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Controllability of second-order neutral functional differential inclusions in Banach spaces

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Abstract

In this paper, we prove the controllability of second-order neutral functional differential inclusions in Banach spaces. The result are obtained by using the theory of strongly continuous cosine families and a fixed point theorem for condensing maps due to Martelli.

Keywords: Neutral functional differential inclusions; Controllability; Fixed points

1. Introduction

In this paper, we study the controllability of second-order neutral functional differential inclusions in Banach spaces. More precisely, we consider the following form:

$$\frac{d}{dt}\big[y'(t) - g(t, y_t)\big] \in Ay(t) + Bu(t) + F(t, y_t), \quad t \in J = [0, T], \tag{1}$$

$$y_0 = \phi, \qquad y'(0) = x_0, \tag{2}$$

where $F : J \times C(J_0, E) \to 2^E$ (here $J_0 = [-r, 0]$) is a bounded, closed, convex multivalued map, $g : J \times C(J_0, E) \to E$ is given function, $\phi \in C(J_0, E)$, $x_0 \in E$, and A is the

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infinitesimal generator of a strongly continuous cosine family $\{C(t)\colon t \in R\}$ in a real Banach space X with the norm $|\cdot|$. Also, the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space. Finally, B is a bounded linear operator from U to E.

For any continuous function y defined on the interval $J_1 = [-r, T]$ and any $t \in J$, we denote by y_t the element of $C(J_0, E)$ defined by

$$y_t(\theta) = y(t+\theta), \quad \theta \in J_0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t.

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional space has been extensively studied. Several authors have extended the concept to infinite-dimensional systems in Banach spaces with bounded operators [1,13,15].

With the help of fixed point theorem several authors have investigated the problem of controllability of nonlinear systems in Banach spaces. Balachandran et al. [1] studied controllability for nonlinear integrodifferential systems in Banach spaces using the Schauder fixed point theorem. Benchohra and Ntouyas [3], using a fixed point theorem for condensing maps, proved controllability of second-order differential inclusions in Banach spaces with nonlocal conditions.

In many cases it is advantageous to treat the second-order abstract differential equations directly rather than to convert them into first-order systems. A useful tool for the study of abstract second-order equations is the theory of strongly continuous cosine families. Here we use of the basic ideas from cosine family theory [16,17].

Motivation for second-order neutral systems can be found in [3,8,12]. The purpose of this paper is to study the controllability of second-order neutral functional differential inclusions (1), (2) relying on a fixed point theorem for condensing maps due to Martelli [11].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C(J, E)$ is the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty := \sup\{|y(t)|\colon t \in J\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E. A measurable function $y\colon J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [18].)

$L^1(J, E)$ denotes the Banach space of continuous functions $y\colon J \to E$ which are Bochner integrable, normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|\,dt, \quad \text{for all } y \in L^1(J, E).$$

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \to 2^X$ is convex (closed) valued, if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(D) = \bigcup_{x \in D} G(x)$ is bounded in X, for any bounded set D of X, i.e.,

$$\sup_{x \in D}\big\{\sup\big\{\|y\|:\ y \in G(x)\big\}\big\} < \infty.$$

G is called upper semicontinuous on X, if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set V of X containing $G(x_0)$, there exists an open neighborhood A of x_0 such that $G(A) \subseteq V$.

G is said to be completely continuous if $G(D)$ is relatively compact for every bounded subset $D \subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is upper semicontinuous if and only if G has a closed graph, i.e.,

$$x_n \to x_*, \qquad y_n \to y_*, \qquad y_n \in Gx_n \quad \text{imply} \quad y_* \in Gx_*.$$

G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following, $BCC(X)$ denotes the set of all nonempty bounded closed and convex subsets of X.

A multivalued map $G : J \to BCC(X)$ is said to be measurable if for each $x \in X$, the distance between x and $G(t)$ is a measurable function on J. For more details on multivalued maps, see the books of Deimling [4] and Hu and Papageorgiou [9].

An upper semicontinuous map $G : X \to 2^X$ is said to be condensing if for any bounded subset $D \subseteq X$, with $\alpha(D) \neq 0$, we have

$$\alpha\big(G(D)\big) < \alpha(D),$$

where α denotes the Kuratowski measure of noncompactness. For properties of the Kuratowski measure, we refer to Banas and Goebel [2].

We remark that a completely continuous multivalued map is the easiest example of a condensing map.

We say that the family $\{C(t):\ t \in R\}$ of operators in $B(E)$ is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operator in E),
(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in R$,
(iii) the map $t \to C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t):\ t \in R\}$, associated to the given strongly continuous cosine family $\{C(t):\ t \in R\}$, is defined by

$$S(t)y = \int_0^t C(s)y\,ds, \qquad y \in E,\ t \in R.$$

The infinitesimal generator $A : E \to E$ of a cosine family $\{C(t):\ t \in R\}$ is defined by

$$Ay = \frac{d^2}{dt^2}C(t)y\bigg|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [7] and to the papers of Fattorini [5,6] and of Travis and Webb [16,17].

Let us list the following hypotheses.

(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators from X into itself.

(H2) $C(t)$, $t > 0$ is compact.

(H3) $F : J \times C(J_0, E) \to BCC(E)$; $(t, u) \to F(t, u)$ is measurable with respect to t for each $u \in C(J_0, E)$, upper semicontinuous with respect to u for each $t \in J$, and for each fixed $u \in C(J_0, E)$, the set

$$S_{F,u} = \{ f \in L^1(J, E) \colon\ f(t) \in F(t, u) \text{ for a.e. } t \in J \}$$

is nonempty.

(H4) The linear operator $W : L^2(J, U) \to E$, defined by

$$Wu = \int_0^T S(T - s)Bu(s)\, ds,$$

induces a bounded invertible operators $\widetilde{W}$ defined on $L^2(J, U)/\ker W$ and there exist positive constants M_1, M_2 such that $|\widetilde{W}| \leqslant M_1$ and $|B| \leqslant M_2$ (see [15]).

(H5) The function $g : J \times C(J_0, E) \to E$ is completely continuous and for any bounded set K in $C(J_1, E)$, the set $\{t \to g(t, y_t) \colon\ y \in K\}$ is equicontinuous in $C(J, E)$.

(H6) There exist constants c_1 and c_2 such that

$$\left|g(t, v)\right| \leqslant c_1\|v\| + c_2, \quad t \in J,\ v \in C(J_0, E).$$

(H7) $\|F(t, u)\| := \sup\{|v| \colon\ v \in F(t, u)\} \leqslant p(t)\Psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, R_+)$ and $\Psi : R_+ \to (0, \infty)$ is continuous and increasing with

$$\int_0^T m(s)\, ds < \int_c^\infty \frac{ds}{s + \Psi(s)},$$

where $c = M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2] + MT^2 M_0$, $m(t) = \max\{Mc_1, MTp(t)\}$, $M = \sup\{|C(t)| \colon\ t \in J\}$, $M_0 = M_1 M_2[\|x_1\| + M\|\phi\| + MT[|y_0| + c_1\|\phi\| + 2c_2] + Mc_1 \int_0^T \|y_s\|\, ds + MT \int_0^T p(s)\Psi(\|y_s\|)\, ds]$.

Remark 2.1. (i) If $\dim E < \infty$, then for each $u \in C(J_0, E)$, $S_{F,u} \neq \emptyset$ (see Lasota and Opial [10]).

(ii) $S_{F,u}$ is nonempty if and only if the function $Y : J \to R$ defined by

$$Y(t) := \inf\{|v| \colon\ v \in F(t, u)\}$$

belongs to $L^1(J, R)$ (see Papageorgiou [14]).

In order to define the concept of mild solution for (1), (2), by comparison with abstract Cauchy problem

$$y''(t) = Ay(t) + h(t),$$
$$y(0) = y_0, \qquad y'(0) = y_1$$

whose properties are well known [16,17], we associate problem (1), (2) to the integral equation

$$y(t) = C(t)\phi(0) + S(t)\big[x_0 - g(0,\phi)\big] + \int_0^t C(t-s)g(s,y_s)\,ds$$
$$+ \int_0^t S(t-s)Bu(s)\,ds + \int_0^t S(t-s)f(s)\,ds, \quad t \in J, \tag{3}$$

where

$$f \in S_{F,y} = \big\{ f \in L^1(J,E)\colon\ f(t) \in F(t,y_t) \text{ for a.e. } t \in J \big\}.$$

Definition 2.1. A function $y:(-r,T) \to E$, $T > 0$ is called a mild solution of the problem (1), (2) if $y(t) = \phi(t)$, $t \in [-r,0]$, and there exists a $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e. on J, and the integral equation (3) is satisfied.

Definition 2.2. The problem (1), (2) is said to be controllable on the interval J if, for every $\phi \in C(J_0,E)$ with $\phi(0) \in D(A)$, $x_0 \in E$, and $x_1 \in E$, there exists a control $u \in L^2(J,U)$ such that the mild solution $y(\cdot)$ of (1), (2) satisfies $y(T) = x_1$.

The following lemmas are crucial in the proof of our main theorem.

Lemma 2.1 [10]. *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying* (H3) *and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$. Then the operator*

$$\Gamma \circ S_F : C(I,X) \to BCC\big(C(I,X)\big), \qquad y \to (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2.2 [11]. *Let X be a Banach space and $N : X \to BCC(X)$ be a condensing map. If the set*

$$\Omega := \{y \in X\colon\ \lambda y \in Ny \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

3. Main result

Now, we are able to state and prove our main theorem.

Theorem 3.1. *Assume that hypotheses* (H1)–(H7) *are satisfied. Then the system* (1), (2) *is controllable on J.*

Proof. Using (H4), for an arbitrary function $y(\cdot)$, we define the control

$$u_y(t) = \widetilde{W}^{-1}\Bigg[x_1 - C(T)\phi(0) - S(T)\big[x_0 - g(0,\phi)\big] - \int_0^T C(T-s)g(s,y_s)\,ds - \int_0^T S(T-s)f(s)\,ds\Bigg](t),$$

where $f \in S_{F,y} = \{f \in L^1(J,E)\colon\ f(t) \in F(t,y_t) \text{ for a.e. } t \in J\}$.

Now we shall show that, when using this control, the operator $N : C(J_1,E) \to 2^{C(J_1,E)}$ defined by

$$Ny = \left\{ h \in C(J_1,E)\colon\ h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0, \\ C(t)\phi(0) + S(t)\big[x_0 - g(0,\phi)\big] + \int_0^t C(t-s)g(s,y_s)\,ds \\ \quad + \int_0^t S(t-s)Bu_y(s)\,ds + \int_0^t S(t-s)f(s)\,ds, & \text{if } t \in J, \end{cases} \right.$$

has a fixed point. This fixed point is then a solution of the problem (1), (2).

Clearly, $x_1 \in (Ny)(T)$.

We shall show that N is completely continuous with bounded closed convex values and it is upper semicontinuous. The proof will be given in several steps.

Step 1. Ny is convex for each $y \in C(J_1,E)$.

Indeed, if h_1, h_2 belong to Ny, then there exist $f_1, f_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$h_i(t) = C(t)\phi(0) + S(t)\big[x_0 - g(0,\phi)\big] + \int_0^t C(t-s)g(s,y_s)\,ds + \int_0^t S(t-s)Bu_y(s)\,ds + \int_0^t S(t-s)f_i(s)\,ds, \quad i = 1,2.$$

Let $0 \leqslant \alpha \leqslant 1$. Then for each $t \in J$, we have

$$\begin{aligned}\big(\alpha h_1 + (1-\alpha)h_2\big)(t) &= C(t)\phi(0) + S(t)\big[x_0 - g(0,\phi)\big] + \int_0^t C(t-s)g(s,y_s)\,ds \\ &\quad + \int_0^t S(t-s)Bu_y(s)\,ds + \int_0^t S(t-s)\big[\alpha f_1(s) + (1-\alpha)f_2(s)\big]\,ds.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$\alpha h_1 + (1-\alpha)h_2 \in Ny$$

completing the proof of Step 1.

Step 2. N maps bounded sets into bounded sets in $C(J_1, E)$.

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in Ny$, $y \in B_q = \{y \in C(J_1, E)\colon \|y\|_\infty \leqslant q\}$, one has $\|h\|_\infty \leqslant \ell$. If $h \in Ny$, then there exists $f \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = C(t)\phi(0) + S(t)\big[x_0 - g(0,\phi)\big] + \int_0^t C(t-s)g(s, y_s)\,ds + \int_0^t S(t-s)Bu_y(s)\,ds + \int_0^t S(t-s)f(s)\,ds.$$

By (H4), (H6), and (H7), we have that, for each $t \in J$,

$$\begin{aligned}\big|h(t)\big| &\leqslant \big|C(t)\phi(0)\big| + \big|S(t)\big[x_0 - g(0,\phi)\big]\big| + \left|\int_0^t C(t-s)g(s, y_s)\,ds\right| \\ &\quad + \left|\int_0^t S(t-s)Bu_y(s)\,ds\right| + \left|\int_0^t S(t-s)f(s)\,ds\right| \\ &\leqslant M\|\phi\| + MT\big[|x_0| + c_1\|\phi\| + 2c_2\big] + Mc_1\int_0^t \|y_s\|\,ds \\ &\quad + MT^2M_0 + MT\sup_{y\in[0,q]}\Psi(y)\left(\int_0^t p(s)\,ds\right).\end{aligned}$$

Then for each $h \in N(B_q)$ we have

$$\begin{aligned}\|h\|_\infty &\leqslant M\|\phi\| + MT\big[|x_0| + c_1\|\phi\| + 2c_2\big] + Mc_1\int_0^T \|y_s\|\,ds \\ &\quad + MT^2M_0 + MT\sup_{y\in[0,q]}\Psi(y)\left(\int_0^T p(s)\,ds\right) := \ell.\end{aligned}$$

Step 3. N maps bounded sets into equicontinuous sets of $C(J_1, E)$.

Let $t_1, t_2 \in J$, $0 < t_1 < t_2$ and $B_q = \{y \in C(J_1, E)\colon \|y\|_\infty \leqslant q\}$ be a bounded set of $C(J_1, E)$. For each $y \in B_q$ and $h \in Ny$, there exists $f \in S_{F,y}$ such that

$$h(t) = C(t)\phi(0) + S(t)\big[x_0 - g(0,\phi)\big] + \int_0^t C(t-s)g(s, y_s)\,ds$$

$$+\int_0^t S(t-s)Bu_y(s)\,ds+\int_0^t S(t-s)f(s)\,ds,\quad t\in J.$$

Thus,

$$\begin{aligned}
&\bigl|h(t_2)-h(t_1)\bigr|\\
&\quad\leqslant \bigl|\bigl[C(t_2)-C(t_1)\bigr]\phi(0)\bigr|+\bigl|\bigl[S(t_2)-S(t_1)\bigr]\bigl[x_0-g(0,\phi)\bigr]\bigr|\\
&\qquad+\left|\int_0^{t_2}\bigl[C(t_2-s)-C(t_1-s)\bigr]g(s,y_s)\,ds\right|\\
&\qquad+\left|\int_{t_1}^{t_2}C(t_1-s)g(s,y_s)\,ds\right|\\
&\qquad+\left|\int_0^{t_2}\bigl[S(t_2-s)-S(t_1-s)\bigr]B\widetilde{W}^{-1}\Bigl[x_1-C(T)\phi(0)\right.\\
&\qquad-S(T)\bigl[x_0-g(0,\phi)\bigr]-\int_0^T C(T-\tau)g(\tau,y_\tau)\,d\tau\\
&\qquad\left.-\int_0^T S(T-\tau)f(\tau)\,d\tau\Bigr](s)\,ds\right|\\
&\qquad+\left|\int_{t_1}^{t_2}S(t_1-s)B\widetilde{W}^{-1}\Bigl[x_1-C(T)\phi(0)-S(T)\bigl[x_0-g(0,\phi)\bigr]\right.\\
&\qquad\left.-\int_0^T C(T-\tau)g(\tau,y_\tau)\,d\tau-\int_0^T S(T-\tau)f(\tau)\,d\tau\Bigr](s)\,ds\right|\\
&\qquad+\left|\int_0^{t_2}\bigl[S(t_2-s)-S(t_1-s)\bigr]f(s)\,ds\right|+\left|\int_{t_1}^{t_2}S(t_1-s)f(s)\,ds\right|\\
&\quad\leqslant \bigl|C(t_2)-C(t_1)\bigr|\|\phi\|+\bigl|S(t_2)-S(t_1)\bigr|\bigl[|x_0|+c_1\|\phi\|+c_2\bigr]\\
&\qquad+\int_0^{t_2}\bigl|C(t_2-s)-C(t_1-s)\bigr|\bigl[c_1\|y_s\|+c_2\bigr]\,ds\\
&\qquad+\int_{t_1}^{t_2}\bigl|C(t_1-s)\bigr|\bigl[c_1\|y_s\|+c_2\bigr]\,ds
\end{aligned}$$

$$+\int_0^{t_2}\big|S(t_2-s)-S(t_1-s)\big|M_1M_2\Bigg[|x_1|+M\|\phi\| + MT\big[|x_0|+c_1\|\phi\|+c_2\big] + M\int_0^T\big[c_1\|y_\tau\|+c_2\big]\,d\tau + MT\int_0^T\big\|f(\tau)\big\|\,d\tau\Bigg]\,ds$$
$$+\int_{t_1}^{t_2}\big|S(t_1-s)\big|M_1M_2\Bigg[|x_1|+M\|\phi\|+MT\big[|x_0|+c_1\|\phi\|+c_2\big] + M\int_0^T\big[c_1\|y_\tau\|+c_2\big]\,d\tau + MT\int_0^T\big\|f(\tau)\big\|\,d\tau\Bigg]\,ds$$
$$+\int_0^{t_2}\big|S(t_2-s)-S(t_1-s)\big|\big\|f(s)\big\|\,ds + \int_{t_1}^{t_2}\big|S(t_1-s)\big|\big\|f(s)\big\|\,ds.$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero. The equicontinuities for the cases $t_1 < t_2 \leqslant 0$ and $t_1 \leqslant 0 \leqslant t_2$ are obvious. As a consequence of Step 2, Step 3, and (H5) together with the Ascoli–Arzela theorem, we can conclude that N is completely continuous, and therefore, a condensing map.

Step 4. N has a closed graph.

Let $y_n \to y_*$, $h_n \in Ny_n$, and $h_n \to h_*$. We shall prove that $h_* \in Ny_*$. $h_n \in Ny_n$ means that there exists $f_n \in S_{F,y_n}$ such that

$$h_n(t) = C(t)\phi(0) + S(t)\big[x_0 - g(0,\phi)\big] + \int_0^t C(t-s)g(s,y_{ns})\,ds + \int_0^t S(t-s)Bu_{y_n}(s)\,ds + \int_0^t S(t-s)f_n(s)\,ds, \quad t \in J,$$

where

$$u_{y_n}(t) = \widetilde{W}^{-1}\Bigg[x_1 - C(T)\phi(0) - S(T)\big[x_0 - g(0,\phi)\big] - \int_0^T C(T-s)g(s,y_{ns})\,ds - \int_0^T S(T-s)f_n(s)\,ds\Bigg](t).$$

We must prove that there exists $f_* \in S_{F,y_*}$ such that

$$h_*(t) = C(t)\phi(0) + S(t)\big[x_0 - g(0,\phi)\big] + \int_0^t C(t-s)g(s, y_{*s})\,ds + \int_0^t S(t-s)Bu_{y_*}(s)\,ds + \int_0^t S(t-s)f_*(s)\,ds, \quad t \in J,$$

where

$$u_{y_*}(t) = \widetilde{W}^{-1}\Bigg[x_1 - C(T)\phi(0) - S(T)\big[x_0 - g(0,\phi)\big] - \int_0^T C(T-s)g(s, y_{*s})\,ds - \int_0^T S(T-s)f_*(s)\,ds\Bigg](t).$$

Set

$$\bar{u}_{y_n}(t) = \widetilde{W}^{-1}\Bigg[x_1 - C(T)\phi(0) - S(T)\big[x_0 - g(0,\phi)\big] - \int_0^T C(T-s)g(s, y_{ns})\,ds\Bigg](t).$$

Since g, $\widetilde{W}^{-1}$ are continuous, then

$$\bar{u}_{y_n}(t) \to \bar{u}_{y_*}(t), \quad \text{for } t \in J.$$

Clearly, we have that

$$\Bigg\|\Bigg(h_n - C(t)\phi(0) - S(T)\big[x_0 - g(0,\phi)\big] - \int_0^T C(T-s)g(s, y_{ns})\,ds - \int_0^t S(t-s)B\bar{u}_{y_n}(s)\,ds\Bigg) - \Bigg(h_* - C(t)\phi(0) - S(T)\big[x_0 - g(0,\phi)\big] - \int_0^T C(T-s)g(s, y_{*s})\,ds - \int_0^t S(t-s)B\bar{u}_{y_*}(s)\,ds\Bigg)\Bigg\|_\infty \to 0,$$

as $n \to \infty$.

Consider the linear continuous operator

$$\Gamma : L^1(J,E) \to C(J,E),$$

$$f \to \Gamma(f)(t) = \int_0^t S(t-s)\Bigg[f(s) - B\widetilde{W}^{-1}\Bigg(\int_0^T S(T-\tau)f(\tau)\,d\tau\Bigg)(s)\Bigg]ds.$$

From Lemma 2.1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have that

$$h_n(t) - C(t)\phi(0) - S(t)\big[x_0 - g(0,\phi)\big] - \int_0^t C(t-s)g(s,y_{ns})\,ds - \int_0^t S(t-s)B\bar{u}_{y_n}(s)\,ds \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 2.1 that

$$\begin{aligned} &h_*(t) - C(t)\phi(0) - S(t)\big[x_0 - g(0,\phi)\big] \\ &\quad - \int_0^T C(t-s)g(s,y_{*s})\,ds - \int_0^t S(t-s)B\bar{u}_{y_*}(s)\,ds \\ &= \int_0^t S(t-s)\left[f_*(s) - B\widetilde{W}^{-1}\left(\int_0^T S(T-\tau)f_*(\tau)\,d\tau\right)(s)\right]ds \end{aligned}$$

for some $f_* \in S_{F,y_*}$.

Therefore N is a completely continuous multivalued map, upper semicontinuous with convex closed values. In order to prove that N has a fixed point, we need one more step.

Step 5. The set

$$\Omega := \big\{y \in C(J_1,E):\ \lambda y \in Ny,\ \text{for some } \lambda > 1\big\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus, there exists $f \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1}C(t)\phi(0) + \lambda^{-1}S(t)\big[x_0 - g(0,\phi)\big] + \lambda^{-1}\int_0^t C(t-s)g(s,y_s)\,ds \\ &\quad + \lambda^{-1}\int_0^t S(t-s)B\widetilde{W}^{-1}\Bigg[x_1 - C(T)\phi(0) - S(T)\big[x_0 - g(0,\phi)\big] \\ &\quad - \int_0^T C(T-\tau)g(\tau,y_\tau)\,d\tau - \int_0^T S(T-\tau)f(\tau)\,d\tau\Bigg](s)\,ds \\ &\quad + \lambda^{-1}\int_0^t S(t-s)f(s)\,ds, \quad t \in J. \end{aligned}$$

This implies by (H4), (H6), and (H7) that for each $t \in J$, we have

$$\bigl|y(t)\bigr| \leqslant M\|\phi\| + MT\bigl[|x_0| + c_1\|\phi\| + 2c_2\bigr] + Mc_1\int_0^t \|y_s\|\,ds + MT^2M_0 + MT\int_0^t p(s)\Psi\bigl(\|y_s\|\bigr)\,ds.$$

We consider the function μ defined by

$$\mu(t) = \sup\bigl\{\bigl|y(s)\bigr|\colon -r \leqslant s \leqslant t\bigr\}, \quad t \in J.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\begin{aligned}\mu(t) &\leqslant M\|\phi\| + MT\bigl[|x_0| + c_1\|\phi\| + 2c_2\bigr] \\ &\quad + Mc_1\int_0^{t^*} \|y_s\|\,ds + MT^2M_0 + MT\int_0^{t^*} p(s)\Psi\bigl(\|y_s\|\bigr)\,ds \\ &\leqslant M\|\phi\| + MT\bigl[|x_0| + c_1\|\phi\| + 2c_2\bigr] \\ &\quad + Mc_1\int_0^{t} \mu(s)\,ds + MT^2M_0 + MT\int_0^{t} p(s)\Psi\bigl(\mu(s)\bigr)\,ds.\end{aligned}$$

If $t^* \in J_0$, then $\mu(t) \leqslant \|\phi\|$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$c = v(0) = M\|\phi\| + MT\bigl[|x_0| + c_1\|\phi\| + 2c_2\bigr] + MT^2M_0,$$

$$\mu(t) \leqslant v(t),$$

$$v'(t) = Mc_1\mu(t) + MTp(t)\Psi\bigl(\mu(t)\bigr), \quad t \in J.$$

Using the nondecreasing character of Ψ, we get

$$v'(t) \leqslant Mc_1v(t) + MTp(t)\Psi\bigl(v(t)\bigr) \leqslant m(t)\bigl[v(t) + \Psi\bigl(v(t)\bigr)\bigr], \quad t \in J.$$

This implies that for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{ds}{s + \Psi(s)} \leqslant \int_0^T m(s)\,ds < \int_{v(0)}^{\infty} \frac{ds}{s + \Psi(s)}.$$

This inequality implies that there exists a constant L such that $v(t) \leqslant L$, $t \in J$, and hence $\mu(t) \leqslant L$, $t \in J$. Since for every $t \in J$, $\|y_t\| \leqslant \mu(t)$, we have

$$\|y\|_\infty := \sup\bigl\{\bigl|y(t)\bigr|\colon -r \leqslant t \leqslant T\bigr\} \leqslant L,$$

where L depends only on T and on the function p and Ψ. This shows that Ω is bounded.

Set $X := C(J_1, E)$. As a consequence of Lemma 2.2, we deduce that N has a fixed point and thus the system (1), (2) is controllable on J. □

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