Quasi-birth-and-death processes, level-geometric distributions. An aggregation/disaggregation approach

Ivo Marek

Katedra Matematiky, Stavební Fakulta Českého vysokého učení technického, Thákurova 7, Praha 1 166 29, Czech Republic

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Abstract

A special class of homogeneous continuous time quasi-birth and death (QBD) Markov chains (MCs) which possess level-geometric (LG) stationary distribution are considered. A functional analytic approach is applied which provides not only a clear analytic interpretation of the concepts introduced elsewhere but also represents a suitable basis for computations. An iterative aggregation/disaggregation method is proposed as tool for numerical computations. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The continuous-time Markov process on the countable state space \( S = \{(l, j): l \geq 0, 1 \leq j \leq m\} \) with block tridiagonal infinitesimal generator

\[
Q = \begin{pmatrix}
\tilde{A} & A_0 \\
A_2 & A_1 & A_0 \\
& A_2 & A_1 & A_0 \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots
\end{pmatrix}
\] (1.1)

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E-mail address: marek@ms.mff.cuni.cz (I. Marek).
having blocks of order \( m \) is called a *homogeneous continuous-time quasi-birth-and-death Markov chain*. The rows sums of \( Q \) are zero meaning

\[(B + A_0)\mathbf{e}(m) = 0\]  

(1.2)
and

\[(A_2 + A_1 + A_0)\mathbf{e}(m) = 0,\]  

(1.3)

where \( \mathbf{e}(m) = (1, \ldots, 1)^T \in \mathbb{R}^m \). We denote by \( \mathbb{R}^m \) the \( m \)-dimensional vector space over the field of real numbers, i.e. \( \mathbb{R}^m = \{ x = ((x)_1, \ldots, (x)_m) : (x)_j \in \mathbb{R} \} \), \( \mathbb{R} = \mathbb{R}^1 \) denotes the field of real numbers. The matrices \( A_0 \) and \( A_2 \) are elementwise nonnegative and the matrices \( \tilde{A} \) and \( A_1 \) have nonnegative off-diagonal elements and strictly negative diagonal elements. The first component \( l \) in the description of \( S \) denotes the level and the second component \( j \) the phase. In homogeneous QBD MCs the elements of the \( m \times m \) matrices \( \tilde{A}, A_0, A_1 \) and \( A_2 \) do not depend on the level number.

A basic assumption in the most of papers devoted to quasi-birth and death (QBD) processes is that the homogeneous continuous-time QBD MC at hand is irreducible and positive recurrent. The concept of a positive recurrent MC is treated as synonymum for existence of a steady-state probability distribution denoted by symbol \( \pi \). Being a stationary distribution \( \pi \) satisfies

\[\pi Q = 0\]

and

\[1 = \sum_{j=1}^{\infty} \pi_j < + \infty.\]

Let \( \pi \) be partitioned by levels into subvectors \( \pi_{\text{sub}(l)} = ((\pi)_{(l-1)m+1}, \ldots, (\pi)_{lm}), \) i.e. \( \pi_{\text{sub}(l)}^T \in \mathbb{R}^m \), \( l = 1, 2, \ldots \).

In terms of functional analysis the concept of a positive recurrent MC can be interpreted as an MC whose transition operator \( T \) produces the averages \( \{C_N(T)x\} \) defined by formula

\[C_N(T)x = \frac{1}{N} \sum_{k=1}^{N} T^k x, \quad x \in \mathcal{E}\]

with the property that \( \{C_N(T)x\} \) is convergent if and only if the sequence \( \{C_N(T)x\} \) is weakly sequentially compact and \( \{(1/N)T^k x\} \) tends to zero-vector [3, VIII.5, Theorem 1, p. 661]. In the above formula symbol \( \mathcal{E} \) denotes a suitable Banach space in which the transition map \( T \) operates.

The paper contains besides the introductory section and the section collecting the definitions and auxiliary results. Section 3 is devoted to a general theory of Markov chains in infinite-dimensional Banach spaces in Sections 3.1 and 3.2 we describe some topics typical for stochastic modeling. Most of the results in stochastic modeling known in the literature are derived via probabilistic methods of investigation. In this contribution, we want to show that an analytic approach may be a reliable alternative. In particular, we show how to make some considerations which seem complicated from the point of view of stochastic modeling clear just by utilizing some basic principles and concepts of functional analysis. In order to derive our results we apply some recent results in multi-level computations in Section 4. A quite natural appears the method of aggregation/disaggregation. The paper concludes with some remarks having computational aspects.
2. Definitions and notation

Let \( E \) be a Banach space over the field of real numbers. Let \( E' \) denote the dual space of \( E \). Let \( F, F' \) be the corresponding complex extensions of \( E, E' \), respectively, and let \( B(E) \) and \( B(F) \) be the spaces of all bounded linear operators mapping \( E \) into \( E \) and \( F \) into \( F \), respectively.

Let \( K \subset E \) be a closed normal and generating cone, i.e. let

(i) \( K + K \subset K \),
(ii) \( aK \subset K \) for \( a \in \mathbb{R}_+ \),
(iii) \( K \cap (-K) = \{0\} \),
(iv) \( K = \overline{K} \),
(v) \( E = K - K \)

where \( \overline{K} \) denotes the norm-closure of \( K \),

(vi) there exists a \( \delta > 0 \) such that \( \|x + y\| \geq \delta \|x\| \), whenever \( x, y \in K \).

Property (vi) is called normality of \( K \).

We let \( x \leq y \) or equivalently \( y \geq x \iff (y - x) \in K \).

(vii) For every pair \( x, y \in K \) there exist \( x \wedge y = \inf\{x, y\} \) and \( x \vee y = \sup\{x, y\} \) as elements of \( K \).

A cone \( K \) satisfying condition (vii) is called a lattice cone and the partial order on \( E \) a lattice order. In the terminology of Schaefer [10] \( E \) is called a Banach lattice.

Let \( K' = \{x' \in E' : x'(x) \geq 0 \text{ for all } x \in K\} \) and \( K^d = \{x \in K : x'(x) > 0 \text{ for all } 0 \neq x' \in K'\} \).

We call \( K' \) the dual cone of \( K \) and \( K^d \) the dual interior of \( K \), respectively. We let \( y > x \) or equivalently \( x < y \iff (y - x) \in K^d \).

It is well known [4] that assuming \( K \) to satisfy (i)–(vi) then the dual cone \( K' \) is normal and generates the dual space \( E' \).

In the following analysis, we assume that the dual interior \( K^d \) is nonempty.

A linear form \( x' \in K' \) is called strictly positive, if \( x'(x) > 0 \) for all \( x \in K \), \( x \neq 0 \).

We write \([x, x']\) in place of \( x'(x) \), where \( x \in E \) and \( x' \in E' \), respectively. If \( E \) happens to be a Hilbert space then \([x, x']\) denotes the appropriate inner product.

If \( T \in B(E) \) then \( T' \) denotes its dual and hence, \( T' \in B(E') \).

**Definition 2.1** (Dunford and Schwartz [3, p. 580]). Let \( F \) be a Banach space over the field of complex numbers. \( T \in B(F) \). The set of all complex \( \lambda \) such that in inverse does not belong to \( B(F) \) is
called spectrum of $T$. In more detail, the part of spectrum $\sigma(T)$ consisting of $\lambda$’s for which $\lambda I - T$ is not one-to-one, is called point spectrum and is denoted by $\sigma_p(T)$. Those $\lambda$’s for which $\lambda I - T$ is one-to-one and $(\lambda I - T)\mathcal{F}$ is dense but $(\lambda I - T)\mathcal{F} \neq \mathcal{F}$, form the continuous spectrum $\sigma_c(T)$. The set of all $\lambda$’s for which $\lambda I - T$ is one-to-one, but such that $(\lambda I - T)\mathcal{F}$ is not dense in $\mathcal{F}$ is called residual spectrum and is denoted by $\sigma_r(T)$.

Further, let $T \in \mathcal{B}(\mathcal{E})$. We introduce the operator $\tilde{T}$ by setting $\tilde{T}z = Tx + iTy$, where $z = x + iy$, $x, y \in \mathcal{E}$ and call it complex extension of $T$.

By definition, we let $\sigma(T) := \sigma(\tilde{T})$. Similarly, we let $r(T) := r(\tilde{T})$, where $r(\tilde{T}) = \max\{||\mu|| : \mu \in \sigma(\tilde{T})\}$ denotes the spectral radius of $\tilde{T}$.

It is well known that \cite[p. 249]{11} \[ r(T) = \lim_{k \to \infty} \|T^k\|^{1/k}. \]

As local spectral radius, we call the quantity defined as \[ \lim_{k \to \infty} \|T^kx\|^{1/k}, \] $x \in \mathcal{E}$.

In order to simplify notation, we will identify $T$ and its complex extension and will thus omit the tilde sign denoting the complex extension.

**Definition 2.2.** Let $\mathcal{E}$ be a Banach space generated by a closed normal cone $\mathcal{E}_+$. Let $T \in \mathcal{B}(\mathcal{E})$ be an operator such that $T\mathcal{E}_+ \subset \mathcal{E}_+$. Let $e' \in \mathcal{E}'_+$ be a strictly positive linear form in the dual cone.

We say that $T$ is a transition operator of Markov type, or, equivalently, a transition operator of a Markov chain (MC), if the following relations \[ [Tx, e'] = [x, e'] \] hold for all $x \in \mathcal{E}_+$.

**Definition 2.3.** A transition operator $T \in \mathcal{B}(\mathcal{E})$ of an MC is said to be irreducible if for every pair $x \in \mathcal{E}_+, x \neq 0, x' \in \mathcal{E}'_+, x' \neq 0$, there exists a positive integer $p = p(x, x')$ such that $[T^px, x'] > 0$.

**Remark 2.4.** The definition of irreducibility just introduced is known to be equivalent with most of definitions of irreducibility \cite{8}. For the purposes of this paper, Definition 2.3 is more suitable than the standard definition using the graph concepts applied in \cite{5,2,1}.

3. Existence results

3.1. Stationary probability distribution

In order to abbreviate our formulations, we introduce the following definition.
Definition 3.1. A Markov chain is said to satisfy the Markov chain average condition (MCAC) if the following requirements (i) and (ii) hold, where

(i) The collection of all $x \in E$ for which the set $E_0 = \{CN(T)x\}$ is weakly sequentially compact in $E$ is nontrivial, i.e. $E_0 \neq \{0\}$

and

(ii) The sequence $\{(1/k)T^kx\}$ tends to the zero-vector as $k$ tends to $\infty$.

Theorem 3.2. $T \in B(E)$ is the transition operator of an $MC(E, T, e)$ satisfying the MCAC. Then, there is a stationary distribution $\pi \in E_+$. 

Proof. According to [3, Theorem 1, VIII.5, p. 661], for every $x \in E_0 \cap E_+, x \neq 0$, there is a vector $vEM$ and a positive real number $vCR(x)$ such that

$$\lim_{N \to \infty} \|SN(T)x - \gamma(x)\pi\| = 0. \quad (3.1)$$

We see that $[CN(T)x, e] = [x, CN(T')e] = [x, e] > 0$

and this implies $\gamma(x)\pi \neq 0$. Obviously, since $(T')^k e = e$, $k = 1, 2, \ldots$, we deduce that $T\pi = \pi$ and also $\pi \in E_+^d$. Choosing $x$ satisfying $[x, e] = 1$ we come up with $\gamma(x) = 1$ and consequently, $[\pi, e] = 1$. $\square$

Theorem 3.3. Assume the space $E$ be generated by a cone $E_+$ having its dual cone with a nonempty interior $Int(\tilde{E}_+)$. Let $T \in B(E)$, the transition operator of an $MC$, be irreducible. Assume

$$\dim Ker(I - T) = \dim Ker(I - T'). \quad (3.2)$$

Then there exists at most one stationary distribution.

Proof. Let $T'y' = vy'$, $y' \in E_+$, $y' \neq 0$, and $T\pi = \pi$, $[\pi, e] = 1$, $\pi \in E_+^d$. We see that

$$0 < [\pi, y'] = [T\pi, y'] = [\pi, T'y'] = v[\pi, y']$$

implying $v = 1$.

Further, there is a $\kappa > 0$ such that $z' \in \tilde{E}_+$, where $z' = e - \kappa y'$ but $z' \notin Int(\tilde{E}_+)$. It follows that there exists an element $u \in E_+$ such that $0 = [u, z']$. However,

$$[u, z'] = [u, (T')^kz'] = [T^k u, z'], \quad k = 1, 2, \ldots$$

and irreducibility of $T$ implies $[u, z'] = [T^p u, z'] > 0$ for some $p = p(u, z')$. This contradiction shows that $z' = 0$ and thus, $\dim Ker(I - T') = 1$. Because of (3.2) the proof is complete. $\square$

Corollary 3.4. Under the hypotheses of Theorems 3.2 and 3.3 there exists a unique stationary distribution.
Remark 3.5. Let $T \in \mathcal{B}(\mathcal{E})$ and let $\|T\| = \|T'e\|$. Then the uniqueness properties of the stationary distributions of the MC at hand together with an obvious estimate
\[
\|(\lambda - 1)(\lambda I - T)^{-1}\| \leq 1, \quad \lambda > 1,
\]
 imply that all the limits obtained by process (3.1) belong to the range of the Perron projection $P$
\[
P x = [x, e] \pi,
\]
mapping the space $\mathcal{E}$ onto $P \mathcal{E}$ — the eigenspace of $T$ corresponding to eigenvalue 1.

3.2. LG-distribution

Definition 3.6. Assume $e \in \text{Int}(\mathcal{E}_+^r)$ and $S \in \mathcal{B}(\mathcal{E})$. Operator $S$ satisfying $S\mathcal{E}_+^r \subset \mathcal{E}_+^r$ and
\[
[Sx, e] \leq [x, e] \quad \forall x \in \mathcal{E}_+^r
\]
is called substochastic.

Lemma 3.7. Let $S \in \mathcal{B}(\mathcal{E})$ be substochastic. Then
\[
\sigma(S) \subset \{ \lambda : |\lambda| \leq 1 \}.
\]
If value $1 \notin \sigma_p(S) \cup \sigma_p(C')$, then $1 \in \sigma_c(S)$ and the inverse $(I - S)^{-1}$ exists on a dense set in $\mathcal{E}$.

Proof. It should be noted that according to the basic theorem of positive operators [9], the spectral radius is always in the spectrum of the operator investigated. Since the first part of Lemma 3.7 is trivial, let us assume value $1 \in \sigma(S)$, leaving the case $1 > r(S)$ aside as being trivial as well. Now, there are two possibilities. First, $1 \in \sigma_c(T)$ and, second, $1 \in \sigma_c(S)$. The first possibility is excluded by observing that the $1 \in \sigma_p(S')$ [3, p. 580], a contradiction. In the remaining case $1 \in \sigma_c(S)$ and the conclusions follow from the definition of continuous spectrum.

Lemma 3.8. Assume $S \in \mathcal{B}(\mathcal{E})$ is a substochastic operator such that $1 \notin \sigma_p(S) \cup \sigma_p(S')$. Then the series $\sum_{k=0}^{\infty} S^k x$ is convergent and
\[
\sum_{k=0}^{\infty} S^k x = (I - S)^{-1} x, \quad (3.3)
\]
w henever $x \in \mathcal{E}_+^r$, $x = y - Sy$, is such that its local spectral radius $r(S; x) < 1$.

Proof. Convergence of the series considered follows by definition of the local spectral radius. Hence, only validity of (3.5) is to be shown.

Convergence of $\sum_{k=0}^{\infty} S^k x$
implies that
\[ \lim_{k \to \infty} S^k x = 0. \]

It follows
\[ (I - S) \sum_{k=0}^{N} S^k x = \sum_{k=0}^{N} S^k (I - S)x = x - S^{N+1}x \to x \quad \text{as } N \to \infty \]
and thus (3.3). \qed

**Remark 3.9.** It is worth noting that assuming all the vectors in \( \mathcal{D} \) to have their local spectral radii smaller than 1, then
\[ (I - S)^{-1}(\mathcal{E}_+ \cap \mathcal{D}) \subset \mathcal{E}_+, \]
where \( \mathcal{D} = \{ y \in \mathcal{E} : y = z - Sz \text{ for some } z \in \mathcal{E} \} \) is the domain of the (generally unbounded) inverse \( (I - S)^{-1} \). In other words, the inverse \( (I - S)^{-1} \) leaves the cone \( \mathcal{E}_+ \) invariant.

**Theorem 3.10.** Let \( \mathcal{E} = l^1 \). Assume \( T \in \mathcal{B}(\mathcal{E}) \) is the transition operator of an MC satisfying the MCAC. Assume further that its infinitesimal generator \( Q \) is given in (1.1).

Then there exists an \( m \times m \) matrix \( R \) such that
\[ \pi_{\text{sub}(k+1)} = \pi_{\text{sub}(k)} R, \quad k \geq 0. \quad (3.4) \]

Moreover, matrix \( R \) is zero convergent, i.e. its spectral radius
\[ r(R) < 1. \quad (3.5) \]

**Proof.** After setting
\[ S_Q = \begin{pmatrix}
A_1 & A_0 & 0 & 0 & \ldots \\
A_2 & A_1 & A_0 & 0 & \ldots \\
0 & A_2 & A_1 & A_0 & \ldots \\
0 & 0 & A_2 & A_1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \]
and

\[
L = \begin{pmatrix}
0 & 0 & 0 & \ldots \\
\vdots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \\
0 & 0 & 0 & \ldots \\
A_0 & 0 & 0 & \ldots 
\end{pmatrix},
\]

we check easily that

\[
(\pi_{\text{sub}(k+1)}, \pi_{\text{sub}(k+2)}, \ldots)(I - S_Q) = (\pi_{\text{sub}(0)}, \pi_{\text{sub}(1)}, \ldots, \pi_{\text{sub}(k)})L.
\]

Now, for any admissible \(x\), we can write

\[
XL(I - SQ)^{-1} = (x_{\text{sub}(k)}A_0, 0, \ldots, 0 \ldots) = (x_{\text{sub}(k)}A_0 N_{11}, x_{\text{sub}(k)}A_0 N_{12}, \ldots),
\]

where the symbols \(N_{ik}\) denote the corresponding blocks in \((I - S_Q)^{-1}\). The desired \(R\) is obtained by putting

\[
x_{\text{sub}(k)}R = x_{\text{sub}(k)}A_0 N_{11}.
\]

Irrespective of value chosen for \(k\), the operator \(S_Q\) is the same independent of the level index \(k\) and hence, the formulas just obtained are valid with the same \(R\) for any level, i.e.

\[
\pi_{\text{sub}(k+1)} = \pi_{\text{sub}(k)}R, \quad k = 1, 2, \ldots.
\]

The validity of (3.5) is a direct consequence of the fact that

\[
\pi_{\text{sub}(k)} = \pi_{\text{sub}(0)}R^k
\]

and

\[
\lim_{k \to \infty} \pi_{\text{sub}(k)} = 0.
\]

The proof is thus complete. \(\square\)

4. Iterative aggregation/disaggregation. Computational aspects

In this section, we are going to show that the problem of finding the necessary and sufficient conditions guaranteeing the Markov chain whose transition operators is given in (1.1), where \(A, A_1, A_2\) are \(m \times m\) matrices, possess an LG stationary distribution can be reduced to the problem of finding the necessary and sufficient conditions of a Markov chain whose transition operator is given by a matrix whose elements are real numbers. A solution to the problem of finding the necessary conditions for the Markov chain of the form (1.1) is given in [2] by using the classical Gauss elimination method.

As we shall see the result of [2] when applied to a particular case of Markov chain with transition operator being an infinite matrix as in (1.1), where the matrices \(A, A_1, A_2\) are just real numbers offers
a key to solving the original problem by our technique—aggregation/disaggregation. We first reduce
the original transition operator with matrices $m \times m$, $\tilde{A}, A_1, A_2$ to an infinite transition matrix with
reals as elements, and then utilize the necessary and sufficient conditions obtained for this particular
case form [2]. In this way, we obtain the necessary and sufficient conditions for the original problem.
It is easy to see that any type of necessary and sufficient conditions for the particular Markov chains
just mentioned can be used to get analogous necessary and sufficient conditions for the general
Markov chains using our technique of reduction aggregation/disaggregation.

First, we formulate the problem.

Problem I.G. Let $E = l^1$ be the Banach space of (infinite) sequences of reals possessing absolutely
convergent series with the norm $\|x\| = \sum_{j=0}^{\infty} |(x)_j| < + \infty$. As well known, the corresponding
dual space is the $l^\infty$-space consisting of uniformly bounded sequences of reals with the norm $\|x\|_{l^\infty} = \sup \{|(x)_j|: j = 0, 1, \ldots\}$, where $(x)_k$ denotes the $k$th component of $x$.

Let us consider a Markov chain whose infinitesimal generator is given in (1.1) satisfying con-
ditions (1.2) and (1.3).

It is to find necessary and sufficient conditions guaranteeing an existence of a stationary distri-
bution $\pi \in l^1$ such that

$$\pi Q = 0, \quad [\pi Q, e(\infty)] = 1, \quad \sum_{j=1}^{\infty} \pi_j < + \infty,$$

and moreover,

$$\pi_{sub(k+1)} = \rho \pi_{sub(k)}, \quad k \geq L,$$

where $\rho \in \mathcal{R}_+, \quad \rho < 1$ and $L$ is a positive integer.

Our approach of analyzing the existence of stationary distributions with level geometric behavior
(LG-distribution) exploits some ideas well known in an approximate method of finding stationary
distributions via iterative aggregation/disaggregation (IAD) [11,6,7].

Let $E = \mathcal{R}^m \times \mathcal{R}^m \times \cdots \times \mathcal{R}^m \ldots$ be interpreted as $l^1$-space in which

$$x = (x_{sub(0)}, \ldots, x_{sub(k)}, \ldots), \quad x_{sub(k)} \in \mathcal{R}^m, \quad k = 0, 1, \ldots$$

Let $F$ be the standard $l^1$-space interpreted as the one introduced above with $m = 1$. We define
communication operators between $E$ and $F$ by setting

$$U = \begin{pmatrix} \hat{U} & \hat{U} & \cdots \\ \hat{U} & \hat{U} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$
where
\[ \hat{U}_{\text{sub}(k)} = \sum_{j=1}^{m} (x_{\text{sub}(k)})_j, \quad x_{\text{sub}(k)} \in \mathbb{R}^m \]
and
\[
V(x) = \begin{pmatrix}
\hat{V}(x_{\text{sub}(0)}) \\
\hat{V}(x_{\text{sub}(1)}) \\
\hat{V}(x_{\text{sub}(2)}) \\
\vdots
\end{pmatrix},
\]
where
\[ \hat{V}(x_{\text{sub}(k)}) z_k = \frac{x_{\text{sub}(k)}}{(\hat{U}_{\text{sub}(k)})} z_k, \quad z_k \in \mathbb{R}^1. \]

In accordance with the IAD methods in finite-dimensional spaces the operators \( U \) and \( V(x) \) are called the reduction and prolongation operator, respectively. In the same spirit we can construct a corresponding aggregated infinitesimal generator by setting
\[
UQV(x) = -I_{\not \exists} + W_{-1} + W_{+1},
\]
where
\[
W_{-1} = \text{diag}^{-1}\{\hat{U}(-A_1)^{-1}A_2 V(x_{\text{sub}(2)}), \hat{U}(-A_1)^{-1}A_2 \hat{V}(x_{\text{sub}(3)}), \ldots\}
\]
and
\[
W_{+1} = \text{diag}^{+1}\{\hat{U}(-\hat{A}_1)^{-1}A_0 \hat{V}(x_{\text{sub}(1)}), \hat{U}(-A_1)^{-1}A_0 \hat{V}(x_{\text{sub}(2)}), \ldots\}
\]
denote the first subdiagonal and the first superdiagonal of the tridiagonal matrix \( UQV(x) \), respectively. The elements of \( UQV(x) \) are just real numbers and hence
\[
UQV(x) = \begin{pmatrix}
-1 & 1 \\
c_{10}(x) & -1 & c_{12}(x) \\
c_{21}(x) & -1 & c_{24}(x) \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix},
\]
where
\[ c_{jj-1}(x) + c_{jj+1}(x) = 1, \quad j = 1, 2, \ldots. \]
Remark 4.1. It is easy to check that the operator $UQV(x)$ is the infinitesimal generator of a Markov chain and is called aggregated infinitesimal generator of $Q$ associated with $U$ and $V(x)$. Moreover, similarly as in the finite-dimensional case, the aggregated transition operator $UTV(x)$ is irreducible whenever is the transition operator $T = W_{-1} + W_{+1}$, where $x = ((x)_0, (x)_1, \ldots)$, $(x)_j > 0$, $j = 0, 1, \ldots$.

The following theorem contains the main result of this section.

**Theorem 4.2.** Let the following hypotheses (i)–(v) hold, where

(i) Operator $Q$, the infinitesimal generator of a Markov chain, satisfies conditions (1.2) and (1.3).

(ii) The Markov chain corresponding to the infinitesimal generator $Q$ satisfies the MCAC.

(iii) The transition operator $T = \text{diag}(-Q)^{-1}(Q + I)$ is irreducible.

(iv) $\dim \ker(I - T) = \dim \ker(I - T)'$.

(v) There exists a positive integer $s \geq 0$ such that the following relations:

$$\alpha z_s^* = z_{s+1}^*, \quad \alpha > 0$$

and

$$\beta \pi_s = \pi_{s+1}, \quad \beta > 0,$$

hold, where $z_k^*$ is the unique stationary distribution of the aggregated operator $UTV(\pi)$, $\pi$ being the unique stationary distribution of $T$.

Then there exists a stationary distribution of $T$ possessing an LG property with $L \leq s$.

**Proof.** We see that

$$\beta \pi_{\text{sub}(k)} = \pi_{\text{sub}(k+1)} = \pi_{\text{sub}(k)}^R$$

and these relations imply that $\beta = r(R) = \rho$. It follows that

$$U\pi_{\text{sub}(k+1)} = z_{k+1}^* = \alpha z_k^* = \alpha U\pi_{\text{sub}(k)}$$

implying $\alpha = \rho$ holds for $k \geq s$. The proof is complete. \hspace{1cm} \Box

The next result shows that the IAD procedures might be very efficient in some particular cases.

**Remark 4.3.** Let the matrix $A_0$ appearing as element in the formula defining the infinitesimal generator $Q$ of a Markov chain in (1.1) be a rank-one matrix, i.e.

$$A_0 = vu^T,$$

where $u, v \in \mathbb{R}^m$ are nonnegative non-zero vectors. Then obviously,

$$xR = xA_0N_{11} = [x, u]vN_{11}.$$
In particular,

$$\pi_{\text{sub}(k)} = [\pi_{\text{sub}(k)}, u]vN_{11} = \gamma_k vN_{11}$$

and the $\gamma$’s can be obtained as solutions of the aggregated Markov chain

$$z^*UQV(\pi) = 0, \quad [z^*, e(m)] = 1.$$  \hfill (4.3)

Because of uniqueness of the solutions of (4.3) we deduce that $\gamma_k = z_k^*$. Thus,

$$\pi = (z_0^* \pi_{\text{sub}(0)}, z_1^* \pi_{\text{sub}(1)}, \ldots).$$

5. Concluding remarks

To find whether a Markov chain at hand possess an LG-distribution some particular IAD methods have been shown to be useful. For some special cases the computational work can be very substantially reduced in comparison with some other methods. In general, IAD methods can be applied in order to obtain an exact stationary distribution vector as a limit vector of such a procedure. Section 4 shows that the IAD methods are competitive and we want to investigate the question of convergence as well as the speed of convergence of IAD methods applied to problems of type (1.1) in a subsequent paper.

References