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ORIGINAL ARTICLE

Fixed point theorems under Pata-type conditions in metric spaces

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Abstract In this paper, we prove a generalization of Chatterjea's fixed point theorem, based on a recent result of Pata. Also, we establish common fixed point results of Pata-type for two maps, as well as a coupled fixed point result in ordered metric spaces. An example is given to show that new results are different from the known ones.

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1. Introduction and preliminaries

Throughout this paper, (X, d) will be a given complete metric space. Let us select an arbitrary point $x_0 \in X$, and call it the "zero of X"; further, denote

$$\|x\| = d(x, x_0), \quad \text{for all } x \in X.$$

It will be clear that the obtained results do not depend on the particular choice of point x_0. Also, $\psi : [0,1] \to [0,\infty)$ will be a fixed increasing function, continuous at zero, with $\psi(0) = 0$.

In a recent paper [1], Pata obtained the following refinement of the classical Banach Contraction Principle.

Theorem 1.1 [1]. *Let $f : X \to X$ and let $\Lambda \geqslant 0, \alpha \geqslant 1$ and $\beta \in [0, \alpha]$ be fixed constants. If the inequality*

$$d(fx, fy) \leqslant (1-\varepsilon)d(x,y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\|]^{\beta} \tag{1.1}$$

is satisfied for every $\varepsilon \in [0,1]$ and all $x, y \in X$, then f has a unique fixed point $z \in X$. Furthermore, the sequence $\{f^n x_0\}$ converges to z.

Chakraborty and Samanta extended in [2] the result of Pata to the case of Kannan-type contractive condition.

In this paper, we prove a further extension of Pata's result, using contractive condition of Chatterjea's type [3,4]. Also, we establish common fixed point results of Pata-type for two maps, as well as a coupled fixed point result in ordered metric spaces. An example is given to show that new results are different from the known ones.

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1.1. An auxiliary result

Assertions similar to the following lemma were used (and proved) in the course of proofs of several fixed point results in various papers.

Lemma 1.1 [5]. *Let* (X, d) *be a metric space and let* $\{y_n\}$ *be a sequence in X such that* $d(y_{n+1}, y_n)$ *is nonincreasing and that*

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0.$$

If $\{y_{2n}\}$ *is not a Cauchy sequence then there exist a* $\delta > 0$ *and two strictly increasing sequences* $\{m_k\}$ *and* $\{n_k\}$ *of positive integers such that the following sequences tend to* δ *when* $k \to \infty$*:*

$$\begin{aligned} &d(y_{2m_k}, y_{2n_k}), \quad d(y_{2m_k}, y_{2n_k+1}), \quad d(y_{2m_k-1}, y_{2n_k}),\\ &d(y_{2m_k-1}, y_{2n_k+1}), \quad d(y_{2m_k+1}, y_{2n_k+1}). \end{aligned} \tag{1.2}$$

2. A Chatterjea-type fixed point result

Theorem 2.1. *Let* $f : X \to X$ *and let* $\Lambda \geqslant 0, \alpha \geqslant 1$ *and* $\beta \in [0, \alpha]$ *be fixed constants. If the inequality*

$$\begin{aligned} d(fx, fy) &\leqslant \frac{1-\varepsilon}{2}(d(x, fy) + d(y, fx))\\ &\quad + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\| + \|fx\| + \|fy\|]^{\beta} \end{aligned} \tag{2.1}$$

is satisfied for every $\varepsilon \in [0, 1]$ *and all* $x, y \in X$*, then* f *has a unique fixed point* $z \in X$*.*

Proof.

1. Uniqueness.For any two fixed $u, v \in X$, we can write (2.1) in the form

   $$d(fu, fv) \leqslant \frac{1-\varepsilon}{2}(d(u, fv) + d(v, fu)) + K\varepsilon\psi(\varepsilon), \qquad K > 0.$$

   If $fu = u$ and $fv = v$ then

   $$d(u, v) \leqslant K\psi(\varepsilon),$$

   for all $\varepsilon \in (0, 1]$, which implies that $d(u, v) = 0$.
2. Existence of z.
   Starting from x_0, we introduce the sequences

   $$x_n = fx_{n-1} = f^n x_0 \text{ and } c_n = \|x_n\|.$$

2.1. First, we have that the sequence $d(x_{n+1}, x_n)$ is nonincreasing, that is

$$d(x_{n+1}, x_n) \leqslant d(x_n, x_{n-1}) \leqslant \cdots \leqslant d(x_1, x_0), \tag{2.2}$$

for all $n \in \mathbb{N}$.
Indeed, putting $\varepsilon = 0, x = x_n, y = x_{n-1}$ in (2.1), we obtain (2.2).

2.2. The sequence $\{c_n\}$ is bounded.
Using (2.2), we deduce the following estimate

$$\begin{aligned} c_n = d(x_n, x_0) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_1) + d(x_1, x_0)\\ &\leqslant d(x_{n+1}, x_1) + 2c_1 = d(fx_n, fx_0) + 2c_1. \end{aligned}$$

Therefore, we infer from (2.1) that

$$\begin{aligned} c_n &\leqslant \frac{1-\varepsilon}{2}[d(x_n, x_1) + d(x_{n+1}, x_0)]\\ &\quad + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x_n\| + \|x_{n+1}\| + \|x_1\|]^{\beta} + 2c_1. \end{aligned}$$

Using $d(x_n, x_1) \leqslant d(x_n, x_0) + d(x_0, x_1), d(x_{n+1}, x_0) \leqslant d(x_{n+1}, x_n) + d(x_n, x_0)$ and (2.2), as $\beta \leqslant \alpha$, the previous inequality implies that

$$c_n \leqslant (1-\varepsilon)(c_n + c_1) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + 2c_n + 2c_1]^{\alpha} + 2c_1$$

Now,

$$[1 + 2c_n + 2c_1]^{\alpha} \leqslant (1 + 2c_n)^{\alpha}(1 + 2c_1)^{\alpha} \leq 2^{\alpha}c_n^{\alpha}(1 + 2c_1)^{\alpha},$$

which implies that

$$c_n \leqslant (1-\varepsilon)c_n + a\varepsilon^{\alpha}\psi(\varepsilon)c_n^{\alpha} + b,$$

for some $a, b > 0$. Hence,

$$\varepsilon c_n \leqslant a\varepsilon^{\alpha}\psi(\varepsilon)c_n^{\alpha} + b.$$

Now, for the same reason as in [1], it follows that the sequence $\{c_n\}$ is bounded.

2.3. $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.
For all $\varepsilon \in (0, 1]$ and for $x = x_n$, $y = x_{n-1}$ we have

$$\begin{aligned} d(x_{n+1}, x_n) = d(fx_n, fx_{n-1}) &\leqslant \frac{1-\varepsilon}{2}(d(x_n, x_n) + d(x_{n-1}, x_{n+1}))\\ &\quad + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + 2\|x_n\| + \|x_{n-1}\|\\ &\quad + \|x_{n+1}\|]^{\beta} \leqslant \frac{1-\varepsilon}{2}(d(x_{n-1}, x_n)\\ &\quad + d(x_n, x_{n+1})) + K\varepsilon\psi(\varepsilon), \quad K > 0. \end{aligned} \tag{2.3}$$

If $\lim_{n\to\infty} d(x_{n+1}, x_n) = d^* > 0$, it follows from (2.3) that

$$d^* \leqslant K\psi(\varepsilon),$$

that is $d^* = 0$. A contradiction.

2.4. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.
If it is not the case, choose $\delta > 0$, $\{m_k\}$ and $\{n_k\}$ as in Lemma 2.1. Putting $x = x_{2m(k)-1}$, $y = x_{2n(k)}$ in (2.1), we obtain

$$\begin{aligned} d(x_{2m(k)}, x_{2n(k)+1}) &\leqslant \frac{1-\varepsilon}{2}(d(x_{2m(k)-1}, x_{2n(k)+1})\\ &\quad + d(x_{2m(k)}, x_{2n(k)})) + K\varepsilon\psi(\varepsilon), \end{aligned} \tag{2.4}$$

where $d(x_{2m(k)}, x_{2n(k)+1}) \to \delta$, $d(x_{2m(k)-1}, x_{2n(k)+1}) \to \delta$ and $d(x_{2m(k)}, x_{2n(k)}) \to \delta$. Letting $k \to \infty$ in (2.4), we obtain

$$\delta \leqslant K\psi(\varepsilon),$$

that is $\delta = 0$, a contradiction.
Taking into account the completeness of (X, d), we can now guarantee the existence of some $z \in X$ to which $\{x_n\}$ converges.Finally, all that remains to show is:

2.5. z is a fixed point for f.
For this we observe that, for all $n \in \mathbb{N}$ and for $\varepsilon = 0$,

$$\begin{aligned} d(fz, z) &\leqslant d(fz, x_{n+1}) + d(x_{n+1}, z) = d(fz, fx_n) + d(x_{n+1}, z)\\ &\leqslant \frac{1}{2}(d(z, x_{n+1}) + d(fz, x_n)) + d(x_{n+1}, z). \end{aligned}$$

Hence, $d(fz, z) \leqslant \frac{1}{2}d(fz, z)$, that is $fz = z$, which is the required result. □

The classical Chatterjea's result [3] is a consequence of Theorem 2.1, since the condition

$$d(fx, fy) \leqslant \frac{\lambda}{2}(d(x, fy) + d(y, fx))$$

for some $\lambda \in [0, 1)$ and all $x, y \in X$, implies condition (2.1). This can be proved in the same way as in [1, Section 3], or [2, Section 3].

3. Common fixed point results

In this section, we deduce some common fixed point results using a Pata-type contractive condition.

Let f and g be two self-mappings of the given metric space (X,d), such that $fX \subset gX$, and at least one of these subspaces of X is complete. Choose arbitrary $x_0 \in X$ and denote $y_0 = fx_0$; this time, for $x \in X$, denote $\|x\| = d(x, y_0)$. Suppose that the function ψ has the same properties as in the previous section.

Theorem 3.1. *Let $\Lambda \geqslant 0$, $\alpha \geqslant 1$ and $\beta \in [1, \alpha]$ be fixed constants such that the inequality*

$$d(fx,fy) \leqslant (1-\varepsilon)\ d(gx,gy) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|gx\| + \|gy\|]^\beta. \tag{3.1}$$

holds for each $\varepsilon \in [0,1]$ and for all $x, y \in X$. Then the pair (f,g) has a unique point of coincidence. If, moreover, the pair (f,g) is weakly compatible then f and g have a unique common fixed point $z \in X$ and for an arbitrary initial point $x \in X$, each corresponding Jungck sequence $y_n = fx_n = gx_{n+1}$ converges to z.

Proof. First of all, note that Λ can be supposed to be positive, otherwise we have the classical Jungck's result.

Starting from the given point x_0, and using that $fX \subset gX$, construct a usual Jungck sequence $\{y_n\}$ by $y_n = fx_n = gx_{n+1}$, $n = 0, 1, 2, \ldots$ We proceed by proving the following steps.

1. If the pair (f,g) has a point of coincidence w then it is unique.
   Indeed, let $w_1 = fu_1 = gu_1$ and $w_2 = fu_2 = gu_2$, and let $\varepsilon \geqslant 0$ be arbitrary. Then

$$\begin{aligned} d(w_1, w_2) &= d(fu_1, fu_2) \\ &\leqslant (1-\varepsilon)d(gu_1, gu_2) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|gu_1\| + \|gu_2\|]^\beta \\ &= (1-\varepsilon)d(w_1, w_2) + K\varepsilon^\alpha\psi(\varepsilon), \end{aligned}$$

   where $K = \Lambda[1 + \|gu_1\| + \|gu_2\|]^\beta > 0$. Hence, $d(w_1, w_2) \leqslant K\varepsilon^{\alpha-1}\psi(\varepsilon)$. Using the properties of function ψ, it follows that $w_1 = w_2$.
2. $d(y_{n+1}, y_n) \downarrow \delta \geqslant 0$.
   This is obtained by putting $\varepsilon = 0$, $x = x_{n+1}, y = x_n$ in (3.1).
3. The sequence $\{c_n\}$, where $c_n = d(y_n, y_0)$, is bounded.
   We have

$$\begin{aligned} c_n = d(y_n, y_0) &\leqslant d(y_n, y_{n+1}) + d(y_{n+1}, y_1) + d(y_1, y_0) \\ &= d(y_n, y_{n+1}) + d(fx_{n+1}, fx_1) + d(y_1, y_0) \\ &\leqslant 2c_1 + (1-\varepsilon)\ d(gx_{n+1}, gx_1) \\ &\quad + \Lambda\varepsilon^\alpha\psi(\varepsilon)\left[1 + \|gx_{n+1}\| + \|gx_1\|\right]^\beta \\ &= 2c_1 + (1-\varepsilon)d(y_n, y_0) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|y_n\| + \|y_0\|]^\beta \\ &\leqslant 2c_1 + (1-\varepsilon)c_n + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + c_n]^\alpha \\ &\leqslant (1-\varepsilon)c_n + a\varepsilon^\alpha\psi(\varepsilon)c_n^\alpha + b, \qquad a > 0, \quad b > 0, \end{aligned}$$

   i.e.

$$\varepsilon c_n \leqslant a\varepsilon^\alpha\psi(\varepsilon)c_n^\alpha + b.$$

   If we suppose that $\{c_n\}$ is not bounded, we obtain a contradiction, similarly as in [1].
4. $\delta = 0$.
   First of all, we have that

$$\begin{aligned} d(y_{n+1}, y_n) &= d(fx_{n+1}, fx_n) \\ &\leqslant (1-\varepsilon)\ d(gx_{n+1}, gx_n) \\ &\quad + \Lambda\varepsilon^\alpha\psi(\varepsilon)\left[1 + \|gx_{n+1}\| + \|gx_n\|\right]^\beta. \end{aligned}$$

   Now, using that $\{c_n\}$ is bounded and modifying the constant Λ, we get that

$$d(y_{n+1}, y_n) \leqslant (1-\varepsilon)\ d(y_n, y_{n-1}) + K\varepsilon^\alpha\psi(\varepsilon),$$

   wherefrom, passing to the limit as $n \to \infty$,

$$\delta \leqslant (1-\varepsilon)\delta + K\varepsilon^\alpha\psi(\varepsilon),$$

   i.e., $\delta \leqslant K\varepsilon^{\alpha-1}\psi(\varepsilon)$, hence $\delta = 0$. (Note that we have taken $\varepsilon \in (0,1]$).
5. Using now Lemma 1.1 in the usual way and taking into account that $\{c_n\}$ is bounded, we can prove that $\{y_n\}$ is a Cauchy sequence.
6. Suppose that gX is a complete subspace of X (the proof when fX is complete is similar). We have that $y_n = gx_{n+1} \to gz$, for some $z \in X$. But then, putting $\varepsilon = 0, x = x_n, y = z$ in (3.1), and passing to the limit, we get that $fz = gz = \lim_{n\to\infty} y_n$. Hence, $fz = gz = w$ is a (unique) point of coincidence of (f,g).
7. If the pair (f,g) is weakly compatible, by a classical result, it follows that z is a unique common fixed point of f and g.

Finally, note that the choice of the initial point x for the Jungck's sequence is irrelevant. □

Putting $g = i_X$ in the previous theorem, we get Theorem 1.1 as a consequence.

Clearly, Theorem 3.1 generalizes the classical Jungck's result [6].

In a very similar way, the following results of Pata–Kannan and Pata–Chatterjea type can be proved for two mappings.

Theorem 3.2. *Let $\Lambda \geqslant 0, \alpha \geqslant 1$ and $\beta \in [1, \alpha]$ be fixed constants such that the inequality*

$$d(fx,fy) \leqslant \frac{1-\varepsilon}{2}(d(fx,gx) + d(fy,gy)) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|fx\| + \|fy\| + \|gx\| + \|gy\|]^\beta.$$

holds for each $\varepsilon \in [0,1]$ and for all $x, y \in X$. Then the pair (f,g) has a unique point of coincidence. If, moreover, the pair (f,g) is weakly compatible then f and g have a unique common fixed point $z \in X$ and for an arbitrary initial point $x \in X$, each corresponding Jungck sequence $y_n = fx_n = gx_{n+1}$ converges to z.

Theorem 3.3. *Let $\Lambda \geqslant 0, \alpha \geqslant 1$ and $\beta \in [1, \alpha]$ be fixed constants such that the inequality*

$$d(fx,fy) \leqslant \frac{1-\varepsilon}{2}(d(fx,gy) + d(gx,fy)) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|fx\| + \|fy\| + \|gx\| + \|gy\|]^\beta.$$

holds for each $\varepsilon \in [0,1]$ and for all $x, y \in X$. Then the pair (f,g) has a unique point of coincidence. If, moreover, the pair (f,g) is weakly compatible then f and g have a unique common fixed point $z \in X$ and for an arbitrary initial point $x \in X$, each corresponding Jungck sequence $y_n = fx_n = gx_{n+1}$ converges to z.

4. Coupled fixed point results

First of all, note that all the obtained results can be easily formulated and proved in versions adapted to ordered metric spaces. For example, the following form can be given to the basic Pata's Theorem 1.1.

Theorem 4.1. *Let $(X, \preceq, d)$ be a complete ordered metric space and let $\Lambda \geqslant 0, \alpha \geqslant 1$ and $\beta \in [0, \alpha]$ be fixed constants. Let $f : X \to X$ be a non-decreasing map such that there exists x_0 satisfying $x_0 \preceq fx_0$. If the inequality*

$$d(fx, fy) \leqslant (1-\varepsilon)d(x, y) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|x\| + \|y\|]^\beta \tag{4.1}$$

is satisfied for every $\varepsilon \in [0, 1]$ and all comparable $x, y \in X$ (i.e., such that either $x \preceq y$ or $y \preceq x$ holds), then f has a fixed point $z \in X$.

The notions of a coupled fixed point and a mixed monotone mapping were introduced and investigated by Guo and Lakshmikantham in [7]. Further, a lot of authors obtained several results of this kind.

Recall the following notions.

Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$.

(1) F is said to have the *mixed monotone property* if the following two conditions are satisfied:

$$(\forall x_1, x_2, y \in X)\ x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$
$$(\forall x, y_1, y_2 \in X)\ y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

(2) A point $(x, y) \in X \times X$ is said to be a *coupled fixed point* of F if $F(x, y) = x$ and $F(y, x) = y$.

Coupled fixed point results under Pata-type contractive conditions were obtained by Eshaghi et al. in [8]. Their basic result was the following.

Theorem 4.2 [8]. *Let $(X, \preceq, d)$ be a complete ordered metric space and let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property. Suppose that there exists $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$; for $x, y \in X$, denote $\|x, y\| = d(x, x_0) + d(y, y_0)$. Let $\Lambda \geqslant 0$, $\alpha \geqslant 1$ and $\beta \in [0, \alpha]$ be fixed constants. If the inequality*

$$d(F(x, y), F(u, v)) \leqslant \frac{1-\varepsilon}{2}[d(x, u) + d(y, v)] + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|x, y\| + \|u, v\|]^\beta \tag{4.2}$$

is satisfied for every $\varepsilon \in [0, 1]$ and all $(x, y), (u, v) \in X \times X$ with $u \preceq x, y \preceq v$, then F has a coupled fixed point in $X \times X$.

A new approach to these problems was initiated by Berinde in [9] and further developed, e.g., in [10–12]. The basic idea is to exploit results for mappings with one variable and apply them to mappings defined on products of spaces. We are going to apply this approach to problems with Pata-type conditions and we are going to show that better results can be obtained in this way than by a classical procedure used, e.g., in [8].

The following lemma is easy to prove.

Lemma 4.1 [11].

(i) *Let $(X, \preceq, d)$ be an ordered metric space. If the relation $\sqsubseteq$ is defined on X^2 by*

$$Y \sqsubseteq V \iff x \preceq u \wedge y \succeq v, \qquad Y = (x, y), V = (u, v) \in X^2,$$

and $D : X^2 \times X^2 \to \mathbb{R}^+$ is given by

$$D(Y, V) = d(x, u) + d(y, v), \qquad Y = (x, y), V = (u, v) \in X^2,$$

then $(X^2, \sqsubseteq, D)$ is an ordered metric spaces. The space $(X^2, \sqsubseteq, D)$ is complete iff $(X, \preceq, d)$ is complete.

(ii) *If $F : X \times X \to X$ has the mixed monotone property, then the mapping $T_F : X^2 \to X^2$ given by*

$$T_F Y = (F(x, y), F(y, x)), \qquad Y = (x, y) \in X^2.$$

is nondecreasing w.r.t. $\sqsubseteq$, i.e.

$$Y \sqsubseteq V \Rightarrow T_F Y \sqsubseteq T_F V.$$

(iii) *$(x, y) \in X \times X$ is a coupled fixed point of F* iff *$Y = (x, y)$ is a fixed point of T_F.*

(iv) *If F is continuous from (X^2, D) to (X, d)* (i.e. *$x_n \to x$ and $y_n \to y$ imply $F(x_n, y_n) \to F(x, y)$*) *then T_F is continuous in (X^2, D).*

Using these results, we can formulate and prove the following coupled fixed point result under Pata-type contractive condition.

Theorem 4.3. *Suppose that all the conditions of Theorem 4.2 hold, except that the condition (4.2) is replaced by*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leqslant (1-\varepsilon)[d(x, u) + d(y, v)] + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|x, y\| + \|u, v\|]^\beta. \tag{4.3}$$

Then F has a coupled fixed point in $X \times X$.

Proof. Consider the space $(X^2, \sqsubseteq, D)$ and the mapping $T_F : X^2 \to X^2$ described in Lemma 4.1. It is easy to show that they satisfy all the conditions of Theorem 4.1; in particular the contractive condition of the form

$$D(T_F Y, T_F V) \leqslant (1-\varepsilon)D(Y, V) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|Y\| + \|V\|]^\beta$$

holds for all comparable (w.r.t. $\sqsubseteq$) $Y, V \in X^2$, where $\|Y\| = \|(x, y)\| = D((x, y), (x_0, y_0))$. Applying Theorem 4.1, we obtain the desired result. □

Remark 4.1. It is easy to show that each example which can be handled using Theorem 4.2 can also be handled using Theorem 4.3. The example that follows this remark will show that the converse is not true.

Indeed, suppose that $(x, y), (u, v) \in X^2$ are comparable w.r.t. $\sqsubseteq$. Applying (4.2) to the pairs (x, y) and (u, v), we get that

$$d(F(x, y), F(u, v)) \leqslant \frac{1-\varepsilon}{2}D(Y, V) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + D(Y, Y_0) + D(V, Y_0)]^\beta. \tag{4.4}$$

Applying the same inequality to the pairs (y,x) and (v,u), we obtain

$$\begin{aligned}d(F(y,x),F(v,u))\\ &\leqslant \frac{1-\varepsilon}{2}D(Y,V)+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+d(y,x_0)+d(x,y_0)+d(v,x_0)+d(u,y_0)]^{\beta}\\ &\leqslant \frac{1-\varepsilon}{2}D(Y,V)+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+D(Y,Y_0)+D(V,Y_0)+4d(x_0,y_0)]^{\beta}.\end{aligned} \tag{4.5}$$

Adding up the inequalities (4.4) and (4.5), and writing temporarily $A=D(Y,Y_0)+D(V,Y_0)$, we get the following estimate:

$$D(T_FY,T_FV)\leqslant(1-\varepsilon)D(Y,V)+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\{[1+A]^{\beta} + [1+A+4d(x_0,y_0)]^{\beta}\}. \tag{4.6}$$

Now,

$$\begin{aligned}[1+A]^{\beta}+[1+A+4d(x_0,y_0)]^{\beta}&=[1+A]^{\beta}\left[1+\left(1+\frac{4d(x_0,y_0)}{1+A}\right)^{\beta}\right]\\ &\leqslant[1+A]^{\beta}[1+(1+4d(x_0,y_0))^{\beta}] \quad =C[1+A]^{\beta},\end{aligned}$$

where C is a constant (not depending on Y, V and ε). Hence, putting $\Lambda_1=\Lambda C$, (4.6) can be written as

$$D(T_FY,T_FV)\leqslant(1-\varepsilon)D(Y,V) + \Lambda_1\varepsilon^{\alpha}\psi(\varepsilon)[1+D(Y,Y_0)+D(V,Y_0)]^{\beta},$$

which means that all the conditions of Theorem 4.3 are fulfilled.

Example 4.1. Let $X=\mathbb{R}$ be equipped with the usual metric and order. The mapping $F:X\times X\to X$ defined by $F(x,y)=\frac{1}{6}(x-4y)$ is obviously mixed monotone. It is easy to obtain that

$$\begin{aligned}d(F(x,y),F(u,v))+d(F(y,x),F(v,u))&=\left|\frac{x-4y}{6}-\frac{u-4v}{6}\right|\\ &\quad+\left|\frac{y-4x}{6}-\frac{v-4u}{6}\right|\\ &\leqslant\frac{1}{6}|x-u|+\frac{4}{6}|y-v|\\ &\quad+\frac{1}{6}|y-v|+\frac{4}{6}|x-u|\\ &=\frac{5}{6}[d(x,u)+d(y,v)],\end{aligned}$$

i.e., $D(T_FY,T_FV)\leqslant\lambda D(Y,V)$, where $\lambda=\frac{5}{6}$. Further, we follow the procedure as in [1, Section 3], only we write it with some details that were skipped in [1].

First of all, for arbitrary $\varepsilon\in[0,1]$, write the obtained inequality in the form

$$\begin{aligned}D(T_FY,T_FV)&\leqslant(1-\varepsilon)D(Y,V)+(\lambda+\varepsilon-1)D(Y,V)\\ &\leqslant(1-\varepsilon)D(Y,V)+(\lambda+\varepsilon-1)(\|Y\|+\|V\|).\end{aligned}$$

We want to prove that there are some $\gamma\geqslant0$ and $\Lambda\geqslant0$ such that

$$(\lambda+\varepsilon-1)(\|Y\|+\|V\|)\leqslant\Lambda\varepsilon^{1+\gamma}(1+\|Y\|+\|V\|),$$

holds for each $\varepsilon\in[0,1]$ and all comparable $Y,V\in X^2$. Indeed, this will be the case if one can find $\Lambda\geqslant0$ such that

$$\Lambda\geqslant\frac{\lambda+\varepsilon-1}{\varepsilon^{1+\gamma}}$$

holds for some $\gamma\geqslant0$ and each $\varepsilon\in[0,1]$. By a routine procedure, it is easy to show that this is the case if we chose γ such that $\frac{\gamma}{1+\gamma}>1-\lambda$ and then

$$\Lambda=\frac{\gamma^{\gamma}}{(1+\gamma)^{1+\gamma}}\frac{1}{(1-\lambda)^{\gamma}}.$$

Hence, we have that, for the chosen γ and Λ,

$$\begin{aligned}d(F(x,y),F(u,v))+d(F(y,x),F(v,u))&\leqslant(1-\varepsilon)[d(x,u)+d(y,v)]\\ &\quad+\Lambda\varepsilon^{1+\gamma}[1+\|x,y\|+\|u,v\|],\end{aligned}$$

for each $\varepsilon>0$ and all $x,y,u,v\in X$ with $u\leqslant x$, $y\leqslant v$. Thus, the conditions of Theorem 4.3 are fulfilled (with $\alpha=\beta=1$), and the mapping F has a coupled fixed point (which is (0, 0)).

On the other hand, suppose that the condition (4.2) of Theorem 4.2 holds, i.e.,

$$\left|\frac{x-4y}{6}-\frac{u-4v}{6}\right|\leqslant\frac{1-\varepsilon}{2}[|x-u|+|y-v|] + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|x,y\|+\|u,v\|]^{\beta}$$

is satisfied for each $\varepsilon\in[0,1]$ and all $x,y,u,v\in X$ with $u\leqslant x$, $y\leqslant v$. Taking $\varepsilon=0$ and $x=u$, we obtain that

$$\frac{2}{3}|y-v|\leqslant\frac{1}{2}|y-v|$$

which obviously cannot hold (except when $y=v$).

5. An open question

The following would be a Pata-version of the well-known Ćirić's result on quasicontractions (see, e.g., [4]).

Question 5.1. Prove or disprove the following. Let $f:X\to X$ and let $\Lambda\geqslant0$, $\alpha\geqslant1$ and $\beta\in[0,\alpha]$ be fixed constants. If the inequality

$$\begin{aligned}d(fx,fy)&\leqslant(1-\varepsilon)\\ &\quad\times\max\{d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx)\}\\ &\quad+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|x\|+\|y\|]^{\beta}\end{aligned}$$

is satisfied for every $\varepsilon\in[0,1]$ and all $x,y\in X$, then f has a unique fixed point $z\in X$. Furthermore, the sequence $\{f^nx_0\}$ converges to z.

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