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Direct Computation of the Simultaneous Stone–Weierstrass Approximation of a Function and Its Partial Derivatives in Banach Spaces, and Combination with Hermite Interpolation

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We present a new variant of the simultaneous Stone–Weierstrass approximation of a function and its partial derivatives, when the function takes its values in a Banach space, and provide an explicit and direct computation of this approximation. In the particular case of approximation by means of polynomials, we show that the simultaneous approximation can be required to be exact at a finite number of prescribed points. © 1994 Academic Press, Inc.

INTRODUCTION

The first extension of the Stone–Weierstrass approximation theorem to the simultaneous approximation of a function and its partial derivatives was established by L. Nachbin in 1949 [11]. Since then, a great deal of work has been carried out on improving and extending this result, notably to mappings from a Banach space into another Banach space. An outstanding synthesis of these works has been achieved in the exhaustive treatise of J. G. Llavona [10], which moreover furnishes an extensive bibliography and a historical account. An extension of some of these results to mappings from locally compact Hausdorff spaces into topological vector spaces has been established in [13, 14]. Recently, extensive studies have also focused on the estimation of the error for the Hermite interpolation of a function (see [1–3] for example). However, the information about the simultaneous approximation of a function and its derivatives has not been well disseminated yet.

In the first part of this paper, we present a new approach to the approximation of a function, which not only furnishes an explicit computation of the approximation, but also extends the approximation of the function to the simultaneous approximation of the function and its partial

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derivatives. In the second part of this paper, we present a new result on the combination of the simultaneous approximation and the Hermite interpolation, in the particular case of approximations by means of polynomials. For information about the Hermite interpolation, see [5] or [9] for example. For the first part of this paper, we apply the method of sequences of kernels, which is very well known in Harmonic Analysis (see [6] for example) to establish the Weierstrass theorem on approximation by means of polynomials or trigonometric polynomials. We extend this result to the approximation by means of a set of functions that is not supposed to be an algebra in general, but which satisfies some conditions that are easy to check, giving a new variant of the Stone–Weierstrass theorem.

Our interest in the simultaneous approximation of a function and its partial derivatives originated from the study of connectedness properties by means of smooth curves in topological vector spaces. As a consequence of the results of this paper a synthesis of the results of the first step of [4] and the results of [12] has been achieved in [4].

Throughout this paper, we will use the following notation and terminology. We let $\mathbb{N} = \{0, 1, \dots\}$. Then

$$\mathbb{N}^n = \{(m_1, \dots, m_n) | m_1, \dots, m_n \in \mathbb{N}\}$$

is the set of all multi-indices. We define

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \alpha! = \alpha_1! \cdots \alpha_n!,$$

$$\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n), \qquad \alpha \leq \beta \Leftrightarrow \alpha_1 \leq \beta_1, \dots, \alpha_n \leq \beta_n,$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n,$$

$$\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

If $\mathbb{E}$ is an $\mathbb{R}$-Banach space and $S_1 \subseteq \mathbb{R}^n$, $S_2 \subseteq \mathbb{E}$, then we denote by $C^m(S_1, S_2)$ the set of all mappings f from an open neighborhood $\mathscr{D}(f) \subseteq \mathbb{R}^n$ of S_1 into S_2 that are m times continuously differentiable, or just continuous when $m = 0$. Besides, if $f \in C^m(S_1, S_2)$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is such that $|\alpha| \leq m$, then we denote by $f^{(\alpha)}$ the mixed derivative

$$f^{(\alpha)}(x) = \frac{\partial^{|\alpha|} f(x)}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \in \mathbb{E} \qquad \forall x = (x_1, \dots, x_n) \in \mathscr{D}(f),$$

where the partial derivatives are defined in the usual elementary way. The use of the Fréchet derivative would be an unnecessary obfuscation in this case.

If $f_1, \ldots, f_n \in C^0(\mathbb{R}, \mathbb{R})$, then $f_1 \otimes \cdots \otimes f_n$ denotes the mapping $f \in C^0(\mathbb{R}^n, \mathbb{R})$ defined by

$$f(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n) \qquad \forall x_1, \ldots, x_n \in \mathbb{R}.$$

If $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ are subsets of $C^0(\mathbb{R}, \mathbb{R})$, then $\mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_n$ denotes the subset of $C^0(\mathbb{R}^n, \mathbb{R})$ of all mappings f of the form

$$f = \sum_{i=1}^{r} c_i f_{i1} \otimes \cdots \otimes f_{in},$$

where $c_i \in \mathbb{R}$, $f_{i1} \in \mathfrak{A}_1, \ldots, f_{in} \in \mathfrak{A}_n$ for every $i \in \{1, \ldots, r\}$. If $\mathbb{E}$ is a topological $\mathbb{R}$-vector space, and $\mathfrak{A} \subseteq C^0(\mathbb{R}^n, \mathbb{R})$, then $\mathfrak{A} \otimes \mathbb{E}$ denotes the set of all mappings $f \in C^0(\mathbb{R}^n, \mathbb{E})$ of the form

$$f(x) = \sum_{i=1}^{r} a_i(x) v_i \qquad \forall x \in \mathbb{R}^n,$$

where $r \in \{1, 2, \ldots, \}$, $a_1, \ldots, a_r \in \mathfrak{A}$, and $v_1, \ldots, v_r \in \mathbb{E}$. Clearly, when $\mathfrak{A}$ is an $\mathbb{R}$-vector subspace of $C^0(\mathbb{R}^n, \mathbb{R})$ and $\mathbb{E} = \mathbb{R}$, then $\mathfrak{A} \otimes \mathbb{R} = \mathfrak{A}$. In the particular case where $\mathfrak{A} = \mathbb{R}[x_1, \ldots, x_n]$ is the set of all real polynomials in n variables, it follows directly from the definition that $\mathfrak{A} \otimes \mathbb{E}$ is the set of all mappings p from $\mathbb{R}^n$ into $\mathbb{E}$ of the form

$$\begin{aligned} p(x_1, \ldots, x_n) &= p_1(x_1, \ldots, x_n) v_1 + \cdots + p_r(x_1, \ldots, x_n) v_r \\ &= \sum_{(i_1, \ldots, i_n) \in \mathcal{I}} x_1^{i_1} \cdots x_n^{i_n} w_{i_1 \cdots i_n} = \sum_{i \in \mathcal{I}} x^i w_i \\ &\qquad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \end{aligned}$$

where $p_1, \ldots, p_r \in \mathfrak{A}$, $v_1, \ldots, v_r \in \mathbb{E}$, $\mathcal{I}$ is a finite subset of $\mathbb{N}^n$, and $w_i = w_{i_1 \cdots i_n} \in \mathbb{E}$ for every $i = (i_1, \ldots, i_n) \in \mathcal{I}$. We will say that every $p \in \mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{E}$ is *an $\mathbb{E}$-polynomial in n variables*. Since $\mathbb{R}[x_1, \ldots, x_n]$ is an $\mathbb{R}$-vector subspace of $C^0(\mathbb{R}^n, \mathbb{R})$, we have

$$\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R} = \mathbb{R}[x_1, \ldots, x_n].$$

Finally, we let

$$C_r = [-r, r]^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | -r \leq x_k \leq r \ \forall k \in \{1, \ldots, n\}\}, \qquad \forall r > 0.$$

A sequence $K_0, K_1, \ldots \in C^0(\mathbb{R}^n, \mathbb{R})$ will be called *a Dirac sequence of* *kernels* normalized with respect to a measurable neighborhood N_0 of 0 in $\mathbb{R}^n$ if and only if

(D1) $K_i(x) \geq 0 \ \forall i \in \mathbb{N}, x \in N_0$.

(D2) $\int_{N_0} K_i(x)\,dx = 1 \ \forall i \in \mathbb{N}$.

(D3) For every $\varepsilon > 0$ and $\delta > 0$ such that $C_\delta \subseteq N_0$, there exists $i_{\varepsilon\delta} \in \mathbb{N}$ such that

$$\int_{C_\delta} K_i(x)\,dx \geq 1 - \varepsilon \qquad \forall i \geq i_{\varepsilon,\delta}.$$

Results

THEOREM 1. (Simultaneous Stone–Weierstrass Approximation of a Function and Its Partial Derivatives). *Let $\mathfrak{A} \subseteq C^m(\mathbb{R}^n, \mathbb{R})$ and $r > 0$. Suppose there exists a Dirac sequence of kernels $K_0, K_1, \ldots \in C^0(\mathbb{R}^n, \mathbb{R})$ normalized with respect to C_{2r} such that for every $i \in \mathbb{N}$, there exist $\alpha_{i1}, \ldots, \alpha_{ip_i} \in \mathfrak{A}$ and $\varphi_{i1}, \ldots, \varphi_{ip_i} \in C^0(\mathbb{R}^n, \mathbb{R})$ such that*

$$K_i(x + y) = \sum_{k=1}^{p_i} \alpha_{ik}(x)\varphi_{ik}(y) \qquad \forall x, y \in \mathbb{R}^n. \tag{1}$$

Then for every $\mathbb{R}$-Banach space $\mathbb{E}$, for every $f \in C^m(\mathbb{R}^n, \mathbb{E})$ with support in C_r, and for every $\varepsilon > 0$, there exists $a \in \mathfrak{A} \otimes \mathbb{E}$ such that

$$\left\| f^{(\alpha)}(x) - a^{(\alpha)}(x) \right\| < \varepsilon \qquad \forall x \in C_r, \alpha \in \mathbb{N}^n, |\alpha| \leq m.$$

Proof. Let $\mathbb{E}$, f, and ε be as above. Let

$$a_i(x) = (-1)^n \int_{\mathbb{R}^n} f(t) K_i(x - t)\,dt = (-1)^n \int_{C_r} f(t) K_i(x - t)\,dt \tag{2}$$

$$= \int_{x - C_r} f(x - s) K_i(s)\,ds \qquad \forall x \in \mathbb{R}^n, i \in \mathbb{N}. \tag{3}$$

Then by (1) and (2), we have for every $i \in \mathbb{N}$,

$$a_i(x) = (-1)^n \int_{C_r} f(t) \sum_{k=1}^{p_i} \alpha_{ik}(x)\varphi_{ik}(-t)\,dt \qquad \forall x \in \mathbb{R}^n,$$

hence

$$a_i = \sum_{k=1}^{p_i} \alpha_{ik} v_{ik} \in \mathfrak{A} \otimes \mathbb{E},$$

where for every $k \in \{1, \ldots, p_i\}$,

$$v_{ik} = (-1)^n \int_{C_r} f(t)\varphi_{ik}(-t)\, dt \in \mathbb{E}.$$

Furthermore, since the support of f is contained in C_r, we have

$$f^{(\alpha)}(x) = 0 \qquad \forall x \in \overline{\mathbb{R}^n \setminus C_r},\ \alpha \in \mathbb{N}^n,\ |\alpha| \le m. \tag{4}$$

Hence by (3) and (4),

$$a_i^{(\alpha)}(x) = \int_{x - C_r} f^{(\alpha)}(x - t) K_i(t)\, dt = \int_{\mathbb{R}^n} f^{(\alpha)}(x - t) K_i(t)\, dt \qquad \forall x \in \mathbb{R}^n,\ \alpha \in \mathbb{N}^n,\ |\alpha| \le m. \tag{5}$$

On the other hand, by (D2),

$$f^{(\alpha)}(x) = \int_{C_{2r}} f^{(\alpha)}(x) K_i(t)\, dt \qquad \forall i \in \mathbb{N},\ x \in \mathbb{R}^n,\ \alpha \in \mathbb{N}^n,\ |\alpha| \le m. \tag{6}$$

Since C_r is compact, there exists

$$M = \sup_{|\alpha| \le m} \sup_{x \in C_r} \| f^{(\alpha)}(x) \| = \sup_{|\alpha| \le m} \sup_{x \in \mathbb{R}^n} \| f^{(\alpha)}(x) \| < \infty. \tag{7}$$

Moreover, the set of functions $\{f^{(\alpha)} | \alpha \in \mathbb{N}^n,\ |\alpha| \le m\}$ is uniformly equicontinuous on C_r, and hence on $\mathbb{R}^n$, because C_r contains the support of $f^{(\alpha)}$ for every $\alpha \in \mathbb{N}^n$, $|\alpha| \le m$. Consequently, there exists $\delta > 0$ such that $\delta < 2r$ and

$$x - y \in C_\delta \Rightarrow \| f^{(\alpha)}(x) - f^{(\alpha)}(y) \| < \frac{\varepsilon}{2} \qquad \forall x, y \in \mathbb{R}^n,\ \alpha \in \mathbb{N}^n,\ |\alpha| \le m. \tag{8}$$

Besides, by (D3), there exists $i_0 \in \mathbb{N}$ such that

$$\int_{C_\delta} K_i(x)\, dx \ge 1 - \frac{\varepsilon}{4M + 1} \qquad \forall i \ge i_0. \tag{9}$$

Let $i \ge i_0$, $\alpha \in \mathbb{N}^n$ with $|\alpha| \le m$, and $x \in C_r$. Then $x - C_r \subseteq C_{2r}$, and by

(5), (6), (7), (8), (9), (D1), and (D2), we have

$$\begin{aligned}\|f^{(\alpha)}(x) - a_i^{(\alpha)}(x)\| &\le \int_{C_{2r}} \|f^{(\alpha)}(x) - f^{(\alpha)}(x-t)\| K_i(t)\,dt \\ &\le \int_{C_{2r}\setminus C_\delta} (\|f^{(\alpha)}(x)\| + \|f^{(\alpha)}(x-t)\|) K_i(t)\,dt \\ &\quad + \int_{C_\delta} \|f^{(\alpha)}(x) - f^{(\alpha)}(x-t)\| K_i(t)\,dt \\ &\le 2M \int_{C_{2r}\setminus C_\delta} K_i(t)\,dt + \frac{\varepsilon}{2}\int_{C_\delta} K_i(t)\,dt \\ &< 2M\frac{\varepsilon}{4M+1} + \frac{\varepsilon}{2}\int_{C_{2r}} K_i(t)\,dt < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare\end{aligned}$$

COROLLARY 2. (Use of Functions Generating a Dirac Sequence). *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be subalgebras of the $\mathbb{R}$-algebra $C^m(\mathbb{R}, \mathbb{R})$. Let $\mathfrak{A} = \mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_n$. Let $r > 0$. Let $g_1, \ldots, g_n \in C^0(\mathbb{R}, \mathbb{R})$ be even on $\mathbb{R}$, and strictly decreasing and nonnegative on $[0, 2r]$. Suppose that for every $k \in \{1, \ldots, n\}$, there exist $a_{k1}, \ldots, a_{ks_k} \in \mathfrak{A}_k$ and $\varphi_{k1}, \ldots, \varphi_{ks_k} \in C^0(\mathbb{R}, \mathbb{R})$ such that*

$$g_k(x+y) = \sum_{i=1}^{s_k} a_{ki}(x)\varphi_{ki}(y) \qquad \forall x, y \in \mathbb{R}. \tag{10}$$

Then for every $\mathbb{R}$-Banach space $\mathbb{E}$, for every $f \in C^m(\mathbb{R}^n, \mathbb{E})$ with support in C_r, and for every $\varepsilon > 0$, there exists $a \in \mathfrak{A} \otimes \mathbb{E}$ such that

$$\|f^{(\alpha)}(x) - a^{(\alpha)}(x)\| < \varepsilon \qquad \forall x \in C_r, \alpha \in \mathbb{N}^n, |\alpha| \le m.$$

Proof. For every $k \in \{1, \ldots, n\}$, the hypothesis implies that $g_k(0) > 0$, and hence, since g_k is continuous, $\int_{-2r}^{2r} g_k(t)\,dt > 0$. Consequently, we can define

$$K_i(x_1, \ldots, x_n) = \frac{g_1^i(x_1) \cdots g_n^i(x_n)}{\left(\int_{-2r}^{2r} g_1^i(u_1)\,du_1\right) \cdots \left(\int_{-2r}^{2r} g_n^i(u_n)\,du_n\right)} \qquad \forall x_1, \ldots, x_n \in \mathbb{R}, i \in \mathbb{N}. \tag{11}$$

It follows directly from the definition that the sequence $(K_i)_{i\in\mathbb{N}}$ satisfies (D1) and (D2) with respect to C_{2r}. Let us check that it satisfies (D3) too. Let $\varepsilon > 0$ and $\delta > 0$ be such that $\delta \le 2r$. Let $k \in \{1, \ldots, n\}$. Since g_k is strictly decreasing and nonnegative on $[0, 2r]$, and since $0 < \delta \le 2r$, we have

$$g_k\left(\frac{\delta}{2}\right) > g_k(\delta) \geq 0, \qquad \lim_{i\to\infty}\left[\frac{g_k(\delta)}{g_k(\delta/2)}\right]^i = 0.$$

Consequently, there exists $i_k \in \mathbb{N}$ such that

$$0 \leq \left[\frac{g_k(\delta)}{g_k(\delta/2)}\right]^i < \frac{\delta}{4r}\varepsilon^{1/n} \qquad \forall i \in \mathbb{N}, i \geq i_k. \tag{12}$$

Let $i_0 = \max\{i_1, \ldots, i_n\}$. Let $i \in \mathbb{N}$ be such that $i \geq i_0$. Then by (11), (12), and since $g_1, \ldots, g_n$ are even on $\mathbb{R}$, and strictly decreasing and nonnegative on $[0, 2r]$, we have

$$\begin{aligned}\int_{C_{2r}\setminus C_\delta} K_i(x)\,dx &= \prod_{k=1}^n \frac{\int_\delta^{2r} g_k^i(t)\,dt}{\int_0^{2r} g_k^i(u)\,du} \leq \prod_{k=1}^n \frac{g_k^i(\delta)(2r-\delta)}{\int_0^{\delta/2} g_k^i(u)\,du}\\ &\leq \prod_{k=1}^n \frac{g_k^i(\delta)2r}{g_k^i(\delta/2)(\delta/2)} = \prod_{k=1}^n \frac{4r}{\delta}\left[\frac{g_k(\delta)}{g_k(\delta/2)}\right]^i < \varepsilon.\end{aligned}$$

Hence, by (D2), $\int_{C_\delta} K_i(x)\,dx \geq 1 - \varepsilon$. Thus the sequence $(K_i)_{i\in\mathbb{N}}$ is a Dirac sequence of kernels normalized with respect to C_{2r}. Let $i \in \mathbb{N}$, and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Let

$$c_0 = \frac{1}{\left(\int_{-2r}^{2r} g_1^i(u_1)\,du_1\right) \cdots \left(\int_{-2r}^{2r} g_n^i(u_n)\,du_n\right)}.$$

Then by (10) and (11), we have

$$\begin{aligned}K_i(x+y) &= c_0 g_1^i(x_1+y_1) \cdots g_n^i(x_n+y_n)\\ &= c_0 \prod_{k=1}^n \left[\sum_{j=1}^{s_k} a_{kj}(x_k)\varphi_{kj}(y_k)\right]^i\\ &= c_0 \prod_{k=1}^n \sum_{\beta_1=1}^{s_k} \cdots \sum_{\beta_i=1}^{s_k} a_{k\beta_1}(x_k) \cdots a_{k\beta_i}(x_k)\varphi_{k\beta_1}(y_k)\\ &\quad \cdots \varphi_{k\beta_i}(y_k)\\ &= c_0 \prod_{k=1}^n \sum_{\beta\in\{1,\ldots,s_k\}^i} a_{k\beta}(x_k)\varphi_{k\beta}(y_k),\end{aligned}$$

where for every $\beta = (\beta_1, \dots, \beta_i) \in \{1, \dots, s_k\}^i$,

$$\varphi_{k\beta} = \varphi_{k\beta_1} \cdots \varphi_{k\beta_i} \in C^0(\mathbb{R}, \mathbb{R}),$$

and

$$a_{k\beta} = a_{k\beta_1} \cdots a_{k\beta_i} \in \mathfrak{A}_k,$$

because $\mathfrak{A}_k$ is an algebra. It follows that

$$\begin{aligned} K_i(x+y) &= c_0 \left[\sum_{\Lambda_1 \in \{1, \dots, s_1\}^i} a_{1\Lambda_1}(x_1) \varphi_{1\Lambda_1}(y_1) \right] \\ &\quad \cdots \left[\sum_{\Lambda_n \in \{1, \dots, s_n\}^i} a_{n\Lambda_n}(x_n) \varphi_{n\Lambda_n}(y_n) \right] \\ &= c_0 \sum_{(\Lambda_1, \dots, \Lambda_n) \in \{1, \dots, s_1\}^i \times \cdots \times \{1, \dots, s_n\}^i} a_{1\Lambda_1}(x_1) \\ &\quad \cdots a_{n\Lambda_n}(x_n) \varphi_{1\Lambda_1}(y_1) \cdots \varphi_{n\Lambda_n}(y_n) \\ &= \sum_{\Lambda \in \{1, \dots, s_1\}^i \times \cdots \times \{1, \dots, s_n\}^i} a_\Lambda(x) \varphi_\Lambda(y), \end{aligned}$$

where

$$a_\Lambda = a_{1\Lambda_1} \otimes \cdots \otimes a_{n\Lambda_n} \in \mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_n = \mathfrak{A},$$

and

$$\varphi_\Lambda = c_0 \varphi_{1\Lambda_1} \otimes \cdots \otimes \varphi_{n\Lambda_n} \in C^0(\mathbb{R}^n, \mathbb{R})$$

for every $\Lambda = (\Lambda_1, \dots, \Lambda_n) \in \{1, \dots, s_1\}^i \times \cdots \times \{1, \dots, s_n\}^i$. Thus $(K_i)_{i \in \mathbb{N}}$ and $\mathfrak{A}$ satisfy the hypotheses of Theorem 1, and the conclusion follows. ∎

COROLLARY 3. (Simultaneous Polynomial Approximation of a Function and Its Partial Derivatives). *For every $\mathbb{R}$-Banach space $\mathbb{E}$, for every $f \in C^m(\mathbb{R}^n, \mathbb{E})$ with compact support $K \subseteq \mathbb{R}^n$, and for every $\varepsilon > 0$, there exists an $\mathbb{E}$-polynomial p: $\mathbb{R}^n \to \mathbb{E}$ such that*

$$\|f^{(\alpha)}(x) - p^{(\alpha)}(x)\| < \varepsilon \qquad \forall x \in K, \alpha \in \mathbb{N}^n, |\alpha| \le m.$$

Proof. Let $\mathfrak{A} = \mathbb{R}[x_1, \dots, x_n]$ and $\mathfrak{A}_1 = \mathbb{R}[x]$. It follows directly from the definitions that $\mathfrak{A} = \mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_1$. Let $r > 0$ be such that C_r contains K. Let

$$g_1(x) = (2r)^2 - x^2 \qquad \forall x \in \mathbb{R}.$$

Then g_1 is even on $\mathbb{R}$, and strictly decreasing and nonnegative on $[0, 2r]$. Furthermore,

$$g_1(x+y) = (2r)^2 - x^2 - 2xy - y^2 \qquad \forall x, y \in \mathbb{R}.$$

Thus $(\mathfrak{A}_1, \dots, \mathfrak{A}_1, g_1, \dots, g_1)$ satisfies the hypotheses of Corollary 2, and the conclusion follows. ∎

Remark. In Corollary 3, the Dirac sequence generated by $(g_1, \dots, g_1)$ through Formula (11) is the well-known Landau sequence of kernels.

COROLLARY 4. (Simultaneous Approximation of a Function and Its Partial Derivatives by Means of Hyperbolic Functions). *Let* $\mathfrak{A}_1$ *be the subalgebra of the* $\mathbb{R}$*-algebra* $C^m(\mathbb{R}, \mathbb{R})$ *generated by the hyperbolic functions*

$$\mathfrak{A}_1 = \left\{ x \mapsto \sum_{k=0}^{s} a_k \cosh kx + b_k \sinh kx \,|\, s \in \mathbb{N} \right.$$
$$\left. \text{and } a_0, \dots, a_s, b_0, \dots, b_s \in \mathbb{R} \right\}.$$

Let $\mathfrak{A} = \mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_1 \subseteq C^m(\mathbb{R}^n, \mathbb{R})$. *Then for every* $\mathbb{R}$*-Banach space* $\mathbb{E}$, *for every* $f \in C^m(\mathbb{R}^n, \mathbb{E})$ *with compact support* $K \subseteq \mathbb{R}^n$, *and for every* $\varepsilon > 0$, *there exists* $q \in \mathfrak{A} \otimes \mathbb{E}$ *such that*

$$\|f^{(\alpha)}(x) - q^{(\alpha)}(x)\| < \varepsilon \qquad \forall x \in K, \alpha \in \mathbb{N}^n, |\alpha| \le m.$$

Proof. Let $r > 0$ be such that C_r contains K. Let $c_0 = \cosh 2r$, and

$$g_1(x) = c_0 - \cosh x \qquad \forall x \in \mathbb{R}.$$

Then g_1 is even on $\mathbb{R}$, and strictly decreasing and nonnegative on $[0, 2r]$. Moreover,

$$g_1(x+y) = c_0 - \cosh(x+y) = c_0 - \cosh x \cosh y - \sinh x \sinh y \qquad \forall x, y \in \mathbb{R}.$$

Thus $(\mathfrak{A}_1, \dots, \mathfrak{A}_1, g_1, \dots, g_1)$ satisfies the hypotheses of Corollary 2, and the conclusion follows. ∎

The following theorem furnishes a generalization of the Hermite interpolation, in vector spaces, by means of polynomials in several real variables. Following the important trend to study some parts of analysis by means of pure algebra (see the extensive work of E. R. Kolchin [7, 8] for example), we present this generalization in a pure algebraic setting. For this, we define the derivative of a general polynomial in the following algebraic way: Let $\mathbb{E}$ be an (algebraic) $\mathbb{R}$-vector space. Let $p\colon \mathbb{R}^n \to \mathbb{E}$ be an $\mathbb{E}$-polynomial, that is,

$$p(x) = \sum_{i \in \mathscr{I}} c_i x^i \qquad \forall x \in \mathbb{R}^n,$$

where $\mathscr{I}$ is a finite subset of $\mathbb{N}^n$, and $c_i \in \mathbb{E}$ for every $i \in \mathscr{I}$. Then for every $\alpha \in \mathbb{N}^n$, we define the derivative $p^{(\alpha)}$ of p by

$$p^{(\alpha)}(x) = \sum_{i \in \mathscr{I} : i \geq \alpha} c_i \frac{i!}{(i-\alpha)!} x^{i-\alpha} \qquad \forall x \in \mathbb{R}^n,$$

with the usual convention $\sum_{i \in \varnothing} y_i = 0$.

THEOREM 5. (Hermite Interpolation in Vector Spaces by Means of Polynomials in Several Variables). *Let $\mathbb{E}$ be an (algebraic) $\mathbb{R}$-vector space of any dimension. Let $a_1, \ldots, a_N \in \mathbb{R}^n$ be distinct. For every $k \in \{1, \ldots, N\}$ and $\alpha \in \{0, \ldots, m\}^n$, let $v_{k\alpha} \in \mathbb{E}$. Then there exists an $\mathbb{E}$-polynomial p: $\mathbb{R}^n \to \mathbb{E}$ such that*

$$p^{(\alpha)}(a_k) = v_{k\alpha} \qquad \forall k \in \{1, \ldots, N\}, \alpha \in \{0, \ldots, m\}^n.$$

Proof. Let $k \in \{1, \ldots, N\}$. Let $(a_{k1}, \ldots, a_{kn}) = a_k$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, \ldots, m\}^n$. Let $i \in \{1, \ldots, n\}$. By Hermite interpolation, there exists a polynomial $p_{k\alpha i}\colon \mathbb{R} \to \mathbb{R}$ such that

$$p_{k\alpha i}^{(\lambda)}(a_{ji}) = \delta_{a_{ji} a_{ki}} \delta_{\lambda \alpha_i} \qquad \forall j \in \{1, \ldots, N\}, \lambda \in \{0, \ldots, m\}.$$

Let $p_{k\alpha} = p_{k\alpha 1} \otimes \cdots \otimes p_{k\alpha n}$. Then

$$\begin{aligned} p_{k\alpha}^{(\beta)}(a_j) &= p_{k\alpha 1}^{(\beta_1)}(a_{j1}) \cdots p_{k\alpha n}^{(\beta_n)}(a_{jn}) \\ &= \delta_{a_{j1} a_{k1}} \delta_{\beta_1 \alpha_1} \cdots \delta_{a_{jn} a_{kn}} \delta_{\beta_n \alpha_n} = \delta_{a_j a_k} \delta_{\beta\alpha} = \delta_{jk} \delta_{\beta\alpha} \\ &\qquad \forall j \in \{1, \ldots, N\}, \beta = (\beta_1, \ldots, \beta_n) \in \{0, \ldots, m\}^n. \end{aligned}$$

Let

$$p(x) = \sum_{k=1}^{N} \sum_{\alpha \in \{0, \ldots, m\}^n} p_{k\alpha}(x) v_{k\alpha} \qquad \forall x \in \mathbb{R}^n.$$

Then we have

$$p^{(\alpha)}(a_k) = \sum_{j=1}^{N} \sum_{\beta \in \{0, \ldots, m\}^n} p_{j\beta}^{(\alpha)}(a_k) v_{j\beta} = \sum_{j=1}^{N} \sum_{\beta \in \{0, \ldots, m\}^n} \delta_{kj} \delta_{\alpha\beta} v_{j\beta} = v_{k\alpha}$$

$$\forall k \in \{1, \ldots, N\}, \alpha \in \{0, \ldots, m\}^n. \quad \blacksquare$$

Remark. We do not need to find a polynomial of minimal degree in Theorem 5. If not known, this may be a hard and interesting open problem.

THEOREM 6. (Simultaneous Polynomial Approximation of a Function and Its Partial Derivatives Which Is Exact at a Finite Number of Prescribed Points). *For every $\mathbb{R}$-Banach space $\mathbb{E}$, for every $f \in C^m(\mathbb{R}^n, \mathbb{E})$ with compact support K, for every $a_1, \dots, a_N \in \mathbb{R}^n$ distinct, and for every $\varepsilon > 0$, there exists an $\mathbb{E}$-polynomial p: $\mathbb{R}^n \to \mathbb{E}$ such that*

$$\|f^{(\alpha)}(x) - p^{(\alpha)}(x)\| < \varepsilon \qquad \forall x \in K, \alpha \in \mathbb{N}^n, |\alpha| \le m,$$

and moreover

$$p^{(\alpha)}(a_k) = f^{(\alpha)}(a_k) \qquad \forall k \in \{1, \dots, N\}, \alpha \in \mathbb{N}^n, |\alpha| \le m.$$

Proof. Clearly, there exists $r > 0$ such that $K \cup \{a_1, \dots, a_N\} \subseteq C_r$. By Theorem 5, for every $k \in \{1, \dots, N\}$ and $\alpha \in \{0, \dots, m\}^n$, there exists a polynomial $p_{k\alpha}$: $\mathbb{R}^n \to \mathbb{R}$ such that

$$p_{k\alpha}^{(\beta)}(a_i) = \delta_{ik}\delta_{\beta\alpha} \qquad \forall i \in \{1, \dots, N\}, \beta = \{0, \dots, m\}^n. \tag{13}$$

Since C_r is compact, there exists

$$M = \max_{k \in \{1, \dots, N\}} \max_{|\alpha| \le m} \max_{|\beta| \le m} \max_{x \in C_r} \left| p_{k\alpha}^{(\beta)}(x) \right|, \tag{14}$$

and moreover, by (13), $1 \le M < \infty$. On the other hand, by Corollary 3, there exists an $\mathbb{E}$-polynomial q: $\mathbb{R}^n \to \mathbb{E}$ such that

$$\| f^{(\alpha)}(x) - q^{(\alpha)}(x) \| < \frac{\varepsilon}{((n+m)!/n!m!)NM + 1} \qquad \forall x \in C_r, \alpha \in \mathbb{N}^n, |\alpha| \le m. \tag{15}$$

Let

$$d_{k\alpha} = q^{(\alpha)}(a_k) - f^{(\alpha)}(a_k) \in \mathbb{E} \qquad \forall k \in \{1, \dots, N\}, \alpha \in \mathbb{N}^n, |\alpha| \le m, \tag{16}$$

and

$$p(x) = q(x) - \sum_{k=1}^{N} \sum_{|\alpha| \le m} p_{k\alpha}(x) d_{k\alpha} \qquad \forall x \in \mathbb{R}^n. \tag{17}$$

Let $k \in \{1, \dots, N\}$ and $\alpha \in \mathbb{N}^n$ be such that $|\alpha| \le m$. Then, by (13), (16), and (17) we have

$$p^{(\alpha)}(a_k) = q^{(\alpha)}(a_k) - \sum_{i=1}^{N} \sum_{|\beta| \le m} p_{i\beta}^{(\alpha)}(a_k) d_{i\beta}$$

$$= q^{(\alpha)}(a_k) - \sum_{i=1}^{N} \sum_{|\beta| \le m} \delta_{ki} \delta_{\alpha\beta} d_{i\beta}$$

$$= q^{(\alpha)}(a_k) - d_{k\alpha} = q^{(\alpha)}(a_k) - q^{(\alpha)}(a_k) + f^{(\alpha)}(a_k) = f^{(\alpha)}(a_k).$$

Furthermore, by (15) and (16), we have

$$\|d_{k\alpha}\| = \left\| q^{(\alpha)}(a_k) - f^{(\alpha)}(a_k) \right\| < \frac{\varepsilon}{((n+m)!/n!m!)NM + 1}. \quad (18)$$

Finally, by (14), (15), (17), and (18), we conclude that

$$\left\| f^{(\alpha)}(x) - p^{(\alpha)}(x) \right\| \le \left\| f^{(\alpha)}(x) - q^{(\alpha)}(x) \right\| + \sum_{k=1}^{N} \sum_{|\beta| \le m} \left| p_{k\beta}^{(\alpha)}(x) \right| \|d_{k\beta}\|$$

$$< \frac{\varepsilon}{((n+m)!/n!m!)NM + 1} + \frac{(n+m)!}{n!m!} NM \frac{\varepsilon}{((n+m)!/n!m!)NM + 1} = \varepsilon. \quad \blacksquare$$

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