Controllability of second-order neutral functional differential inclusions in Banach spaces

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Abstract

In this paper, we prove the controllability of second-order neutral functional differential inclusions in Banach spaces. The result are obtained by using the theory of strongly continuous cosine families and a fixed point theorem for condensing maps due to Martelli.

Keywords: Neutral functional differential inclusions; Controllability; Fixed points

1. Introduction

In this paper, we study the controllability of second-order neutral functional differential inclusions in Banach spaces. More precisely, we consider the following form:

\[
\frac{d}{dt} \left[ y'(t) - g(t, y_t) \right] \in Ay(t) + Bu(t) + F(t, y_t), \quad t \in J = [0, T],
\]

\[
y_0 = \phi, \quad y'(0) = x_0,
\]

where \( F : J \times C(J_0, E) \to 2^E \) (here \( J_0 = [-r, 0] \)) is a bounded, closed, convex multivalued map, \( g : J \times C(J_0, E) \to E \) is given function, \( \phi \in C(J_0, E), \ x_0 \in E, \) and \( A \) is the

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infinitesimal generator of a strongly continuous cosine family \( \{ C(t) : t \in \mathbb{R} \} \) in a real Banach space \( X \) with the norm \( | \cdot | \). Also, the control function \( u(\cdot) \) is given in \( L^2(J, U) \), a Banach space of admissible control functions with \( U \) as a Banach space. Finally, \( B \) is a bounded linear operator from \( U \) to \( E \).

For any continuous function \( y \) defined on the interval \( J_1 = [-r, T] \) and any \( t \in J \), we denote by \( y_t \) the element of \( C(J_0, E) \) defined by

\[
y_t(\theta) = y(t + \theta), \quad \theta \in J_0.
\]

Here \( y_t(\cdot) \) represents the history of the state from time \( t - r \), up to the present time \( t \).

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional space has been extensively studied. Several authors have extended the concept to infinite-dimensional systems in Banach spaces with bounded operators [1,13,15].

With the help of fixed point theorem several authors have investigated the problem of controllability of nonlinear systems in Banach spaces. Balachandran et al. [1] studied controllability for nonlinear integrodifferential systems in Banach spaces using the Schauder fixed point theorem. Benchohra and Ntouyas [3], using a fixed point theorem for condensing maps, proved controllability of second-order differential inclusions in Banach spaces with nonlocal conditions.

In many cases it is advantageous to treat the second-order abstract differential equations directly rather than to convert them into first-order systems. A useful tool for the study of abstract second-order equations is the theory of strongly continuous cosine families. Here we use of the basic ideas from cosine family theory [16,17].

Motivation for second-order neutral systems can be found in [3,8,12]. The purpose of this paper is to study the controllability of second-order neutral functional differential inclusions \((1), (2)\) relying on a fixed point theorem for condensing maps due to Martelli [11].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

\( C(J, E) \) is the Banach space of continuous functions from \( J \) into \( E \) with the norm

\[
\| y \|_{\infty} := \sup \{ |y(t)| : t \in J \}.
\]

\( B(E) \) denotes the Banach space of bounded linear operators from \( E \) into \( E \). A measurable function \( y : J \to E \) is Bochner integrable if and only if \( |y| \) is Lebesgue integrable. (For properties of the Bochner integral see Yosida [18].)

\( L^1(J, E) \) denotes the Banach space of continuous functions \( y : J \to E \) which are Bochner integrable, normed by

\[
\| y \|_{L^1} = \int_0^T |y(t)| \, dt, \quad \text{for all } y \in L^1(J, E).
\]
Let \((X, \| \cdot \|)\) be a Banach space. A multivalued map \(G : X \to 2^X\) is convex (closed) valued, if \(G(x)\) is convex (closed) for all \(x \in X\). \(G\) is bounded on bounded sets if \(G(D) = \bigcup_{x \in D} G(x)\) is bounded in \(X\), for any bounded set \(D\) of \(X\), i.e.,

\[
\sup_{x \in D} \left\{ \sup\{ \|y\| : y \in G(x)\} \right\} < \infty.
\]

\(G\) is called upper semicontinuous on \(X\), if for each \(x_0 \in X\), the set \(G(x_0)\) is a nonempty closed subset of \(X\), and if for each open set \(V\) of \(X\) containing \(G(x_0)\), there exists an open neighborhood \(A\) of \(x_0\) such that \(G(A) \subseteq V\).

\(G\) is said to be completely continuous if \(G(D)\) is relatively compact for every bounded subset \(D \subseteq X\). If the multivalued map \(G\) is completely continuous with nonempty compact values, then \(G\) is upper semicontinuous if and only if \(G\) has a closed graph, i.e.,

\[
x_n \to x_*, \quad y_n \to y_*, \quad y_n \in Gx_n \quad \text{imply} \quad y_* \in Gx_*.
\]

\(G\) has a fixed point if there is \(x \in X\) such that \(x \in Gx\).

In the following, \(BCC(X)\) denotes the set of all nonempty bounded closed and convex subsets of \(X\).

A multivalued map \(G : J \to BCC(X)\) is said to be measurable if for each \(x \in X\), the distance between \(x\) and \(G(t)\) is a measurable function on \(J\). For more details on multivalued maps, see the books of Deimling [4] and Hu and Papageorgiou [9].

An upper semicontinuous map \(G : X \to 2^X\) is said to be condensing if for any bounded subset \(D \subseteq X\), with \(\alpha(D) \neq 0\), we have

\[
\alpha(G(D)) < \alpha(D),
\]

where \(\alpha\) denotes the Kuratowski measure of noncompactness. For properties of the Kuratowski measure, we refer to Banas and Goebel [2].

We remark that a completely continuous multivalued map is the easiest example of a condensing map.

We say that the family \(\{C(t) : t \in \mathbb{R}\}\) of operators in \(B(E)\) is a strongly continuous cosine family if

(i) \(C(0) = I\) (\(I\) is the identity operator in \(E\)),
(ii) \(C(t + s) + C(t - s) = 2C(t)C(s)\) for all \(s, t \in \mathbb{R}\),
(iii) the map \(t \to C(t)y\) is strongly continuous for each \(y \in E\).

The strongly continuous sine family \(\{S(t) : t \in \mathbb{R}\}\), associated to the given strongly continuous cosine family \(\{C(t) : t \in \mathbb{R}\}\), is defined by

\[
S(t)y = \int_0^t C(s)y\, ds, \quad y \in E, \quad t \in \mathbb{R}.
\]

The infinitesimal generator \(A : E \to E\) of a cosine family \(\{C(t) : t \in \mathbb{R}\}\) is defined by

\[
Ay = \left. \frac{d^2}{dt^2} C(t)y \right|_{t=0}.
\]
For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [7] and to the papers of Fattorini [5, 6] and of Travis and Webb [16, 17].

Let us list the following hypotheses.

(H1) \( A \) is the infinitesimal generator of a strongly continuous cosine family \( C(t), t \in \mathbb{R} \), of bounded linear operators from \( X \) into itself.

(H2) \( C(t), t > 0 \) is compact.

(H3) \( F : J \times C(J_0, E) \to BCC(E); (t, u) \to F(t, u) \) is measurable with respect to \( t \) for each \( u \in C(J_0, E) \), upper semicontinuous with respect to \( u \) for each \( t \in J \), and for each fixed \( u \in C(J_0, E) \), the set
\[
S_{F,u} = \{ f \in L^1(J, E): f(t) \in F(t, u) \text{ for a.e. } t \in J \}
\]
is nonempty.

(H4) The linear operator \( W : L^2(J, U) \to E \), defined by
\[
Wu = \int_0^T S(T - s)Bu(s) \, ds,
\]
induces a bounded invertible operators \( \tilde{W} \) defined on \( L^2(J, U)/\ker W \) and there exist positive constants \( M_1, M_2 \) such that \( |\tilde{W}| \leq M_1 \) and \( |B| \leq M_2 \) (see [15]).

(H5) The function \( g : J \times C(J_1, E) \to E \) is completely continuous and for any bounded set \( K \) in \( C(J_1, E) \), the set \( \{ t \to g(t, y_t) : y \in K \} \) is equicontinuous in \( C(J, E) \).

(H6) There exist constants \( c_1 \) and \( c_2 \) such that
\[
|g(t, v)| \leq c_1 \|v\| + c_2, \quad t \in J, \quad v \in C(J_0, E).
\]

(H7) \( \|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)\Psi(\|u\|) \) for almost all \( t \in J \) and all \( u \in C(J_0, E) \), where \( p \in L^1(J, R^+) \) and \( \Psi : R^+ \to (0, \infty) \) is continuous and increasing with
\[
\int_0^T m(s) \, ds < \int \frac{ds}{s + \Psi(s)}.
\]

where \( c = M\|\phi\| + MT[|y_0| + c_1 \|\phi\| + 2c_2] + M T^2 M_0, \) \( M(t) = \max\{Mc_1, MT p(t)\}, \) \( M = \sup\{|C(t)| : t \in J\}, \) \( M_0 = M_1 M_2[\|x_1\| + M \|\phi\| + MT[|y_0| + c_1 \|\phi\| + 2c_2] + M c_1 \int_0^T \|y_s\| \, ds + MT \int_0^T p(s)\Psi(\|y_s\|) \, ds\} \).

**Remark 2.1.** (i) If \( \dim E < \infty \), then for each \( u \in C(J_0, E) \), \( S_{F,u} \neq \emptyset \) (see Lasota and Opial [10]).

(ii) \( S_{F,u} \) is nonempty if and only if the function \( Y : J \to R \) defined by
\[
Y(t) := \inf\{|v| : v \in F(t, u)\}
\]
belongs to \( L^1(J, R) \) (see Papageorgiou [14]).

In order to define the concept of mild solution for (1), (2), by comparison with abstract Cauchy problem
\[ y''(t) = Ay(t) + h(t), \]
\[ y(0) = y_0, \quad y'(0) = y_1 \]
whose properties are well known [16,17], we associate problem (1), (2) to the integral equation
\[
y(t) = C(t)\phi(0) + S(t)\left[x_0 - g(0, \phi)\right] + \int_0^t C(t-s)g(s, y_s) \, ds \\
+ \int_0^t S(t-s)Bu(s) \, ds + \int_0^t S(t-s)f(s) \, ds, \quad t \in J,
\]
where
\[
f \in S_{F,Y} = \{ f \in L^1(J,E): f(t) \in F(t, y_t) \text{ for a.e. } t \in J \}.
\]

**Definition 2.1.** A function \( y: (-r, T) \to E \), \( T > 0 \) is called a mild solution of the problem (1), (2) if
\[ y(t) = \phi(t), \quad t \in [-r, 0], \]
and there exists a \( v \in L^1(J,E) \) such that \( v(t) \in F(t, y_t) \) a.e. on \( J \), and the integral equation (3) is satisfied.

**Definition 2.2.** The problem (1), (2) is said to be controllable on the interval \( J \) if, for every \( \phi \in C(J_0, E) \) with \( \phi(0) \in D(A), x_0 \in E, \) and \( x_1 \in E \), there exists a control \( u \in L^2(J, U) \) such that the mild solution \( y(\cdot) \) of (1), (2) satisfies \( y(T) = x_1 \).

The following lemmas are crucial in the proof of our main theorem.

**Lemma 2.1** [10]. Let \( I \) be a compact real interval and \( X \) be a Banach space. Let \( F \) be a multivalued map satisfying (H3) and let \( \Gamma \) be a linear continuous mapping from \( L^1(I, X) \) to \( C(I, X) \). Then the operator
\[
\Gamma \circ SF: C(I, X) \to BCC(C(I, X)), \quad y \mapsto (\Gamma \circ SF)(y) := \Gamma(SF, y)
\]
is a closed graph operator in \( C(I, X) \times C(I, X) \).

**Lemma 2.2** [11]. Let \( X \) be a Banach space and \( N: X \to BCC(X) \) be a condensing map. If the set
\[
\Omega := \{ y \in X: \lambda y \in Ny \text{ for some } \lambda > 1 \}
\]
is bounded, then \( N \) has a fixed point.

### 3. Main result

Now, we are able to state and prove our main theorem.

**Theorem 3.1.** Assume that hypotheses (H1)–(H7) are satisfied. Then the system (1), (2) is controllable on \( J \).
Proof. Using (H4), for an arbitrary function \( y(\cdot) \), we define the control
\[
u_y(t) = \tilde{W}^{-1} \left[ x_1 - C(T)\phi(0) - S(T)[x_0 - g(0, \phi)] \right.\\
- \int_0^T C(T - s)g(s, y_s) ds - \int_0^T S(T - s)f(s) ds \left. \right] (t),
\]
where \( f \in S_{F,y} = \{ f \in L^1(J, E) : f(t) \in F(t, y_t) \text{ for a.e. } t \in J \} \).

Now we shall show that, when using this control, the operator \( N : C(J_1, E) \to 2^{C(J_1, E)} \)
defined by
\[
N_y = \begin{cases} 
    h \in C(J_1, E) : h(t) = \begin{cases} 
    \phi(t), & \text{if } t \in J_0, \\
    C(t)\phi(0) + S(t)[x_0 - g(0, \phi)] + \int_0^t C(t - s)g(s, y_s) ds \\
    + \int_0^t S(t - s)Bu_y(s) ds + \int_0^t S(t - s)f(s) ds, & \text{if } t \in J,
    \end{cases}
\end{cases}
\]
has a fixed point. This fixed point is then a solution of the problem (1), (2).

Clearly, \( x_1 \in (N_y)(T) \).

We shall show that \( N \) is completely continuous with bounded closed convex values and it is upper semicontinuous. The proof will be given in several steps.

Step 1. \( N_y \) is convex for each \( y \in C(J_1, E) \).

Indeed, if \( h_1, h_2 \) belong to \( N_y \), then there exist \( f_1, f_2 \in S_{F,y} \) such that for each \( t \in J \), we have
\[
h_i(t) = C(t)\phi(0) + S(t)[x_0 - g(0, \phi)] + \int_0^t C(t - s)g(s, y_s) ds \\
+ \int_0^t S(t - s)Bu_y(s) ds + \int_0^t S(t - s)f_i(s) ds, \quad i = 1, 2.
\]

Let \( 0 \leq \alpha \leq 1 \). Then for each \( t \in J \), we have
\[
\alpha h_1 + (1 - \alpha)h_2(t) = C(t)\phi(0) + S(t)[x_0 - g(0, \phi)] + \int_0^t C(t - s)g(s, y_s) ds \\
+ \int_0^t S(t - s)Bu_y(s) ds + \int_0^t S(t - s)f_1(s) ds + (1 - \alpha) \int_0^t S(t - s)f_2(s) ds.
\]

Since \( S_{F,y} \) is convex (because \( F \) has convex values), then
\[
\alpha h_1 + (1 - \alpha)h_2 \in N_y
\]
completing the proof of Step 1.
Step 2. \( N \) maps bounded sets into bounded sets in \( C(J_1, E) \).

Indeed, it is enough to show that there exists a positive constant \( \ell \) such that for each \( h \in Ny, y \in B_q = \{ y \in C(J_1, E): \| y \|_{\infty} \leq q \} \), one has \( \| h \|_{\infty} \leq \ell \). If \( h \in Ny \), then there exists \( f \in SF,y \) such that for each \( t \in J \) we have

\[
h(t) = C(t)\phi(0) + S(t)[x_0 - g(0, \phi)] + \int_0^t C(t - s)g(s, y_s) \, ds \\
+ \int_0^t S(t - s)Bu_s(s) \, ds + \int_0^t S(t - s)f(s) \, ds.
\]

By (H4), (H6), and (H7), we have that, for each \( t \in J \),

\[
|h(t)| \leq |C(t)\phi(0)| + |S(t)[x_0 - g(0, \phi)]| + \left| \int_0^t C(t - s)g(s, y_s) \, ds \right| \\
+ \left| \int_0^t S(t - s)Bu_s(s) \, ds \right| + \left| \int_0^t S(t - s)f(s) \, ds \right| \\
\leq M\|\phi\| + MT\|x_0\| + c_1\|\phi\| + 2c_2 + MC_1 \int_0^t \|y_s\| \, ds \\
+ MT^2M_0 + MT \sup_{y \in [0,q]} \Psi(y) \left( \int_0^t p(s) \, ds \right).
\]

Then for each \( h \in N(B_q) \) we have

\[
\|h\|_{\infty} \leq M\|\phi\| + MT\|x_0\| + c_1\|\phi\| + 2c_2 + MC_1 \int_0^T \|y_s\| \, ds \\
+ MT^2M_0 + MT \sup_{y \in [0,q]} \Psi(y) \left( \int_0^T p(s) \, ds \right) := \ell.
\]

Step 3. \( N \) maps bounded sets into equicontinuous sets of \( C(J_1, E) \).

Let \( t_1, t_2 \in J \), \( 0 < t_1 < t_2 \) and \( B_q = \{ y \in C(J_1, E): \| y \|_{\infty} \leq q \} \) be a bounded set of \( C(J_1, E) \). For each \( y \in B_q \) and \( h \in Ny \), there exists \( f \in SF,y \) such that

\[
h(t) = C(t)\phi(0) + S(t)[x_0 - g(0, \phi)] + \int_0^t C(t - s)g(s, y_s) \, ds
\]
Thus,

\[
|h(t_2) - h(t_1)| \\
\leq \left| \left[ (C(t_2) - C(t_1))\varphi(0) \right] + \left[ (S(t_2) - S(t_1))\left[ x_0 - g(0, \phi) \right] \right] \right| \\
+ \left| \int_0^{t_2} \left[ (C(t_2) - C(t_1))g(s, y_s) \right] ds \right| \\
+ \left| \int_{t_1}^{t_2} C(t_1 - s)g(s, y_s) ds \right| \\
+ \left| \int_0^{t_2} \left[ (S(t_2) - S(t_1))B\tilde{W}^{-1} \right] x_1 - C(T)\varphi(0) \\
- S(T)\left[ x_0 - g(0, \phi) \right] - \int_0^T C(T - \tau)g(\tau, y_\tau) d\tau \\
- \int_0^T S(T - \tau)f(\tau) d\tau \right| (s) ds \right| \\
+ \left| \int_{t_1}^{t_2} \left[ (S(t_2) - S(t_1))B\tilde{W}^{-1} \right] x_1 - C(T)\varphi(0) \\
- S(T)\left[ x_0 - g(0, \phi) \right] - \int_0^T C(T - \tau)g(\tau, y_\tau) d\tau \\
- \int_0^T S(T - \tau)f(\tau) d\tau \right| (s) ds \right| \\
+ \left| \int_0^{t_2} \left[ (S(t_2) - S(t_1))f(s) \right] ds \right| \\
+ \left| \int_{t_1}^{t_2} S(t_1 - s)f(s) ds \right| \\
\leq \left| C(t_2) - C(t_1)\|\varphi\| \right| + \left| S(t_2) - S(t_1)\left[ |x_0| + c_1\|\phi\| + c_2 \right] \right| \\
+ \left| \int_0^{t_2} \left[ C(t_2) - C(t_1)\right] \left[ c_1\|y_s\| + c_2 \right] ds \right| \\
+ \left| \int_{t_1}^{t_2} C(t_1 - s)\left[ c_1\|y_s\| + c_2 \right] ds \right|
\]
\[ + \int_0^{t_2} |S(t_2 - s) - S(t_1 - s)| M_1 M_2 \left[ |x_1| + M \| \phi \| \right] \]
\[ + MT \left[ |x_0| + c_1 \| \phi \| + c_2 \right] \]
\[ + M \int_0^T \left[ c_1 \| y_\tau \| + c_2 \right] d\tau + MT \int_0^T \| f(\tau) \| d\tau \] \[ ds \]
\[ + \int_{t_1}^{t_2} \left| S(t_1 - s) \right| M_1 M_2 \left[ |x_1| + M \| \phi \| + MT \left[ |x_0| + c_1 \| \phi \| + c_2 \right] \right] \]
\[ + M \int_0^T \left[ c_1 \| y_\tau \| + c_2 \right] d\tau + MT \int_0^T \| f(\tau) \| d\tau \] \[ ds \]
\[ + \int_0^{t_2} \left| S(t_2 - s) - S(t_1 - s) \right| \| f(s) \| ds + \int_{t_1}^{t_2} \left[ S(t_1 - s) \right] \| f(s) \| ds. \]

As \( t_2 \to t_1 \) the right-hand side of the above inequality tends to zero. The equicontinuities for the cases \( t_1 < t_2 \leq 0 \) and \( t_1 \leq 0 \leq t_2 \) are obvious. As a consequence of Step 2, Step 3, and (H5) together with the Ascoli–Arzela theorem, we can conclude that \( N \) is completely continuous, and therefore, a condensing map.

**Step 4.** \( N \) has a closed graph.

Let \( y_n \to y_* \), \( h_n \in N_{y_n} \), and \( h_n \to h_* \). We shall prove that \( h_* \in N_{y_*} \). \( h_n \in N_{y_n} \) means that there exists \( f_n \in SF_{y_n} \) such that

\[
\begin{align*}
    h_n(t) &= C(t) \phi(0) + S(t) \left[ x_0 - g(0, \phi) \right] + \int_0^t C(t - s) g(s, y_n(s)) ds \\
    &\quad + \int_0^t S(t - s) Bu_n(s) ds + \int_0^t S(t - s) f_n(s) ds, \quad t \in J,
\end{align*}
\]

where

\[
u_{y_n}(t) = W^{-1} \left[ x_1 - C(T) \phi(0) - S(T) \left[ x_0 - g(0, \phi) \right] \right]
- \int_0^T C(T - s) g(s, y_n(s)) ds - \int_0^T S(T - s) f_n(s) ds \] \[ (t). \]

We must prove that there exists \( f_* \in SF_{y_*} \) such that
\[ h_n(t) = C(t)\phi(0) + S(t)\left[ x_0 - g(0, \phi) \right] + \int_0^t \int C(t-s)g(s, y_n) \, ds \]

\[ + \int_0^t S(t-s)Bu_{y_n}(s) \, ds + \int_0^t S(t-s)f_\tau(s) \, ds, \quad t \in J, \]

where

\[ u_{y_n}(t) = \tilde{W}^{-1} \left[ x_1 - C(T)\phi(0) - S(T)\left[ x_0 - g(0, \phi) \right] \right. \]

\[ - \int_0^T C(T-s)g(s, y_n) \, ds - \int_0^T S(T-s)f_\tau(s) \, ds \right](t). \]

Set

\[ \tilde{u}_{y_n}(t) = \tilde{W}^{-1} \left[ x_1 - C(T)\phi(0) - S(T)\left[ x_0 - g(0, \phi) \right] \right. \]

\[ - \int_0^T C(T-s)g(s, y_n) \, ds \right](t). \]

Since \( g, \tilde{W}^{-1} \) are continuous, then

\[ \tilde{u}_{y_n}(t) \to \tilde{u}_{y_n}(t), \quad \text{for } t \in J. \]

Clearly, we have that

\[ \left\| \left( h_n - C(t)\phi(0) - S(T)\left[ x_0 - g(0, \phi) \right] \right. \right. \]

\[ - \int_0^T C(T-s)g(s, y_n) \, ds - \int_0^T S(T-s)f_\tau(s) \, ds \right) \right. \]

\[ - \int_0^T C(T-s)g(s, y_n) \, ds - \int_0^T S(T-s)f_\tau(s) \, ds \right) \right\| \to 0, \]

as \( n \to \infty \).

Consider the linear continuous operator

\[ \Gamma : L^1(J, E) \to C(J, E), \]

\[ f \to \Gamma(f)(t) = \int_0^t S(t-s) \left[ f(s) - B\tilde{W}^{-1} \left( \int_0^T S(T-\tau)f(\tau) \, d\tau \right)\right](s) \, ds. \]
From Lemma 2.1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have that
\[
h_n(t) - C(t)\phi(0) - S(t)\left[x_0 - g(0, \phi)\right] - \int_0^t C(t - s)g(s, y_{ns})\, ds - \int_0^t S(t - s)B\bar{u}_{y_n}(s)\, ds \in \Gamma(S_F, y_n).
\]
Since $y_n \to y_*$, it follows from Lemma 2.1 that
\[
h_*(t) - C(t)\phi(0) - S(t)\left[x_0 - g(0, \phi)\right] - \int_0^T C(T - s)g(s, y_{*s})\, ds - \int_0^T S(T - s)B\bar{u}_{y_*}(s)\, ds
\]
for some $f_* \in S_{F, y_*}$.

Therefore $N$ is a completely continuous multivalued map, upper semicontinuous with convex closed values. In order to prove that $N$ has a fixed point, we need one more step.

**Step 5.** The set
\[
\Omega := \{ y \in C(J_1, E): \lambda y \in Ny, \text{ for some } \lambda > 1 \}
\]
is bounded.

Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus, there exists $f \in S_{F, y}$ such that
\[
y(t) = \lambda^{-1}C(t)\phi(0) + \lambda^{-1}S(t)\left[x_0 - g(0, \phi)\right] + \lambda^{-1} \int_0^t C(t - s)g(s, y_s)\, ds
\]
\[
+ \lambda^{-1} \int_0^t S(t - s)B\bar{W}^{-1}\left[ x_1 - C(T)\phi(0) - S(T)\left[x_0 - g(0, \phi)\right] - \int_0^T C(T - \tau)g(\tau, y_\tau)\, d\tau - \int_0^T S(T - \tau)f(\tau)\, d\tau \right](s)\, ds
\]
\[
+ \lambda^{-1} \int_0^t S(t - s)f(s)\, ds, \quad t \in J.
\]

This implies by (H4), (H6), and (H7) that for each $t \in J$, we have
\[ |y(t)| \leq M\|\phi\| + MT\left[|x_0| + c_1\|\phi\| + 2c_2\right] + Mc_1 \int_0^t \|y_s\| \, ds + MT^2 M_0 + MT \int_0^t p(s)\Psi(\|y_s\|) \, ds. \]

We consider the function \( \mu \) defined by
\[ \mu(t) = \sup\{|y(s)|: -r \leq s \leq t\}, \quad t \in J. \]
Let \( t^* \in [-r, t] \) be such that \( \mu(t) = |y(t^*)| \). If \( t^* \in J \), by the previous inequality we have for \( t \in J \)
\[ \mu(t) \leq M\|\phi\| + MT\left[|x_0| + c_1\|\phi\| + 2c_2\right] + Mc_1 \int_0^{t^*} \|y_s\| \, ds + MT\int_0^{t^*} p(s)\Psi(\|y_s\|) \, ds \]
\[ \leq M\|\phi\| + MT\left[|x_0| + c_1\|\phi\| + 2c_2\right] + Mc_1 \int_0^t \mu(s) \, ds + MT^2 M_0 + MT \int_0^t p(s)\Psi(\mu(s)) \, ds. \]

If \( t^* \in J_0 \), then \( \mu(t) \leq \|\phi\| \) and the previous inequality holds.

Let us take the right-hand side of the above inequality as \( v(t) \). Then, we have
\[ c = v(0) = M\|\phi\| + MT\left[|x_0| + c_1\|\phi\| + 2c_2\right] + MT^2 M_0, \]
\[ \mu(t) \leq v(t), \]
\[ v'(t) = Mc_1\mu(t) + MTp(t)\Psi(\mu(t)), \quad t \in J. \]
Using the nondecreasing character of \( \Psi \), we get
\[ v'(t) \leq Mc_1v(t) + MTp(t)\Psi(v(t)) \leq m(t)\left[v(t) + \Psi(v(t))\right], \quad t \in J. \]
This implies that for each \( t \in J \) that
\[ \int_{v(0)}^{v(t)} \frac{ds}{s + \Psi(s)} \leq \int_0^T m(s) \, ds < \int_{v(0)}^{\infty} \frac{ds}{s + \Psi(s)}. \]
This inequality implies that there exists a constant \( L \) such that \( v(t) \leq L, \, t \in J \), and hence \( \mu(t) \leq L, \, t \in J \). Since for every \( t \in J \), \( \|y_t\| \leq \mu(t) \), we have
\[ \|y\|_\infty := \sup\{|y(t)|: -r \leq t \leq T\} \leq L, \]
where \( L \) depends only on \( T \) and on the function \( p \) and \( \Psi \). This shows that \( \Omega \) is bounded.

Set \( X := C(J_1, E) \). As a consequence of Lemma 2.2, we deduce that \( N \) has a fixed point and thus the system (1), (2) is controllable on \( J \). \( \square \)
References