesz if for every
Every pre-Riesz space is directed and every integrally closed directed partially ordered vector space is pre-Riesz ([7, Theorem 1.7(ii)]). The next theorem is contained in [7, Corollary 4.9-11, Theorems 3.5, 3.7, 4.13].

**Theorem 5.4 (Van Haandel).** Let $E$ be a directed partially ordered vector space. The following statements are equivalent:

(i) $E$ is pre-Riesz.
(ii) There exist a Riesz space $F$ and a bijective Riesz* homomorphism $i: E \to F$.
(iii) There exist a Riesz space $F$ and a bijective linear map $i: E \to F$ such that $i(E)$ is order dense in $F$.
(iv) There exist a Riesz space $F$ and a bijective linear map $i: E \to F$ such that $i(E)$ is order dense in $F$ and generates $F$ as a Riesz space.

Moreover, all spaces $F$ as in (iv) are Riesz isomorphic.

A pair $(F, i)$ as in (iv) of the above theorem is called a Riesz completion of $E$. The theorem states that it is essentially unique and we will therefore speak of the Riesz completion of a pre-Riesz space. Our definition is not appropriate from the point of view of categories. Van Haandel in [7] defines the Riesz completion as the ‘smallest larger’ Riesz space in the sense that it has a factorization property with respect to a certain kind of ‘Riesz homomorphisms’. If one adds to Van Haandel’s definition the condition that there exists a linear, bijective map from the space to its Riesz completion, then the resulting notion is equivalent to the one presented above, as follows from Van Haandel’s results. For integrally closed spaces the ingredients of the Riesz completion are also in [2].

We are now in a position to establish property (c) of Section 1. In view of Theorem 5.4 we have to restrict our scope to pre-Riesz spaces. The next theorem follows from Theorem 5.4 together with Theorem 3.4 and Proposition 3.5, and is the reason for the name ‘pre-Riesz seminorm’.

**Theorem 5.5.** Let $E$ be a pre-Riesz space with a seminorm $p$. The following statements are equivalent:

(i) $p$ is pre-Riesz.
(ii) $p$ is the restriction of a Riesz seminorm on the Riesz completion of $E$.
(iii) $p$ can be extended to a Riesz seminorm on every Riesz space in which $E$ can be embedded as an order dense linear subspace.

**Corollary 5.6.** Let $E$ be a pre-Riesz space. If $p$ and $q$ are pre-Riesz seminorms on $E$, then $p \vee q$ and $p + q$ are pre-Riesz as well.

Finally, we show that the restriction of a pre-Riesz seminorm is pre-Riesz if the inclusion map is a Riesz* homomorphism.
Proposition 5.7. Let $E$ and $F$ be directed partially ordered vector spaces and let $h: E \to F$ be a bipositive linear map. If $F$ is pre-Riesz and $h$ is a Riesz* homomorphism, then $E$ is pre-Riesz.

Proof. Let $x, y, z \in E$ be such that every upper bound of $\{x + y, x + z\}$ is an upper bound of $\{y, z\}$, or, in other words, $y, z \preceq u$ for every upper bound $u$ of $\{x + y, x + z\}$. Because $h$ is a Riesz* homomorphism, it follows that $h(y), h(z) \preceq v$ for every upper bound $v$ of $\{h(x) + h(y), h(x) + h(z)\}$. The assumption that $F$ is pre-Riesz yields that $h(x) \preceq 0$, so $x \preceq 0$, by bipositivity of $h$. Thus, $E$ is pre-Riesz. \qed

In view of this proposition, we will call a directed linear subspace $E_0$ of a directed partially ordered vector space $E$ a pre-Riesz subspace if the inclusion map is a Riesz* homomorphism.

Lemma 5.8. Let $E$ and $F$ be directed partially ordered vector spaces and let $h: E \to F$ be a Riesz* homomorphism. Let $S$ be a finite subset of $E$ and let $x \in E$. If $x \preceq S$ in $E$, then $h(x) \preceq h(S)$ in $F$.

Proof. Assume that $x \preceq S$. Then $x, -x \preceq u$ for every upper bound $u$ of $S \cup -S$, hence $h(x), h(-x) \preceq v$ for every upper bound $v$ of $h(S \cup -S)$, because $h$ is a Riesz* homomorphism. From this it follows that $h(x) \preceq h(S)$. \qed

Theorem 5.9. Let $E$ and $F$ be directed partially ordered vector spaces and let $h: E \to F$ be a Riesz* homomorphism. If $p$ is a pre-Riesz seminorm on $F$, then $p \circ h$ is a pre-Riesz seminorm on $E$.

Proof. Observe that $p \circ h$ is a seminorm on $E$. It remains to be shown that its unit ball $B$ is solvex. Let $B'$ be the unit ball of $p$, then $B = \{x \in E : h(x) \in B'\}$. Let $x \in E$, $x_1, \ldots, x_n \in B$, and $\lambda_1, \ldots, \lambda_n \in (0, 1]$ with $\sum \lambda_k = 1$ be such that $x \preceq \{\sum \varepsilon_k \lambda_k x_k : \varepsilon_k = \pm 1\}$. According to the previous lemma, it follows that $h(x) \preceq \{\sum \varepsilon_k \lambda_k h(x_k) : \varepsilon_k = \pm 1\}$. Furthermore, $h(x_k) \in B'$ for $k = 1, \ldots, n$ and $B'$ is solvex, so $h(x) \in B'$. Hence $x \in B$ and thus $B$ is solvex. \qed

Corollary 5.10. Let $E$ be a directed partially ordered vector space with a pre-Riesz subspace $E_0$. If $p$ is a pre-Riesz seminorm on $E$, then its restriction to $E_0$ is pre-Riesz as well.

Acknowledgements

The results presented in this paper are a refinement of a part of the author's Ph.D. thesis [5]. The author wants to thank A.C.M. van Rooij and C.D. Aliprantis for their help and encouragement.
REFERENCES


(Received 20 September 2002)