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On Some Classes of Continuable Solutions of a Nonlinear Differential Equation

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INTRODUCTION

The aim of this paper is to present a global qualitative analysis for the asymptotic behavior of the solutions of the nonlinear equation

$$[r(t)\, x']' + q(t)\, f(x) = 0, \qquad \left(' = \frac{d}{dt} \right) \tag{1}$$

where

(H) r, q: $[0, +\infty) \to \mathbb{R}$, f: $\mathbb{R} \to \mathbb{R}$ are continuous functions, $r(t) > 0$, $q(t) > 0$ and $u \cdot f(u) > 0$ for $u \neq 0$.

When the function r does not have a continuous derivative, equation (1) may be interpreted as the first order differential system:

$$\begin{aligned} x' &= \frac{1}{r(t)}\, y \\ y' &= -q(t)\, f(x) \end{aligned} \tag{1'}$$

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for the vector $(x, y) = (x, rx')$. For the linear equation

$$[r(t)\, x']' + q(t)\, x = 0, \tag{2}$$

with the same assumptions, a similar investigation has already been completed by the authors in [8] based on the integral behavior in $[0, +\infty)$ of the functions $1/r$ and q.

For the nonlinear equation (1) partial results may be found in the literature with additional assumptions on the function f, such as, for instance, monotonicity, global superlinearity or sublinearity. It seems interesting to study whether a situation similar to the one of the linear case is still valid in the nonlinear case, i.e., to study whether the existence of solutions of given asymptotic behavior may be based again on the integrals of the functions $1/r$ and q in $[0, +\infty)$.

The plan of the paper is the following: in Section 1 we present some preliminary results and recall the global analysis of the linear case. In Section 2 we give necessary and sufficient conditions for the existence of solutions with given asymptotic behavior. In Sections 3 (resp. 4) sufficient conditions for the existence of eventually bounded (resp. unbounded) monotone solutions are presented. Finally in Section 5 we give a global theorem and we compare our results with those in the linear case.

The obtained results will be compared with the ones in the literature in the framework of the paper. We now just recall that a wide investigation on the asymptotic behavior of solutions of (1) which are eventually monotone has been presented in [7, 15, 16] for the case $q < 0$. For further investigations and for a comprehensive list of references concerning this kind of problems the reader is referred to [4, 11, 13, 17, 18, 20, 22] and to the surveys [23, 24].

1

A continuable solution of (1) is said to be *oscillatory* if there exists a sequence $\{t_k\}$, $t_k \to +\infty$, such that $x(t_k) = 0$, and it is said to be *nonoscillatory* otherwise. A continuable nonoscillatory solution is said to be *weakly oscillatory* if x' changes sign for arbitrarily large values of t (see, e.g., [14]). As already has been done in [8], in the sequel we will use the notations:

$$I_r = \int_0^{+\infty} \frac{1}{r(s)}\, ds \qquad I_q = \int_0^{+\infty} q(s)\, ds$$

$$I_{rq} = \int_0^{+\infty} \frac{1}{r(\tau)} \int_0^{\tau} q(s)\, ds\, d\tau \qquad I_{qr} = \int_0^{+\infty} q(\tau) \int_0^{\tau} \frac{1}{r(s)}\, ds\, d\tau$$

Using the following implications:

$$I_r = +\infty \Rightarrow I_{rq} = +\infty$$

$$I_q = +\infty \Rightarrow I_{qr} = +\infty$$

$$I_q < +\infty \text{ and } I_r < +\infty \Rightarrow I_{qr} < +\infty \text{ and } I_{rq} < +\infty$$

we still can conclude that the mutual behavior of I_r, I_q, I_{rq}, I_{qr} is completely described by the following six cases:

(C_1) $I_q = I_r = I_{qr} = I_{rq} = +\infty$

(C_2) $I_q < +\infty$, $I_r = I_{rq} = I_{qr} = +\infty$

(C_3) $I_q < +\infty$, $I_{qr} < +\infty$, $I_r = I_{rq} = +\infty$

(C_4) $I_r < +\infty$, $I_q = I_{qr} = I_{rq} = +\infty$

(C_5) $I_r < +\infty$, $I_{rq} < +\infty$, $I_q = I_{qr} = +\infty$

(C_6) $I_q < +\infty$, $I_r < +\infty$, $I_{qr} < +\infty$, $I_{rq} < +\infty$.

The following Lemma will be useful for the study of the asymptotic behavior of solutions of (1):

LEMMA 1.1. *Let x be a solution of (1) that has not arbitrarily large zeros. Then x' cannot have arbitrarily large zeros.*

Proof. For the proof see, e.g., [10]. ∎

From the previous Lemma it follows that (1) cannot have weakly oscillatory solutions. Let S denote the set of all nontrivial continuable solutions of (1). As in the linear case, solutions in S may be *a priori* divided in the following classes which are mutually disjoint:

$$\mathbb{O} = \{x \in S \colon \exists \{t_k\},\ t_k \to +\infty,\ x(t_k) = 0\}$$

$$M_\infty^+ = \{x \in S \colon \exists t_x \geq 0 : x(t)\, x'(t) > 0 \text{ for } t \geq t_x \text{ and } \lim_{t \to +\infty} |x(t)| = +\infty\}$$

$$M_B^+ = \{x \in S \colon \exists t_x \geq 0 : x(t)\, x'(t) > 0 \text{ for } t \geq t_x \text{ and } \lim_{t \to +\infty} |x(t)| = L_x < +\infty\}$$

$$M_B^- = \{x \in S \colon \exists t_x \geq 0 : x(t)\, x'(t) < 0 \text{ for } t \geq t_x \text{ and } \lim_{t \to +\infty} x(t) = l_x \neq 0\}$$

$$M_0^- = \{x \in S \colon \exists t_x \geq 0 : x(t)\, x'(t) < 0 \text{ for } t \geq t_x \text{ and } \lim_{t \to +\infty} x(t) = 0\}.$$

In other words if $x \in M_\infty^+$, then x is eventually either positive increasing or negative decreasing and it is unbounded; if $x \in M_B^+$, then x is eventually either positive increasing or negative decreasing and bounded. On the other hand solutions in M_B^- and M_0 are always bounded because they are eventually either positive decreasing or negative increasing. Moreover solutions in M_0^- tend to zero when $t \to +\infty$. Finally we set

$$M^+ = M_B^+ \cup M_\infty^+ \qquad M^- = M_B^- \cup M_0^-$$

The following crucial result was proved in [8] for the linear case:

THEOREM 1.1. (a) *If* (C_2) *holds, then every solution of* (2) *is either oscillatory or of class* M_∞^+.

(b) *If* (C_3) *holds, then* (2) *is nonoscillatory. Moreover every solution belongs to the class* M^+ *and the set of bounded solutions is a subspace of dimension one.*

(c) *If* (C_4) *holds, then every solution of* (2) *is either oscillatory or of class* M_0^-.

(d) *If* (C_5) *holds, then* (2) *is nonoscillatory. Moreover every solution belongs to the class* M^- *and the set of solutions which tend to zero as* $t \to +\infty$ *is a subspace of dimension one.*

(e) *If* (C_6) *holds, then* (2) *is nonoscillatory. Moreover all solutions are bounded and there are solutions belonging to* M_B^+, *solutions belonging to* M_B^- *and solutions belonging to* M_0^-. *Finally the set* M_0^- *is a subspace of dimension one.*

We recall that in the linear case the Sturm Theorem ensures that if a solution of (2) is oscillatory, then all solutions are oscillatory: hence either $\mathbb{O} = S$ or $\mathbb{O} = \emptyset$. Moreover a well-known result of Leighton (see, e.g., [21, Chap. 2, Sect. 6]) states that, if (C_1) holds, then $\mathbb{O} = S$. In the nonlinear case the Sturm Theorem fails: hence oscillatory solutions and nonoscillatory solutions may coexist (see [10, 24]). However, in [1] it is proved that if f verifies (H) and $df/du \geq 0$, then even in the nonlinear case assumption (C_1) implies that all continuable solutions of (1) are oscillatory. A generalization of such result is presented in the next section.

2

In this section we give necessary conditions for the existence of eventually monotone solutions of (1). Part of such results were proved in [10] and will be only stated here. The following holds:

THEOREM 2.1. *If* $I_q = +\infty$, *then* $M_B^+ = \emptyset$. *If, in addition,*

$$(H_1) \qquad \lim_{|u| \to +\infty} \inf |f(u)| > 0$$

then $M_\infty^+ = \emptyset$.

Proof. See [10]. ∎

The following example shows that assumption (H_1) cannot be omitted. Consider the equation

$$x'' + \frac{1}{4t+4} f(x) = 0 \qquad t \in [0, +\infty)$$

where $f(x) = 1/x$ for $x \geqslant 1$ and satisfies condition (H). We have $I_q = +\infty$, but such equation has the positive increasing unbounded function $x(t) = \sqrt{t+1}$ as a solution.

For the existence of solutions of (1) in the class M^- the following holds:

THEOREM 2.2. *If $I_r = +\infty$, then $M^- = \varnothing$.*

Proof. See [10]. ∎

From the above mentioned results we obtain the following theorem on the oscillation of solutions of (1). Such result generalizes the one in [1] which was quoted above (see, also, [10]).

THEOREM 2.3. *Let $I_q = I_r = +\infty$. If condition (H_1) holds, then all the continuable solutions of (1) are oscillatory.*

Proof. The assertion follows immediately from Theorems 2.1 and 2.2. ∎

The reader is referred to [2 and 3] for the problem of continuability of solutions. The following results give sufficient conditions in order that the classes M_B^+, M_B^- are empty. Such results can be considered as particular cases of recent ones proved in [6]. Moreover it is emphasized that while the monotonicity of function f is required for the functional equation considered in [6], in this case such assumption is not necessary. The following holds:

THEOREM 2.4. *If $I_{qr} = +\infty$, then $M_B^+ = \varnothing$.*

Proof. Let $x \in M_B^+$, such that $x(t) > 0$, $x'(t) > 0$ for $t \geqslant t_0$ and let $x(+\infty) = L_x < +\infty$. From (1) it follows $[r(t)\, x'(t)]' < 0$ for $t \geqslant t_0$. Hence $r(t)\, x'(t)$ is positive decreasing for $t \geqslant t_0$. Let $l_x = \lim_{t \to +\infty} r(t)\, x'(t)$. Clearly $l_x \geqslant 0$. Integrating (1) in $(t, +\infty)$, $t > t_0$, we have

$$-r(t)\, x'(t) = -\int_t^{+\infty} q(s)\, f(x(s))\, ds - l_x$$

and therefore

$$r(t)\, x'(t) \geqslant \int_t^{+\infty} q(s)\, f(x(s))\, ds.$$

Let $m = \min_{u \in [u(t_0), l_x]} f(u)$. Then $m > 0$, and from the above inequalities we get

$$r(t)\, x'(t) \geqslant \frac{m}{r(t)} \int_t^{+\infty} q(s)\, ds.$$

Integrating again in $[t_0, t]$ we obtain

$$\begin{aligned} x(t) &\geqslant x(t_0) + m \int_{t_0}^{t} \frac{1}{r(\tau)} \int_\tau^{+\infty} q(s)\, ds\, d\tau \\ &= x(t_0) + m \int_{t_0}^{t} \frac{1}{r(\tau)} \int_\tau^{t} q(s)\, ds\, d\tau + m \left(\int_{t_0}^{t} \frac{1}{r(\tau)}\, d\tau \right) \left(\int_t^{+\infty} q(s)\, ds \right) \\ &= x(t_0) + m \int_{t_0}^{t} q(\tau) \int_{t_0}^{\tau} \frac{1}{r(s)}\, ds\, d\tau + m \left(\int_{t_0}^{t} \frac{1}{r(\tau)}\, d\tau \right) \left(\int_t^{+\infty} q(s)\, ds \right). \end{aligned}$$

When $t \to +\infty$ one has $x(+\infty) = +\infty$, which is a contradiction. The case $x(t) < 0$, $x'(t) < 0$ for $t \geqslant t_0$ is treated in the same way. ▮

THEOREM 2.5. *If $I_{rq} = +\infty$, then $M_B^- = \varnothing$.*

Proof. Let $x \in M_B^-$ and let $x(t) > 0$, $x'(t) < 0$ for $t \geqslant t_0$, $x(+\infty) = l_x > 0$. Define $w(t) = r(t)\, x'(t)/x(t)$ and

$$m = \min_{u \in [l_x, x(t_0)]} \frac{f(u)}{u}.$$

Taking into account that $m > 0$, for $t \geqslant t_0$, we have

$$w'(t) = -\frac{w^2(t)}{r(t)} - q(t) \frac{f(x(t))}{x(t)} \leqslant -\frac{w^2(t)}{r(t)} - m\, q(t) \leqslant -m\, q(t).$$

Integrating in $[t_0, t]$ we obtain

$$w(t) \leqslant w(t_0) - m \int_{t_0}^{t} q(s)\, ds,$$

hence

$$\frac{x'(t)}{x(t)} \leqslant \frac{w(t_0)}{r(t)} - \frac{m}{r(t)} \int_{t_0}^{t} q(s)\, ds.$$

Integrating again in $[t_0, t]$ we obtain

$$\log \frac{x(t)}{x(t_0)} \leqslant w(t_0) \int_{t_0}^{t} \frac{1}{r(s)}\, ds - m \int_{t_0}^{t} \frac{1}{r(\tau)} \int_{t_0}^{\tau} q(s)\, ds\, d\tau;$$

since $w(t_0) < 0$, we get that $\log(x(t)/x(t_0))$ tends to $-\infty$ as $t \to +\infty$. Hence $x(+\infty) = 0$, which is a contradiction. If $x(t) < 0$, $x'(t) > 0$ for $t \geqslant t_0$, a similar argument holds. ∎

For the existence of unbounded solutions, the following holds:

THEOREM 2.6. *If $I_r < +\infty$, then $M_\infty^+ = \emptyset$.*

Proof. Let $x \in M_\infty^+$ and let $x(t) > 0$, $x'(t) > 0$, for $t \geqslant t_0$. From (1) it follows that the function $r(\cdot)\, x'(\cdot)$ is positive decreasing for $t \geqslant t_0$. Hence

$$r(t)\, x'(t) < r(t_0)\, x'(t_0).$$

Integrating in $[t_0, t]$ we have

$$x(t) < x(t_0) + r(t_0)\, x'(t_0) \int_{t_0}^{t} \frac{1}{r(s)}\, ds,$$

therefore x is a bounded solution, which is a contradiction. If $x(t) < 0$, $x'(t) < 0$ for $t \geqslant t_0$, a similar argument holds. ∎

3

In this section we present sufficient conditions for the existence of eventually monotone and bounded solutions of (1), that is solutions belonging to the classes M_B^-, M_B^+, M_0^-. In order to prove such results, we use a topological tool and employ a result on continuity and compactness of operators associated to boundary value problems in noncompact intervals ([5]; see also [7], pp. 22–23, for more details). The following holds:

THEOREM 3.1. *If $I_{rq} < +\infty$, $M_B^- \neq \emptyset$.*

Proof. We prove the existence of eventually positive decreasing solutions of (1) which approach a nonzero limit as $t \to +\infty$. We observe that in the same way the existence of eventually negative increasing solutions of (1), which approach a nonzero limit as $t \to +\infty$, can be obtained.

Let $K = \max_{u \in [1/2,\, 1]} f(u)$ and t_0 such that

$$K \int_{t_0}^{+\infty} \frac{1}{r(\tau)} \int_{t_0}^{\tau} q(s)\, ds\, d\tau \leqslant \frac{1}{2}.$$

Define $\Omega = \{u \in C([t_0, +\infty), \mathbb{R}) : \frac{1}{2} \leqslant u(t) \leqslant 1\}$, where $C([t_0, +\infty), \mathbb{R})$ denotes the Fréchet space of continuous real functions defined on $[t_0, +\infty)$, endowed with the topology of the uniform convergence on the compacts of

$[t_0, +\infty)$. For every $u \in \Omega$ let $x_u = Tu$ the unique solution, defined in $[t_0, +\infty)$, of the Cauchy problem:

$$[r(t)\,x'(t)]' + q(t)\,f(u(t)) = 0$$
$$x(t_0) = 1, \qquad x'(t_0) = 0.$$

It is easy to see that $x_u'(t) < 0$ for $t > t_0$. Hence

$$x_u(t) = (Tu)(t) < x(t_0) = 1 \qquad \text{for} \quad t > t_0.$$

From $[r(t)\,x_u'(t)]' = -q(t)\,f(u(t)) \geqslant -K\,q(t)$, integrating twice in $[t_0, t]$ we get

$$x_u(t) \geqslant 1 - K \int_{t_0}^{t} \frac{1}{r(\tau)} \int_{t_0}^{\tau} q(s)\,ds\,d\tau,$$

hence $x_u(+\infty) \geqslant \frac{1}{2}$ and so $T(\Omega) \subset \Omega$. The above quoted result in [5] gives continuity and compactness for the operator T in Ω. From Schauder–Tychonov Theorem we have now the existence of a fixed point x for the operator T. Clearly x is an eventually positive decreasing solution of (1) such that $x(+\infty) = l_x$ with $l_x \neq 0$, hence $M_B^- \neq \emptyset$. ∎

From this result and Theorem 2.5 one gets immediately the following:

THEOREM 3.2. *Assumption $I_{rq} = +\infty$ is a necessary and sufficient condition for having $M_B^- = \emptyset$.*

We now prove the existence of solutions in M_B^+. The following holds:

THEOREM 3.3. *If $I_{qr} < +\infty$, then $M_B^+ \neq \emptyset$.*

Proof. Again we prove the existence of eventually positive increasing bounded solutions. The case of eventually negative decreasing bounded solutions can be treated in a similar way.

Let $K = \max_{u \in [1,2]} f(u)$ and let t_0 be such that

$$K \int_{t_0}^{+\infty} q(\tau) \int_{t_0}^{\tau} \frac{1}{r(s)}\,ds\,d\tau \leqslant \frac{1}{2}.$$

Define $\Omega = \{u \in C([t_0, +\infty), \mathbb{R}) : 1 \leqslant u(t) \leqslant 2\}$, and for every $u \in \Omega$ let $x_u = Tu$ be the function given by

$$x_u(t) = 1 + \int_{t_0}^{t} q(\tau)\,f(u(\tau)) \int_{t_0}^{\tau} \frac{1}{r(s)}\,ds\,d\tau + \left(\int_{t_0}^{t} \frac{1}{r(s)}\,ds\right)\left(\int_{t}^{+\infty} q(s)\,f(u(s))\,ds\right).$$

Clearly $x_u(t) \geqslant 1$. Moreover one has

$$\begin{aligned} x_u(t) &\leqslant 1 + K\int_{t_0}^{+\infty} q(\tau)\int_{t_0}^{\tau}\frac{1}{r(s)}\,ds\,d\tau + K\int_{t}^{+\infty} q(\tau)\int_{t_0}^{t}\frac{1}{r(s)}\,ds\,d\tau \\ &\leqslant 1 + K\int_{t_0}^{+\infty} q(\tau)\int_{t_0}^{\tau}\frac{1}{r(s)}\,ds\,d\tau + K\int_{t}^{+\infty} q(\tau)\int_{t_0}^{\tau}\frac{1}{r(s)}\,ds\,d\tau \\ &\leqslant 1 + 2K\int_{t_0}^{+\infty} q(\tau)\int_{t_0}^{\tau}\frac{1}{r(s)}\,ds\,d\tau \leqslant 2. \end{aligned}$$

Hence $T(\Omega) \subset \Omega$. In order to prove continuity and compactness for the operator T, we notice that from $I_{qr} < +\infty$ we have $I_q < +\infty$. Hence

$$\int_{t_0}^{+\infty} q(s)\,f(u(s))\,ds < +\infty \qquad \forall u \in \Omega.$$

Continuity and compactness of T follow now easily from the quoted result in [5]. Again, Schauder–Tychonov Theorem gives the existence of a fixed point x for the operator T, that is the existence of $x \in \Omega$ such that

$$x = 1 + \int_{t_0}^{t} q(\tau)\,f(x(\tau))\int_{t_0}^{\tau}\frac{1}{r(s)}\,ds\,d\tau + \left(\int_{t_0}^{t}\frac{1}{r(s)}\,ds\right)\left(\int_{t}^{+\infty} q(s)\,f(x(s))\,ds\right).$$

As

$$x'(t) = \frac{1}{r(t)}\int_{t}^{+\infty} q(s)\,f(x(s))\,ds > 0$$

we get $x \in M^+$ and so, being $1 \leqslant x(t) \leqslant 2$, we obtain $x \in M_B^+$. ∎

From this result and Theorem 2.4 one gets immediately the following:

THEOREM 3.4. *Assumption $I_{qr} = +\infty$ is a necessary and sufficient condition for having $M_B^+ = \emptyset$.*

For the existence of solutions in M_0^- we just recall the following result which was proved in [9]:

THEOREM 3.5. *If $I_{rq} < +\infty$, then $M_0^- \neq \emptyset$.*

Remark 3.1. From Theorem 3.5 we get that $M_0^- = \emptyset$ implies $I_{rq} = +\infty$. However such condition is only necessary for having $M_0^- = \emptyset$, and results similar to the previous Theorems 3.2 and 3.4 do not hold, as the equation with constant coefficients

$$x'' + ax' + bx = 0 \qquad (a, b > 0,\ a^2 > 4b)$$

shows.

Furthermore we emphasize that, for all the results previously stated in this section, assumption $u \cdot f(u) > 0$ for $u \neq 0$ is the only requirement for the forcing term f. Assuming additional conditions on the nonlinearity, we have the following result which still deals with the existence of solutions in M_0^-.

THEOREM 3.6. *Let $I_r < +\infty$. If f is nondecreasing and such that*

$$(\mathrm{H}_2) \quad \int_0^{+\infty} q(\tau) f\left(\int_\tau^{+\infty} \frac{1}{r(s)} ds\right) d\tau < +\infty,$$

then $M_0^- \neq \emptyset$.

Proof. Let t_0 be such that

$$\int_{t_0}^{+\infty} q(\tau) f\left(\int_\tau^{+\infty} \frac{1}{r(s)} ds\right) d\tau < 1$$

and define $\Omega = \{u \in C([t_0, +\infty), \mathbb{R}) : 0 \leqslant u(t) \leqslant \int_t^{+\infty} (1/r(s))\, ds\}$. For every $u \in \Omega$, let T be the operator $T: \Omega \to C([t_0, +\infty))$ given by

$$x_u = (Tu)(t) = \int_t^{+\infty} \frac{1}{r(\tau)} \int_{t_0}^{\tau} q(s) f(u(s))\, ds\, d\tau$$

Taking into account that f is nondecreasing, we have

$$\begin{aligned} x_u &= \int_t^{+\infty} \frac{1}{r(\tau)} \int_{t_0}^{\tau} q(s) f(u(s))\, ds\, d\tau \leqslant \int_t^{+\infty} \frac{1}{r(\tau)} \int_{t_0}^{+\infty} q(s) f(u(s))\, ds\, d\tau \\ &\leqslant \int_t^{+\infty} \frac{1}{r(\tau)} \int_{t_0}^{+\infty} q(s) f\left(\int_s^{+\infty} \frac{1}{r(\theta)} d\theta\right) ds\, d\tau \\ &= \left(\int_{t_0}^{+\infty} q(s) f\left(\int_s^{+\infty} \frac{1}{r(\theta)} d\theta\right) ds\right)\left(\int_t^{+\infty} \frac{1}{r(\tau)} d\tau\right) \leqslant \int_t^{+\infty} \frac{1}{r(\tau)} d\tau. \end{aligned}$$

Since $x_u(t) \geqslant 0$, we obtain that $T(\Omega) \subset \Omega$. The compactness and the continuity of the operator T in Ω follows easily once again from the above mentioned result in [5]. Hence there exists $x \in \Omega$ such that $x = Tx$. Clearly x is an eventually positive decreasing solution of (1) which tends to zero as $t \to +\infty$. The proof is now complete. ∎

Remark 3.2. We observe that in the linear case assumption (H_2) becomes $I_{rq} < +\infty$. However, (H_2) may be satisfied with $I_{rq} = +\infty$. Consider, for instance, the equation

$$[(1+t)^2 x']' + x^2 \operatorname{sgn} x = 0;$$

here (H_2) is satisfied and $I_r < +\infty$, $I_q = I_{qr} = I_{rq} = +\infty$. From Theorem 3.6 such an equation has solutions in the class M_0^-. The same result does not hold for the corresponding linear equation

$$[(1+t)^2 x']' + x = 0 \tag{3}$$

which oscillates, as can be seen using, for example, the following argument. One compares (3) with the "dual" equation

$$x'' + \frac{1}{(1+t)^2} x = 0; \tag{4}$$

it is easy to see that (4) oscillates being the classical Euler equation with $\gamma = 1$ (see, e.g., [21]). From the duality principle (see, e.g., [8; 19, p. 474]), (3) and (4) have the same oscillatory behavior.

Remark 3.3. Theorem 3.6 is still valid if we have, instead of (H_2), the following assumption

$$(H_{2'}) \quad \int_0^{+\infty} q(\tau) f\left(-\int_\tau^{+\infty} \frac{1}{r(s)} ds\right) d\tau > -\infty.$$

Indeed in this case, with a similar argument, we get the existence of eventually negative increasing solutions of (1) which tend to zero as $t \to +\infty$.

Remark 3.4. If f is a Lipschitz function, i.e. $|f(u)| \leqslant K|u|$, and $I_r < +\infty$, $I_{qr} < +\infty$, then assumptions (H_2)–$(H_{2'})$ hold, since

$$\int_0^{+\infty} q(\tau) f\left(\int_\tau^{+\infty} \frac{1}{r(\theta)} d\theta\right) d\tau \leqslant K \int_0^{+\infty} q(\tau) \int_\tau^{+\infty} \frac{1}{r(\theta)} d\theta\, d\tau$$

$$= K \int_0^{+\infty} \frac{1}{r(\tau)} \int_0^\tau q(s)\, ds\, d\tau < +\infty.$$

4

In this section we consider the problem of existence of eventually monotone and unbounded solutions of (1). Hence we deal with the class M_∞^+.

We notice that in this case, as in the case of the existence of solutions in the class M_0^-, some other assumption of integral type, concerning the forcing term f and the functions r, q, will be necessary. The following holds:

THEOREM 4.1. *Let f be nondecreasing and such that*

$$(\mathrm{H}_3) \quad \int_0^{+\infty} q(\tau)\, f\left(\int_0^{\tau} \frac{1}{r(s)}\, ds\right) d\tau < +\infty.$$

If $I_r = +\infty$, then $M_\infty^+ \neq \emptyset$.

Proof. Let t_0 such that

$$\int_{t_0}^{+\infty} q(\tau)\, f\left(\int_{t_0}^{\tau} \frac{1}{r(s)}\, ds\right) d\tau < \frac{1}{2},$$

and define $\Omega = \{u \in C([t_0, +\infty), \mathbb{R}) : 0 \leqslant u(t) \leqslant \int_{t_0}^{t} (1/r(s))\, ds\}$. For every $u \in \Omega$ consider the problem

$$\begin{aligned} &[r(t)\, x'(t)]' + q(t)\, f(u(t)) = 0 \\ &x(t_0) = 0 \\ &x'(t_0) = \frac{1}{r(t_0)} \int_{t_0}^{+\infty} q(\tau)\, f\left(\int_{t_0}^{\tau} \frac{1}{r(s)}\, ds\right) d\tau + \frac{1}{2r(t_0)}. \end{aligned} \tag{5}$$

From variation of constants formula, the unique solution of (5) is given by

$$\begin{aligned} x_u(t) = {} & \left(\int_{t_0}^{+\infty} q(\tau)\, f\left(\int_{t_0}^{\tau} \frac{1}{r(s)}\, ds\right) d\tau\right) \cdot \int_{t_0}^{t} \frac{1}{r(s)}\, ds \\ & - \int_{t_0}^{t} \frac{1}{r(\tau)} \int_{t_0}^{\tau} q(s)\, f(u(s))\, ds\, d\tau + \frac{1}{2} \int_{t_0}^{t} \frac{1}{r(s)}\, ds. \end{aligned}$$

Let T be the operator which associates to every $u \in \Omega$ the unique solution x_u of (5). We have

$$\begin{aligned} x_u(t) = {} & \int_{t_0}^{t} \frac{1}{r(s)} \left[\int_{t_0}^{+\infty} q(\tau)\, f\left(\int_{t_0}^{\tau} \frac{1}{r(\theta)}\, d\theta\right) d\tau - \int_{t_0}^{s} q(\tau)\, f(u(\tau))\, d\tau\right] ds \\ & + \frac{1}{2} \int_{t_0}^{t} \frac{1}{r(s)}\, ds \\ \geqslant {} & \int_{t_0}^{t} \frac{1}{r(s)} \left[\int_{t_0}^{+\infty} q(\tau)\, f\left(\int_{t_0}^{\tau} \frac{1}{r(\theta)}\, d\theta\right) d\tau - \int_{t_0}^{s} q(\tau)\, f\left(\int_{t_0}^{\tau} \frac{1}{r(\theta)}\, d\theta\right) d\tau\right] ds \\ & + \frac{1}{2} \int_{t_0}^{t} \frac{1}{r(s)}\, ds \\ = {} & \int_{t_0}^{t} \frac{1}{r(s)} \int_{s}^{+\infty} q(\tau)\, f\left(\int_{t_0}^{\tau} \frac{1}{r(\theta)}\, d\theta\right) d\tau\, ds + \frac{1}{2} \int_{t_0}^{t} \frac{1}{r(s)}\, ds \geqslant 0. \end{aligned}$$

Moreover

$$x_u(t)=\int_{t_0}^{t}\frac{1}{r(s)}\left[\int_{t_0}^{+\infty}q(\tau)\,f\left(\int_{t_0}^{\tau}\frac{1}{r(\theta)}\,d\theta\right)d\tau-\int_{t_0}^{s}q(\tau)\,f(u(\tau))\,d\tau\right]ds$$
$$+\frac{1}{2}\int_{t_0}^{t}\frac{1}{r(s)}\,ds$$
$$\leqslant\int_{t_0}^{t}\frac{1}{r(s)}\int_{t_0}^{+\infty}q(\tau)\,f\left(\int_{t_0}^{\tau}\frac{1}{r(\theta)}\,d\theta\right)d\tau\,ds+\frac{1}{2}\int_{t_0}^{t}\frac{1}{r(s)}\,ds$$
$$\leqslant\frac{1}{2}\int_{t_0}^{t}\frac{1}{r(s)}\,ds+\frac{1}{2}\int_{t_0}^{t}\frac{1}{r(s)}=\int_{t_0}^{t}\frac{1}{r(s)}\,ds.$$

Hence $T(\Omega)\subset\Omega$. The compactness and the continuity of the operator T in Ω follow easily once again from the above mentioned result in [5]. Hence there exists $x\in\Omega$ such that $x=Tx$.

Let us prove that $x\in M_{\infty}^{+}$. We have

$$r(t)\,x'(t)=\int_{t_0}^{+\infty}q(s)\,f\left(\int_{t_0}^{s}\frac{1}{r(\theta)}\,d\theta\right)ds-\int_{t_0}^{t}q(s)\,f(x(s))\,ds+\frac{1}{2}$$
$$\geqslant\int_{t_0}^{+\infty}q(s)\,f(x(s))\,ds-\int_{t_0}^{t}q(s)\,f(x(s))\,ds+\frac{1}{2}$$
$$=\int_{t}^{+\infty}q(s)\,f(x(s))\,ds+\frac{1}{2}>0.$$

Integrating in $[t_0, t]$ we obtain

$$x(t)=\int_{t_0}^{t}\frac{1}{r(s)}\left[\int_{t_0}^{+\infty}q(\tau)\,f\left(\int_{t_0}^{\tau}\frac{1}{r(\theta)}\,d\theta\right)d\tau-\int_{t_0}^{s}q(\tau)\,f(x(\tau))\,d\tau\right]ds$$
$$+\frac{1}{2}\int_{t_0}^{t}\frac{1}{r(s)}\,ds$$
$$\geqslant\int_{t_0}^{t}\frac{1}{r(s)}\left[\int_{t_0}^{+\infty}q(\tau)\,f(x(\tau))\,d\tau-\int_{t_0}^{s}q(\tau)\,f(x(\tau))\,d\tau\right]ds+\frac{1}{2}\int_{t_0}^{t}\frac{1}{r(s)}\,ds$$
$$=\int_{t_0}^{t}\frac{1}{r(s)}\int_{t_0}^{+\infty}q(\tau)\,f(x(\tau))\,d\tau\,ds+\frac{1}{2}\int_{t_0}^{t}\frac{1}{r(s)}\,ds\geqslant\frac{1}{2}\int_{t_0}^{t}\frac{1}{r(s)}\,ds$$

and so $x(+\infty)=+\infty$. The proof is now complete. ∎

Remark 4.1. We observe that in the linear case assumption (H_3) becomes $I_{qr}<+\infty$. Therefore Theorem 4.1 extends to Eq. (1) a previous result given in [8] for the linear equation (2). However, in our case, (H_3) may be satisfied with $I_{qr} = +\infty$. Consider, for instance, the equation

$$[e^{-t}x']' + 4e^{-t}f(x) = 0,$$

where $f(u) = \sqrt{u}$ for u large enough; here (H_3) is satisfied and $I_{qr} = +\infty$. From the previous result such equation has solutions in the class M_∞^+. As can easily be seen, the same result does not hold for the corresponding linear equation

$$[e^{-t}x']' + 4e^{-t}x = 0$$

which oscillates.

Remark 4.2. Theorem 4.1 is still valid if we have, instead of (H_3), the following assumption

$$(H_{3'}) \quad \int_0^{+\infty} q(\tau)\, f\left(-\int_0^\tau \frac{1}{r(s)}\, ds\right) d\tau > -\infty.$$

Indeed in this case, with a similar argument, we get the existence of eventually negative decreasing and unbounded solutions of (1).

Remark 4.3. If f is a Lipschitz function, i.e., $|f(u)| \leqslant K\,|u|$, and $I_{qr} < +\infty$, then assumptions (H_3)–$(H_{3'})$ hold, since

$$\int_0^{+\infty} q(\tau)\, f\left(\int_0^\tau \frac{1}{r(\theta)}\, d\theta\right) d\tau \leqslant K \int_0^{+\infty} q(\tau) \int_0^\tau \frac{1}{r(\theta)}\, d\theta\, d\tau < +\infty.$$

From Theorems 2.1–4.1 and from Remark 4.2 we get the following:

THEOREM 4.2. *Let f be nondecreasing and assumptions (H_3)–$(H_{3'})$ hold. If $I_q < +\infty$, then assumption $I_r < +\infty$ is a necessary and sufficient condition for having $M_\infty^+ = \varnothing$.*

Proof. The assertion follows easily taking into account that assumption (H_1) holds since f is nondecreasing. ∎

5

We recall again that in case (C_1), if f satisfies assumption (H_1), then as a consequence of Theorem 2.3 all continuable solutions of (1) are oscillatory. For the other cases (C_2), ..., (C_6), as was done for the linear equation in [8], all the previous results can be summarized in the following way:

THEOREM 5.1. (a) *If* (C_2) *holds, then every continuable solution of* (1) *is oscillatory or belongs to the class* M_{∞}^{+}. *If moreover f is nondecreasing and assumption* (H_3) *or* $(H_{3'})$ *holds, then the class* M_{∞}^{+} *is not empty.*

(b) *If* (C_3) *holds, then every nonoscillatory solution of* (1) *belongs to the class* M^{+} *and there exist solutions in* M_{B}^{+}. *If moreover f is nondecreasing and assumption* (H_3) *or* $(H_{3'})$ *holds, then the class* M_{∞}^{+} *is not empty.*

(c) *If* (C_4) *holds, then every continuable solution of* (1) *is oscillatory or in the class* M_0^{-}. *If moreover f is nondecreasing and assumption* (H_2) *or* $(H_{2'})$ *holds, then the class* M_0^{-} *is not empty.*

(d) *If* (C_5) *holds, then every nonoscillatory solution of* (1) *belongs to the class* M^{-}. *Moreover both classes* M_B^{-} *and* M_0^{-} *are not empty.*

(e) *If* (C_6) *holds, then every nonoscillatory solution of* (1) *is bounded. Moreover none of the classes* M_B^{-}, M_0^{-}, M_B^{+} *is empty.*

Proof. Claim (a) comes from Theorems 2.2, 2.4, 4.1 and Remark 4.2. Claim (b) comes from Theorems 2.2, 3.3, 4.1 and Remark 4.2. Claim (c) comes from Theorems 2.1, 2.5, 2.6, 3.6 and Remark 3.3. Claim (d) comes from Theorems 2.1, 2.6, 3.1, 3.5. Claim (e) comes from Theorems 2.6, 3.1, 3.3, 3.5. ∎

Under rather mild assumptions on the forcing term f, the problem of existence of solutions of (1) belonging to the classes M^{-}, M^{+} is indeed similar to the corresponding linear case, as shown by a comparison between Theorem 1.1 and 5.1. On the other hand, as Sturm Theorem fails in the nonlinear case, the existence of a single nonoscillatory solution does not ensure, in general, that every solution is nonoscillatory. In this light, it is exhibited in Tables I and II a summary which is valid for the linear case and the nonlinear one respectively (where necessary, it is assumed that f satisfies the assumptions (H_1), (H_2), $(H_{2'})$, (H_3), $(H_{3'})$ required in the previous theorems). We stress the fact that, as was already mentioned in [8] for Table I, the results obtained in Table II are mutually exclusive and exhaustive, as is easily checked.

Finally from Theorem 5.1 the following results, which generalize to the nonlinear case previous results in [12, p. 354; 19], easily follow:

THEOREM 5.2. *Let f be nondecreasing and suppose that assumption* (H_3) *or* $(H_{3'})$ *holds. If* $I_r = +\infty$ *and all the solutions of* (1) *are bounded, then every continuable solution of* (1) *is oscillatory.*

THEOREM 5.3. *Let f be nondecreasing and suppose that assumption* (H_3) *or* $(H_{3'})$ *holds. Then the assumption* $I_r < +\infty$ *is a necessary and sufficient condition for having that all the continuable nonoscillatory solutions of* (1) *are bounded.*

TABLE I

$$[r(t)\,x'(t)]' + q(t)\,x(t) = 0$$

(C_1)	$\begin{bmatrix} I_q = I_r = +\infty \\ I_{qr} = I_{rq} = +\infty \end{bmatrix}$	$\Leftrightarrow$	$[S = \mathbb{O}]$
(C_2)	$\begin{bmatrix} I_q < +\infty,\ I_r = +\infty \\ I_{qr} = I_{rq} = +\infty \end{bmatrix}$	$\Leftrightarrow$	$\begin{bmatrix} M^- = \varnothing \\ M_B^+ = \varnothing \end{bmatrix}$
(C_3)	$\begin{bmatrix} I_q < +\infty,\ I_r = +\infty \\ I_{qr} < +\infty,\ I_{rq} = +\infty \end{bmatrix}$	$\Leftrightarrow$	$\begin{bmatrix} M^- = \varnothing \quad \mathbb{O} = \varnothing \\ S = M_B^+ \cup M_\infty^+,\ M_B^+ \neq \varnothing \quad M_\infty^+ \neq \varnothing \end{bmatrix}$
(C_4)	$\begin{bmatrix} I_q = +\infty,\ I_r < +\infty \\ I_{qr} = I_{rq} = +\infty \end{bmatrix}$	$\Leftrightarrow$	$[M_B^- = \varnothing \quad M^+ = \varnothing]$
(C_5)	$\begin{bmatrix} I_q = +\infty,\ I_r < +\infty \\ I_{qr} = +\infty,\ I_{rq} < +\infty \end{bmatrix}$	$\Leftrightarrow$	$\begin{bmatrix} M^+ = \varnothing \quad \mathbb{O} = \varnothing \\ S = M_B^- \cup M_0^-,\ M_B^- \neq \varnothing \quad M_0^- \neq \varnothing \end{bmatrix}$
(C_6)	$\begin{bmatrix} I_q < +\infty,\ I_r < +\infty \\ I_{qr} < +\infty,\ I_{rq} < +\infty \end{bmatrix}$	$\Leftrightarrow$	$\begin{bmatrix} M_\infty^+ = \varnothing \quad \mathbb{O} = \varnothing \\ M_B^- \neq \varnothing \quad M_0^- \neq \varnothing \quad M_B^+ \neq \varnothing \end{bmatrix}$

TABLE II

$$[r(t)\,x'(t)]' + q(t)\,f(x(t)) = 0$$

(C_1)	$\begin{bmatrix} I_q = I_r = +\infty \\ I_{qr} = I_{rq} = +\infty \end{bmatrix}$	$\Leftrightarrow$	$[S = \mathbb{O}]$
(C_2)	$\begin{bmatrix} I_q < +\infty,\ I_r = +\infty \\ I_{qr} = I_{rq} = +\infty \end{bmatrix}$	$\Leftrightarrow$	$[M^- = \varnothing \quad M_B^+ = \varnothing \quad M_\infty^+ \neq \varnothing]$
(C_3)	$\begin{bmatrix} I_q < +\infty,\ I_r = +\infty \\ I_{qr} < +\infty,\ I_{rq} = +\infty \end{bmatrix}$	$\Leftrightarrow$	$[M_B^+ \neq \varnothing \quad M_\infty^+ \neq \varnothing \quad M^- = \varnothing]$
(C_4)	$\begin{bmatrix} I_q = +\infty,\ I_r < +\infty \\ I_{qr} = I_{rq} = +\infty \end{bmatrix}$	$\Leftrightarrow$	$[M_B^- = \varnothing \quad M_0^- \neq \varnothing \quad M^+ = \varnothing]$
(C_5)	$\begin{bmatrix} I_q = +\infty,\ I_r < +\infty \\ I_{qr} = +\infty,\ I_{rq} < +\infty \end{bmatrix}$	$\Leftrightarrow$	$[M_B^- \neq \varnothing \quad M_0^- \neq \varnothing \quad M^+ = \varnothing]$
(C_6)	$\begin{bmatrix} I_q < +\infty,\ I_r < +\infty \\ I_{qr} < +\infty,\ I_{rq} < +\infty \end{bmatrix}$	$\Leftrightarrow$	$\begin{bmatrix} M_B^- \neq \varnothing \quad M_0^- \neq \varnothing \\ M_B^+ \neq \varnothing \quad M_\infty^+ = \varnothing \end{bmatrix}$

Note. When it is necessary, we assume that f satisfies the assumptions (H_i), $i = 1, 2, 3$.

REFERENCES

1. N. P. Bhatia, Some oscillation theorems for second order differential equation, *J. Math. Anal. Appl.* **15** (1966), 442–446.
2. T. Burton and R. Grimmer, On continuability of solutions of second order differential equations, *Proc. Amer. Math. Soc.* **29** (1971), 277–289.
3. G. J. Butler, The existence of continuable solutions of a second order differential equation, *Can. J. Math.* **29** (1977), 472–479.
4. G. J. Butler and L. H. Erbe, A generalization of Olech-Opial-Wazewski oscillation criteria to second order nonlinear equations, *Nonlinear Anal. T.M.A.* **11**, No. 2 (1987), 207–219.
5. M. Cecchi, M. Furi, and M. Marini, On continuity and compactness of some nonlinear operator associated with differential equations in non compact intervals, *Nonlinear Anal. T.M.A.* **9**, No. 2 (1985), 171–180.
6. M. Cecchi and M. Marini, Oscillatory and nonoscillatory behavior of a second order functional differential equation, *Rocky Mountain J. Math.* **22**(4) (1992), 1259–1275.
7. M. Cecchi, M. Marini, and Gab. Villari, On the monotonicity property for a certain class of second order differential equations, *J. Differential Equations* **82** (1989), 15–27.
8. M. Cecchi, M. Marini, and Gab. Villari, Integral criteria for a classification of solutions of linear differential equations, *J. Differential Equations* **99** (1992), 381–397.
9. M. Cecchi, M. Marini, and Gab. Villari, On the qualitative behavior of convergent solutions of a second order nonlinear differential equation, *in* "Ordinary and Delay Differential Equations" (J. Wiener and J. K. Hale, Eds.), Pitman Research Notes in Math. Series, Vol. 272, pp. 259–263, Longman, Harlow/New York, 1992.
10. M. Cecchi, M. Marini, and Gab. Villari, Topological and variational approaches for nonlinear oscillation: An extension of a Bhatia result, *in* "Proc. of World Congress on Nonlinear Analysts, Tampa Aug. 1992," de Gruyter, Berlin, to appear.
11. L. H. Erbe and H. Lu, Nonoscillation theorems for second order nonlinear differential equations, *Funk. Ekv.* **33** (1990), 227–244.
12. P. Hartman, "Ordinary Differential Equations," Wiley, New York, 1964.
13. P. Hartman and A. Wintner, Linear differential and difference equations with monotone solutions, *Amer. J. Math.* **75** (1953), 731–743.
14. V. Liberto Jannelli, Proprietà di parziale e completa oscillatorietà per le soluzioni di equazioni differenziali lineari ordinarie del secondo e del terzo ordine, *Boll. Un. Mat. Ital. C (6)* **3** (1984), 171–187.
15. M. Marini, On nonoscillatory solutions of a second order nonlinear differential equation, *Boll. Un. Mat. Ital. C (6)* **3** (1984), 189–202.
16. M. Marini, Monotone solutions of a class of second order nonlinear differential equations, *Nonlinear Anal.* **8** (1984), 261–271.
17. M. Marini and P. Zezza, On the asymptotic behavior of the solutions of a class of second order linear differential equations, *J. Differential Equations* **28** (1978), 1–17.
18. M. Naito, Asymptotic behavior of solutions of second order differential equations with integrable coefficients, *Trans. Amer. Math. Soc.* **282** (1984), 577–588.
19. R. L. Potter, On self-adjoint differential equations of second order, *Pacific J. Math.* **3** (1953), 467–491.
20. P. Pucci and J. Serrin, Continuation and limit properties for solutions of strongly nonlinear second order differential equations, *Asymptotic Anal.* **4**, No. 2 (1991), 97–160.
21. C. A. Swanson, "Comparison and Oscillation Theory of Linear Differential Equations," Academic Press, New York, 1968.
22. D. Willett and J. S. W. Wong, Some properties of the solutions of $(p(t)\,x')' + q(t)\,f(x) = 0$, *J. Math. Anal. Appl.* **23** (1968), 15–24.
23. J. S. W. Wong, On second order nonlinear oscillation, *Funk. Ekv.* **11** (1969), 207–234.
24. J. S. W. Wong, On the generalized Emden–Fowler equation, *SIAM Rev.* **17** (1975), 339–360.