On the existence of solutions for strongly nonlinear differential equations

Kamel Al-Khaled \textsuperscript{a}, Mohamed Ali Hajji \textsuperscript{b,\*}

\textsuperscript{a} Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan
\textsuperscript{b} Department of Mathematical Sciences, United Arab Emirates University, PO Box 17551, Al Ain, United Arab Emirates

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Abstract
The objectives of this paper are twofold. Firstly, to prove the existence of an approximate solution in the mean for some nonlinear differential equations, we also investigate the behavior of the class of solutions which may be associated with the differential equation. Secondly, we aim to implement the homotopy perturbation method (HPM) to find analytic solutions for strongly nonlinear differential equations.

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1. Introduction

Differential equations are of interest to both scientists and mathematicians since the solutions often provide information about the behavior of the physical world. For many linear and some nonlinear differential equations, solutions are available. When no solution is available there are two basic approaches to the problem. The first is an investigation of approximate solutions or methods of approximating a numerical solution for a given value of the independent variable. The second is to investigate the behavior of the class of solutions which may be associated with the differential equation.

We shall be concerned with finding an approximate solution to the differential equation

\[ \ddot{x}(t) + ax^2(t) + bx(t) = F(t) \]  

where \( a \) and \( b \) represent real constants, \( t \) is a real parameter, and the dots represent differentiation with respect to \( t \). Such equations arise in the study of cyclic accelerations where the orbits of the particles are periodically perturbed. Three questions which arise in connection with this problem are:

* Corresponding author.
E-mail addresses: kamel@just.edu.jo, kamelasel@yahoo.com (K. Al-Khaled), mahajji@uaeu.ac.ae (M.A. Hajji).

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1. What type of solution should be considered?
2. What are the restrictions on any particular solution? and
3. What restrictions must be imposed on the generalized differential equation in order that the solution will have meaning?

For the first question we will consider a “solution in the mean” which will be defined later. For the last two questions we will make a few basic assumptions and demonstrate a solution in the mean to Eq. (1.1). It is of interest to note that under certain conditions Eq. (1.1) is a special case of Hill’s [14] differential equation

\[ \ddot{y}(t) + \left[ \phi(t) + \lambda \right] y(t) = 0 \]  

where \( \phi(t) \) is periodic. This equation has been studied extensively. To see this relation, let \( F(t) \) be periodic and let \( x(t) = y(t)/a \). If we substitute this into Eq. (1.1), we have

\[ \ddot{y}(t) + \dot{y}^2(t) + by(t) = aF(t). \]  

(1.3)

Now multiply both sides by \( e^{y(t)} \) to obtain

\[ \frac{d^2 e^{y(t)}}{dt^2} + \left[ by(t) - aF(t) \right] e^{y(t)} = 0. \]  

(1.4)

Then let \( z(t) = e^{y(t)} \), or \( y(t) = \ln z(t) \), so that we have

\[ \ddot{z}(t) + [b \ln z(t) - aF(t)] z(t) = 0. \]  

(1.5)

Eq. (1.5) is a special case of Hill’s equation when \( b = 0 \). Thus we see that Eq. (1.1) may be more complex than Hill’s equation. In recent years the existence of solutions for nonlinear ordinary differential equations (1.1), and their special cases have been studied by many researchers. In [16] the existence of periodic solutions for nonlinear differential equations with a \( p \)-Laplacian-like operator is studied by applying a new generalization of polar coordinates. In [14], the existence of nontrivial solutions of a general family of second order equations, whose main model is a Hill’s equation with a cubic nonlinear term arising in different physical applications is proved. While in [18], upper bounds for the amplitudes and lower bounds for the periods of periodic solutions of coupled second-order nonlinear differential equations are found.

Numerous methods are developed for analytic solving of strongly nonlinear differential equations describing the vibrations of the oscillator. For example, the harmonic balance method [15] which leads to algebraic equations has been used. In [4], the harmonic balance method involving the Jacobian elliptic function is applied. An analytical approximate approach for determining periodic solutions of nonlinear jerk equations involving third-order time-derivative is presented in [17]. In [2], the authors used the Adomian decomposition method (ADM) to find approximate solutions for general form of second order ordinary differential equations.

All of the above mentioned methods are based on the perturbation of the nonlinear system and certain difficulties appear. Furthermore, ADM needs to be modified in order to work for ordinary differential equations where their solutions consists of a rapidly and slowly oscillating function. In [6,8,9], He adopted the Homotopy technique which is widely applied in differential topology for obtaining the approximate analytic solution of the second order strongly nonlinear differential equations. The Homotopy method does not depend on a small parameter in the equation, and was successfully applied to nonlinear oscillators with discontinuities [9]. Recent trends and developments in the use of the method are reviewed in [6]. For strongly nonlinear equations without small parameter, a transformation \( u = \beta v \) was introduced in [7] for the nonlinear equation \( L(u) + N(u) = 0 \), where \( L \) and \( N \) are general linear and nonlinear differential operators respectively, \( \beta \) is sufficiently small. Therefore, a strongly nonlinear system was transformed into a small parameter system with respect to the new introduced parameter \( \beta \). The homotopy perturbation method was also adopted to solve several types of strongly nonlinear equations [5,10,13].

The objectives of this paper are twofold. Firstly, to prove the existence of an approximate solution in the mean for Eq. (1.1) which contains our previous work [1]. Secondly, we aim to implement the homotopy perturbation method (HPM) to find analytic solutions for some strongly nonlinear differential equations.

2. Solution in the mean

We are concerned with finding an approximate solution to Eq. (1.1). We will base our investigation on the following:
**Definition 2.1.** Given an equation \( f[x(t)] = F(t), a \leq t \leq b \), where \( f[x(t)] \) may be a function involving derivatives of \( x(t) \) with respect to \( t \). Let \( \{\phi_n(t)\}, n = 0, 1, 2, \ldots \), be an orthogonal system of functions over \([a, b]\). Then

\[
x(t) = \sum_{n=0}^{\infty} a_n \phi_n(t)
\]

is a solution in the mean over the interval \([a, b]\) with respect to the system \( \{\phi_n(t)\} \) if and only if

\[
\lim_{k \to \infty} \int_a^b \left( f \left[ \sum_{n=0}^{k} a_n \phi_n(t) \right] - F(t) \right) dt = 0.
\]

(2.1)

The orthogonal system we will use to study Eq. (1.1) is \( \{\sin nt\} \) which is orthogonal over \([0, 2\pi]\). Let us assume that:

1. The differential equation (1.1) has a periodic solution in the mean of the form

\[
x(t) = \sum_{n=1}^{\infty} a_n \sin nt.
\]

(2.2)

2. The second derivative of \( x(t) \) with respect to \( t \) exists for all \( t \).

3. \( F(t) \) is an odd periodic function of period \( 2\pi \) not identically zero which may be expanded in a Fourier sine series such that

\[
F(t) = \sum_{n=1}^{\infty} b_n \sin nt.
\]

(2.3)

Substituting (2.2) and (2.3) into (1.1), and integrating from 0 to \( \pi \) we obtain

\[
-\int_0^\pi \sum_{n=1}^{\infty} n^2 a_n \sin nt \, dt + a \int_0^\pi \left( \sum_{n=1}^{\infty} n a_n \cos nt \right)^2 \, dt + b \int_0^\pi \sum_{n=1}^{\infty} a_n \sin nt \, dt = \int_0^\pi \sum_{n=1}^{\infty} b_n \sin nt \, dt
\]

(2.4)

or, if we assume that termwise integration is permissible,

\[
-\sum_{n=1}^{\infty} n^2 a_n r_n + a \sum_{n=1}^{\infty} n^2 a_n^2 s_n + b \sum_{n=1}^{\infty} a_n r_n = \sum_{n=1}^{\infty} b_n r_n
\]

(2.5)

where

\[
r_n = \int_0^\pi \sin nt \, dt = \begin{cases} 0, & n \text{ even}, \\ 2/n, & n \text{ odd} \end{cases}
\]

and

\[
s_n = \int_0^\pi \cos^2 nt \, dt = \pi/2, \quad \forall n.
\]

A sufficient condition that Eq. (2.5) be satisfied is that we have term by term equality. With this condition, we have for \( n \) even \( an^2 a_n^2 \pi^2/2 = 0 \), and therefore \( a_n = 0, n = 2, 4, 6, \ldots \). Similarly, for \( n \) odd we have

\[
-n^2 a_n \frac{2}{n} + an^2 a_n \frac{\pi}{2} + ba_n \frac{2}{n} = b_n \frac{2}{n}.
\]

This last equation is quadratic in \( a_n \). In standard form it reads

\[
an^3 \pi a_n^2 + 4(b - n^2)a_n - 4b_n = 0
\]
with the solutions

\[ a_n = \frac{2(n^2 - b) \pm \sqrt{(n^2 - b)^2 + ab\pi n^3}}{an^3\pi}, \quad n = 1, 3, 5, \ldots \tag{2.6} \]

Eq. (2.4) is equivalent to condition (2.1) for our particular problem. Hence, Eq. (2.2) must be a solution in the mean of Eq. (1.1) for at least one of the two above values of the \(a_n\), if Eq. (2.2) satisfies assumptions 2 and 3 above. We will investigate the validity of these assumptions and the two values of the \(a_n\) in the next section.

3. Investigation of restrictions

We will now show that the existence of a solution in the mean to Eq. (1.1) is mainly dependent on the function \(F(t)\) and that for sufficient restrictions on \(F(t)\) one of the \(a_n\) in (2.5) is a solution in the mean to Eq. (1.1). For the purposes of this investigation we will need the following theorems.

**Theorem 3.1.** (See [3].) If \(f(t)\) is bounded and continuous, and otherwise satisfies Dirichlet’s conditions in \(-\pi < x < \pi\), while \(f(\pi - 0) = f(-\pi + 0)\), and if \(f'(t)\) is bounded and otherwise satisfies Dirichlet’s conditions in the same interval, the coefficients in the Fourier’s series for \(f(t)\) are less in absolute value than \(K/n^2\), where \(K\) is some positive number independent of \(n\).

**Theorem 3.2.** (See [12].) When \(f(t)\) is an arbitrary function satisfying Dirichlet’s conditions in the interval \((0, \pi)\), the sum of the sine series,

\[ \frac{2}{\pi} \sum_{n=1}^{\infty} \sin nt \int_{0}^{\pi} f(t') \sin nt' dt', \]

is equal to \(\frac{1}{2} [f(t + 0) + f(t - 0)]\) at every point between 0 and \(\pi\), where \(f(t + 0)\) and \(f(t - 0)\) exist; and, when \(t = 0\) or \(t = \pi\), the sum is zero.

**Theorem 3.3.** (See [11].) If \(\{f_n\}\) is a sequence of continuous functions on \([a, b]\) and \(f_n \to f\) uniformly on \([a, b]\), then \(f\) is continuous on \([a, b]\).

We are now ready to proceed with the argument. Consider first the value of \(a_n\) from \(a_n\) Eq. (2.6) with the positive sign, namely

\[ a_n^+ = \frac{2(n^2 - b) + 2\sqrt{(n^2 - b)^2 + ab\pi n^3}}{an^3\pi} = \frac{1}{n} \left[ \frac{2}{a\pi} - \frac{2b}{an^2\pi} + \frac{2}{a\pi} \sqrt{\left(1 - \frac{b}{n^2}\right)^2 + \frac{ab\pi n}{n}} \right] \]

Now from the algebra of limits, we have

\[ \lim_{n \to \infty} \left[ \frac{2}{a\pi} - \frac{2b}{an^2\pi} + \frac{2}{a\pi} \sqrt{\left(1 - \frac{b}{n^2}\right)^2 + \frac{ab\pi n}{n}} \right] = \frac{4}{a\pi} \]

Because of convergence and since the \(a_n^+\) exist for all positive odd integers \(n\), there exists an \(M\) such that

\[ |a_n^+| \leq M \frac{1}{n}, \quad n = 1, 3, \ldots \]

Therefore, the \(a_n^+ = O(1/n)\) and the termwise differentiation of (2.2) is not justifiable. This follows from the second derivative

\[ \frac{d^2x(t)}{dt^2} = -\sum_{n=1}^{\infty} n^2 a_n^+ \sin nt \tag{3.1} \]

and since

\[ \lim_{n \to \infty} n^2 a_n = \frac{4}{a\pi} \lim_{n \to \infty} n = \infty \]
the series of Eq. (3.1) is a divergent series. If we take the negative sign of Eq. (2.6) we have

\[ a_n^- = \frac{2(n^2 - b) - 2\sqrt{(n^2 - b)^2 + ab_n n^3\pi}}{an^3\pi} \]

which may be written in the form

\[ a_n^- = \frac{1}{n^3} \left( \frac{2bn}{1 - (b/n^2) + \sqrt{(1 - (b/n^2))^2 + (ab_n \pi/n)}} \right). \]

Again, from the algebra of limits

\[ \lim_{n\to\infty} \left( 1 - \frac{b}{n^2} + \frac{1}{n} \right)^2 + \frac{ab_n \pi}{n} = 2 \]

and therefore

\[ a_n^- = O \left( \frac{b_n}{n^3} \right). \]

Thus the behavior of the \( a_n^- \) as \( n \) becomes large is dependent on the behavior of \( b_n \). From assumption 3 in Section 2, we know that \( b_n \to 0 \) as \( n \to \infty \). If we restrict \( F(t) \) to a class of functions such that the \( b_n = O(1/n^{1+B}) \), where \( B > 0 \), then the \( [a_n^-] = O(1/n^{3+B}) \) or \( a_n^- \leq M/n^{3+B} \). If we consider the well-known Weierstrass M-test and let the \( M_n = M/n^{3+B}, M/n^{2+B}, M/n^{1+B} \) and the \( f_n = a_n^- \sin(nt), na_n^- \cos(nt), -n^2a_n^- \sin(nt) \) respectively, we see that the representations of \( x(t), dx(t)/dt, \) and \( d^2x(t)/dt^2 \) are uniformly convergent on \( [0, \pi] \). Therefore the termwise differentiation and integration are justifiable. If we restrict \( F(t) \) to satisfy the conditions of Theorem 3.1, then the \( b_n = O(1/n^2) \). This will be sufficient to permit term-wise differentiation and integration. We have also assumed that \( F(t) \) is an odd periodic function. Therefore, by Theorem 3.2, these restrictions will guarantee that \( F(t) \) is equal to its Fourier sine series representation for all \( t \).

We have further assumed that \( F(t) \) is not identically zero in order to guarantee that the \( a_n^- \) are not identically zero. Since we seek a real solution it is necessary that the \( a_n^- \) be real for all positive odd integers \( n \). Therefore, it is necessary that the discriminant \( (n^2 - b)^2 + ab_n n^3 \pi \geq 0 \) or that

\[ -ab_n \leq \frac{(n^2 - b)^2}{n^3\pi} = \frac{n}{\pi} - \frac{2b}{n\pi} + \frac{b^2}{n^2\pi} \tag{3.2} \]

for all positive odd integers \( n \). If \( ab_n > 0 \) for odd, Eq. (3.2) holds. If \( ab_n < 0 \) for some \( n \), there exists an \( N \) such that (3.2) holds for all \( n \geq N \) since the right-hand side is dominated by \( n/\pi \) and the \( \lim_{n \to \infty} b_n = 0 \). Therefore we need only to make restrictions for \( n < N \). These restrictions may take one of the two forms. We may restrict \( F(t) \) so that the \( b_n \) have the same sign as \( a \) for all values of \( n < N \), or we may restrict the values of \( a \) and \( b \) such that we may take \( N = 1 \). A more precise restriction on the relationship between \( a \) and \( b \) does not appear to be tractable for this solution.

Although the discussion and restrictions of \( F(t) \) apply for all \( t \), the solution in the mean, involving the \( a_n^- \), which we have obtained is defined only in the interval \( [0, \pi] \). To extend the solution to the interval \( [0, 2\pi] \) we integrate (2.4) from \( \pi \) to \( 2\pi \) and replace \( a_n \) by \( c_n \) to obtain \( c_n = 0 \) for \( n \) even, and for \( n \) odd we have

\[ c_n = \frac{2(b - n^2) \pm 2\sqrt{(b - n^2)^2 - ab_n n^3\pi}}{an^3\pi}. \tag{3.3} \]

From an analysis similar to that for the \( a_n^- \) we deduce that the orders of the two coefficients in (3.3) are \( c_n^- = O(1/n) \), for \( n \) odd, and \( c_n^- = O(1/n^{3+B}) \), for \( n \) odd. Therefore the \( c_n^- \) do not yield a solution in the mean over \( [\pi, 2\pi] \) when substituted for the \( a_n \) in (2.2). If we consider the \( c_n^+ \), however, we see that it satisfies the same conditions as the \( a_n^- \) except that for the \( c_n^+ \) to be real we must have \( (b - n^2)^2 - ab_n n^3 \pi \geq 0 \), or

\[ ab_n \leq \frac{(n^2 - b)^2}{n^3\pi} = \frac{n}{\pi} - \frac{2b}{n\pi} + \frac{b^2}{n^2\pi}. \tag{3.4} \]

To have a solution that is defined over \( (0, 2\pi) \), both (3.2) and (3.4) must be true. Therefore for all positive odd integers \( n \) we must have

\[ |ab_n| \leq \frac{(n^2 - b)^2}{n^3\pi}. \]
By Theorem 3.3 we have a continuous solution in the mean over \([0, \pi]\) using the \(a_n^+\) in (2.2) and over \([\pi, 2\pi]\) using the \(c_n^+\) in (2.2). At \(t = 0, \pi\) and \(2\pi\), (2.2) is equal to zero and therefore we have a continuous solution defined for all \(t\) since all of the functions are periodic. We may summarize the results to this point in the following theorem.

**Theorem 3.4.** Let \(F(t)\) be an odd periodic function of period \(2\pi\) which is bounded and continuous and otherwise satisfies Dirichlet’s conditions. Let \(F'(t)\) be bounded and otherwise satisfies Dirichlet’s conditions and let the sine series expansion of \(F(t)\) be

\[
F(t) = \sum_{n=1}^{\infty} b_n \sin nt.
\]

Then

\[
x(t) = \begin{cases} 
\sum_{k=1}^{\infty} a_{(2k-1)}^- \sin(2k-1)t, & 0 \leq t \leq \pi, \\
\sum_{k=1}^{\infty} a_{(2k-1)}^+ \sin(2k-1)t, & \pi < t \leq 2\pi,
\end{cases}
\]

(3.5)

where

\[
a_n^- = \frac{2(n^2 - b) - 2\sqrt{(n^2 - b)^2 + ab_n n^3 \pi}}{an^3 \pi}
\]

and

\[
c_n^+ = \frac{2(b - n^2) + 2\sqrt{(n^2 - b)^2 - ab_n n^3 \pi}}{an^3 \pi}
\]

is a solution in the mean with respect to \[[\sin nt]\] to the differential equation (1.1), if \(a\) and \(b_n\) are such that

\[
|ab_n| \leq \frac{(n^2 - b)^2}{n^3 \pi}
\]

(3.6)

for all positive odd integers \(n\). Furthermore (3.5) is continuous.

It should be noted that these results hold for intervals of arbitrary length. The solution in the mean with respect to \(\{\sin(n\pi/\ell)t\}\) over \([0, 2\ell]\) to Eq. (1.1) can be immediately obtained from (3.5) by substituting \(\ell\) for \(\pi\) and \(n\pi/\ell\) for \(n\) except where \(n\) is a subscript. In this case we assume \(F(t)\) has period \(2\ell\).

4. Restrictions on \(F(t)\)

In obtaining the solution in the mean given by Eq. (3.5), it was convenient to impose restrictions on the function \(F(t)\). These restrictions may be more severe than is necessary to guarantee that (3.5) is a solution in the mean. We now consider possible restrictions. For this purpose we will need the following theorem.

**Theorem 4.1.** (See [3].) If \(f(t)\) is bounded and otherwise satisfies Dirichlet’s conditions in the interval \((-\pi, \pi)\), the coefficients in the Fourier series for \(f(t)\) are less in absolute value than \(K/n\), where \(K\) is some positive number independent of \(n\).

We will still assume that \(F(t)\) is an odd periodic function of period \(2\pi\). In our solution, Theorem 3.4, the function \(F(t)\) was restricted such that the \(b_n = \mathcal{O}(1/n^2)\). Earlier in Section 3 it was shown that a sufficient condition on the \(b_n\), to guarantee that (3.5) is a solution in the mean to Eq. (1.1), is that the \(b_n = \mathcal{O}(1/n^{1+B})\). If \(F(t)\) satisfies only the conditions of Theorem 4.1 rather than those of Theorem 3.1, the \(b_n = \mathcal{O}(1/n)\). It seems that it might be possible to impose conditions on \(F(t)\) between those in Theorem 4.1 and those in Theorem 3.1 such that the \(b_n = \mathcal{O}(1/n^{1+B})\). This would lead to a stronger form of Theorem 3.4 in terms of these less severe restrictions.

We have required that the \(b_n = \mathcal{O}(1/n^{1+B})\) in order to guarantee that \(x(t)\) is uniformly convergent and therefore the termwise integration and differentiation are permissible. In some cases it may be possible to relax the conditions on \(F(t)\) such that the \(b_n = \mathcal{O}(1/n)\). The solution in the mean, (3.5) which we have demonstrated does not satisfy Eq. (1.1) in the ordinary sense; it is an approximation in the mean and may differ considerably from a solution from a
solution in the usual sense at particular points of the domain. There is one case, however, in which our solution in the mean does approach the solution to Eq. (1.1). This occurs when the coefficients, $a$, of the nonlinear term approaches zero. To examine this situation in detail, let us assume that in the sine series expansion of $F(t)$, the $b_n = 0$ for $n$ even, and that $b < 1$ in Eq. (1.1). Then if we assume that (2.2) is a solution to the differential equation

$$\ddot{x}(t) + bx(t) = F(t),$$

which may be obtained from (1.1) by letting $a \to 0$, and solve for the $a_n$, we obtain $a_n = 0$ for $n$ even, and $a_n = b_n/(b - n^2)$ for $n$ odd. Now, if we consider the limit of the solution in the mean (3.5) as $a \to 0$, then for $n$ even we have

$$\lim_{a \to 0} a_n^- = \lim_{a \to 0} c_n^- = 0$$

and for $n$ odd we have

$$\lim_{a \to 0} a_n^- = \lim_{a \to 0} \left( \frac{-2b_n}{(n^2 - b) + \sqrt{(n^2 - b)^2 + ab_n n^2 \pi}} \right) = \frac{-2b_n}{(n^2 - b) + |n^2 - b|}$$

and similarly

$$\lim_{n \to 0} c_n^+ = \frac{2b_n}{(b - n^2) - |n^2 - b|}.$$

Since it was assumed that $b < 1$, both of these will reduce to

$$\lim_{a \to 0} a_n^- = \lim_{a \to 0} c_n^+ = -\frac{b_n}{b - n^2}, \quad n = 1, 3, \ldots.$$

Therefore, (3.5) approaches the solution to Eq. (1.1) as $a \to 0$. This suggests that (3.5) may be a better approximation to the solution of Eq. (1.1) if $b_n = 0$ for $n$ even, and if $b < 1$, although these two conditions are not necessary to guarantee that (3.5) is a solution in the mean.

For $\phi_n(t)$ in Definition 2.1 we choose $\{\sin nt\}$. Other choices for orthogonal functions might be $\{\cos nt\}$, or $\{\sin nt, \cos nt\}, n = 1, 2, \ldots$. If we try to construct a solution from $\{\cos nt\}$ we see that $\int_0^\pi \cos nt \, dt = 0$ for all $n$. However, we might choose the interval $[-\pi/2, \pi/2]$ and $[\pi/2, 3\pi/2]$, but since $\cos(t - \pi/2) = \sin t$, this offers no advantage if we choose $\{\cos nt, \sin nt\}$ for the orthogonal system, we assume that

$$x(t) = a_0 + \sum_{n=1}^\infty (a_n \cos nt + b_n \sin nt)$$

and that $F(t)$ is periodic of period $2\pi$ and can be expanded into the Fourier series

$$F(t) = c_0 + \sum_{n=1}^\infty (c_n \cos nt + d_n \sin nt).$$

If we substitute these values into Eq. (1.1) and integrate from 0 to $2\pi$ we obtain $a_n = 0$ and $b_n = 0$ for all $n$. If we integrate from 0 to $\pi$ we obtain

$$-\sum_{n=1}^\infty n^2 b_n r_n + a \sum_{n=1}^\infty n^2 (a_n^2 + b_n^2) \frac{\pi}{2} - a \sum_{m=1}^\infty \sum_{n=1}^\infty mn a_m b_n (r_{m+n} + r_{m-n}) + b a_0 \pi + b \sum_{n=1}^\infty b_n r_n$$

$$= c_0 \pi + \sum_{n=1}^\infty d_n r_n$$

where

$$r_n = \begin{cases} 
0, & n \text{ even}, \\
2/\pi, & n \text{ odd}.
\end{cases}$$

This leads to no straightforward method for obtaining the $a_n$ and $b_n$ independently due to the double summation in the third term. If we integrate from $-\pi/2$ to $\pi/2$ we obtain the same result that we obtain by using $\{\cos nt\}$ since all involving $\sin nt$ vanish.
5. Further discussion

We have considered one method of deriving a solution in the mean to a non-homogeneous differential equation. For this purpose we defined a solution in the mean by Definition 2.1, and then considered under what conditions we could derive a solution in the mean. There are other types of non-homogeneous differential equations for which we can obtain solutions in the mean by using Definition 2.1. It is not convenient to deal with differential equations with terms of the third degree or higher by this mean, because we have various recursion problems similar to that encountered in Eq. (5.1), then we used the functions \( \{\cos nt, \sin nt\} \) over \([0, \pi]\). However, there seems to be no restriction on the order of the differential equation or on the number of terms of second order. Of course the problem of justifying the termwise differentiation becomes greater as the order of the equation increases. Further investigation might be directed toward relaxing the restrictions of \( F(t) \) in Theorem 3.4. One procedure in the latter direction might be to consider a generalized convergence in the mean of order \( p \) of the form:

**Definition 5.1.** If \( \{f[p|\sum_{k=1}^{n} a_k \phi_k(t)]\} \) is a sequence of Riemann integrable functions on \([a, b]\), then \( \{f[p|\sum_{k=1}^{n} a_k \phi_k(t)]\} \) is said to converge in the mean of order \( p \) on \([a, b]\) to \( F(t) \), if

\[
\lim_{n \to \infty} \int_{a}^{b} \left| f[p|\sum_{k=1}^{n} a_k \phi_k(t)] - F(t) \right|^p dt = 0. \tag{5.1}
\]

If we let \( p = 1 \), (5.1) is similar to (2.1) except that (5.1) involves an absolute value. Unfortunately, a direct integration does not provide a straightforward technique for evaluating the \( a_n \) an in the usual mean square case for linear functions. If we consider minimizing the integral

\[
\int_{a}^{b} \left| f[p|\sum_{n=1}^{\infty} a_n \sin nt] - F(t) \right| dt = 0 \tag{5.2}
\]

as a function of \( a_k \), a necessary condition is that the partial derivative of (5.2) with respect to \( a_n \) must vanish. It is then possible to integrate and solve for the \( a_n \), but the resulting solution does not involve the function \( F(t) \) and does not satisfy (5.1).

6. Homotopy perturbation method

In this section, we employ the homotopy perturbation method to seek an approximate solution to the nonlinear differential equation

\[
\ddot{x}(t) + a\dot{x}^2(t) + bx(t) = F(t) \tag{6.1}
\]

with the initial conditions \( x(0) = \alpha_1 \) and \( \dot{x}(0) = \alpha_2 \).

Eq. (6.1) can be written in a more general form

\[
\mathcal{L}(x) + \mathcal{N}(x) = 0, \tag{6.2}
\]

where \( \mathcal{L}(x) = \ddot{x}(t) + bx(t) \) is the linear part of the equation and \( \mathcal{N}(x) = a\dot{x}^2(t) - F(t) \) is the nonlinear part.

The homotopy perturbation method starts by constructing the homotopy

\[
(1 - p)[\mathcal{L}(u(t; p)) - \mathcal{L}(x_0(t))] = -p[\mathcal{L}(u(t; p)) + \mathcal{N}(u(t; p))], \tag{6.3}
\]

which can be written as

\[
\mathcal{L}(u(t; p)) - (1 - p)\mathcal{L}(x_0(t)) = -p\mathcal{N}(u(t; p)), \tag{6.4}
\]

where \( x_0(t) \) is an initial approximation to the solution of (6.1), satisfying the initial conditions. Eq. (6.3) is called the perturbation equation with the parameter \( p \) called the \textit{embedding} parameter. Note that when \( p = 0 \), (6.3) reduces to \( \mathcal{L}(u(t; 0)) - \mathcal{L}(x_0(t)) = 0 \), and when \( p = 1 \), (6.3) reduces to \( \mathcal{L}(u(t; 1)) + \mathcal{N}(u(t; 1)) = 0 \). Hence, \( u(t; 0) = x_0(t) \) and the required solution of (6.1) is \( x(t) = u(t; 1) \). The basic idea behind the homotopy perturbation method is that
as the embedding parameter \( p \) continuously increases from 0 to 1, the approximate solution \( x_0(t) \) is continuously deformed to approach the solution of (6.1). We shall illustrate this method on Eq. (6.1).

Suppose that the solution, \( u(t; p) \), of the homotopy (6.3), has a series expansion in the form

\[
    u(t; p) = \sum_{k=0}^{\infty} p^k u_k(t),
\]

where \( u_k(t), k \geq 0, \) are solution components. Since the solution \( x(t) = u(t; 1) \), we have

\[
    x(t) = u(t; 1) = \sum_{k=0}^{\infty} u_k(t).
\]

Thus it remains to find the components \( u_k(t) \). To this end, we substitute (6.5) into (6.4) to obtain

\[
    \sum_{k=0}^{\infty} p^k \mathcal{L}(u_k) - (1 - p)\mathcal{L}(x_0) = -a \left[ \sum_{k=0}^{\infty} p^{k+1} \left( \sum_{m=0}^{k} \dot{u}_m u_{k-m} \right) \right] + p F(t).
\]

Then equating terms of like powers of \( p \), we obtain

\[
    \mathcal{L}(u_0) - \mathcal{L}(x_0) = 0, \quad (6.6)
\]

\[
    \mathcal{L}(u_1) + \mathcal{L}(x_0) = -a \ddot{u}_0^2 + F(t), \quad (6.7)
\]

\[
    \mathcal{L}(u_k) = -a \sum_{m=0}^{k-1} \dot{u}_m \ddot{u}_{k-m-1}, \quad k \geq 2. \quad (6.8)
\]

It follows from (6.6) that the zeroth component \( u_0(t) = x_0(t) \) and the higher order components \( u_k(t), k \geq 1, \) are given recursively by Eqs. (6.7) and (6.8), and satisfying \( u_k(0) = u_k(0) = 0 \).

We shall apply the above homotopy perturbation technique to particular examples of (6.1).

**Example 6.1.** Consider the special type of Eq. (6.1) when \( F(t) = 0, a = 2, b = 4, \) i.e.,

\[
    \ddot{x}(t) + 2 \dot{x}^2(t) + 4x(t) = 0 \quad (6.9)
\]

with initial conditions \( x(0) = 0, \dot{x}(0) = 1 \).

For this example, the linear operator is \( \mathcal{L}(x) = \ddot{x}(t) + 4x(t) \) and the nonlinear operator is \( \mathcal{N}(x) = 2 \dot{x}^2(t) \). We take as initial approximate solution \( x_0(t) \) the solution to the linear form of (6.9), i.e.,

\[
    \ddot{x}(t) + 4x(t) = 0.
\]

Then, we have

\[
    x_0(t) = c_1 \cos(2t) + c_2 \sin(2t) \quad (6.10)
\]

with \( c_1 \) and \( c_2 \) are arbitrary constants. Satisfying the initial conditions \( x(0) = 0 \) and \( \dot{x}(0) = 1 \), we obtain \( c_1 = 0 \) and \( c_2 = 1/2 \). Thus,

\[
    x_0(t) = \frac{1}{2} \sin(2t). \quad (6.11)
\]

From (6.6), we have the zeroth component \( u_0(t) = x_0(t) = \frac{1}{2} \sin(2t) \). From (6.7), we find that

\[
    \ddot{u}_1 + 4u_1 = -2\ddot{u}_0^2
\]

whose solution, satisfying \( u_1(0) = \dot{u}_1(0) = 0 \), is

\[
    u_1(t) = -\frac{1}{3} \left[ \cos(2t) + 2 \right] \sin^2(t).
\]

Similarly from (6.8), we obtain higher order components \( u_k, k \geq 2 \). Here, we list the first few for brevity:
Table 1
The error $|\bar{x}_{\text{approx}}(t_j) + 2\dot{x}_{\text{approx}}(t_j) + 4x_{\text{approx}}(t_j)|$ in the approximate solution for Example 1

<table>
<thead>
<tr>
<th>$t_j \backslash N$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.01</td>
<td>1.57401E–6</td>
<td>3.77502E–8</td>
<td>8.80391E–10</td>
<td>2.01183E–11</td>
<td>4.5052E–13</td>
<td>1.22125E–14</td>
</tr>
<tr>
<td>0.02</td>
<td>2.4756E–5</td>
<td>5.3166E–8</td>
<td>1.01804E–7</td>
<td>2.03202E–8</td>
<td>4.0644E–9</td>
<td>9.1308E–10</td>
</tr>
<tr>
<td>0.03</td>
<td>1.23106E–4</td>
<td>3.84401E–5</td>
<td>6.19043E–5</td>
<td>2.03202E–8</td>
<td>4.0644E–9</td>
<td>9.1308E–10</td>
</tr>
<tr>
<td>0.05</td>
<td>9.1440E–4</td>
<td>1.09299E–4</td>
<td>1.27118E–5</td>
<td>1.44898E–6</td>
<td>2.2884E–7</td>
<td>3.225E–8</td>
</tr>
<tr>
<td>0.06</td>
<td>1.85821E–3</td>
<td>2.66012E–4</td>
<td>3.71338E–5</td>
<td>5.05763E–6</td>
<td>6.83188E–7</td>
<td>9.0874E–8</td>
</tr>
<tr>
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<td>5.62727E–3</td>
<td>1.07302E–3</td>
<td>1.99153E–4</td>
<td>3.07563E–6</td>
<td>6.83188E–7</td>
<td>9.0874E–8</td>
</tr>
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<td>0.10</td>
<td>1.31227E–2</td>
<td>3.12079E–3</td>
<td>7.22514E–4</td>
<td>1.6401E–4</td>
<td>3.66675E–5</td>
<td>8.09966E–6</td>
</tr>
</tbody>
</table>

An approximate solution is obtained by adding up a finite number of $u_k$:

$$x_{\text{approx}}(t) = \sum_{k=0}^{N} u_k(t).$$

With $N = 4$, we have the approximate solution to (6.9) as

$$x_{\text{approx}}(t) = u_0(t) + u_1(t) + u_2(t) + u_3(t) + u_4(t)$$

$$= \frac{1}{51840}[-1440 \sin(2t)^2 - 1200 \cos(2t) + 1920 \cos(4t) + 1080 \cos(6t) + 14251 \sin(2t) - 4384 \sin(4t) - 1044 \sin(6t) + 416 \sin(8t) + 97 \sin(10t)].$$

Table 1 shows $|\bar{x}_{\text{approx}}(t_j) + 2\dot{x}_{\text{approx}}(t_j) + 4x_{\text{approx}}(t_j)|$ for different levels $N$ at selected $t$ values. It is clear from the table that the error decreases as more and more terms are included.

References