The double mass hierarchy pattern: Simultaneously understanding quark and lepton mixing

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Abstract

The charged fermion masses of the three generations exhibit the two strong hierarchies $m_3 \gg m_2 \gg m_1$. We assume that also neutrino masses satisfy $m_{\nu 3} > m_{\nu 2} > m_{\nu 1}$ and derive the consequences of the hierarchical spectra on the fermionic mixing patterns. The quark and lepton mixing matrices are built in a general framework with their matrix elements expressed in terms of the four fermion mass ratios, m_u/m_c, m_c/m_t, m_d/m_s and m_s/m_b, and m_e/m_μ, m_μ/m_τ, $m_{\nu 1}/m_{\nu 2}$ and $m_{\nu 2}/m_{\nu 3}$, for the quark and lepton sector, respectively. In this framework, we show that the resulting mixing matrices are consistent with data for both quarks and leptons, despite the large leptonic mixing angles. The minimal assumption we take is the one of hierarchical masses and minimal flavor symmetry breaking that strongly follows from phenomenology. No special structure of the mass matrices has to be assumed that cannot be motivated by this minimal assumption. This analysis allows us to predict the neutrino mass spectrum and set the mass of the lightest neutrino well below 0.01 eV. The method also gives the 1σ allowed ranges for the leptonic mixing matrix elements. Contrary to the common expectation, leptonic mixing angles are found to be determined solely by the four leptonic mass ratios without any relation to symmetry considerations as commonly used in flavor model building. Still, our formulae can be used to build up a flavor model that predicts the observed hierarchies in the masses — the mixing follows then from the procedure which is developed in this work.
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1. Introduction

The Standard Model of particle physics (SM) describes the interactions among elementary particles at high energies with great success. In spite of this, the setup of the SM lacks an explanation of the origin of fermion masses and mixing. In particular, for the quark sector, one observes six masses, three mixing angles and one phase. It is a simple exercise to relate the quark mixing matrix to the fundamental parameters of the theory, the Yukawa couplings. Generally, however, it is said that mixing angles as well as the masses are *independent* free parameters. Is there really no functional relation between the quark masses and the corresponding mixing matrix elements? There are many models in the literature that try to give an explanation of the mixing matrix elements in terms of the masses [1–28]. Most of them put assumptions on a specific texture in the original mass matrices. We shall show, by contrast, that the pure phenomenological observation of strong hierarchies in the quark masses leads to a functional description of the mixing matrix elements in terms of mass ratios. The consequences in the mixing of this phenomenological observation have already been studied [15,20,26,29–33]. Our approach differs from the previous ones in many aspects: (i) we take the Singular Value Decomposition of the complex mass matrices as a starting point offering a generic treatment for both quarks and leptons; (ii) by means of an approximation theorem we mathematically formulate the steps to build the reparametrization of the mixing matrix in terms of the singular values (fermion masses); (iii) we rotate the mass matrices in all three planes of family space whereas before, the 1–3 rotation was neglected; (iv) as the two unitary rotations in the 2–3 and 1–3 planes involve an approximation ($m_{f,1} = 0$ and $m_{f,2} = 0$, respectively) we consider for the first time a modified method of perturbation theory to add the effect of the terms neglected; (v) we do not consider the complex CP phases as free parameters and show that a minimal choice is sufficient to explain CP data; (vi) we provide explicit formulae for the mixing angles in terms of only mass ratios.

The applicability of this formulation to the leptonic mixing is not clear *a priori*. First, neutrino masses do not show any strong hierarchy, at best a very mild one. Second, the leptonic mixing matrix exhibits large mixing, while the one in the quark sector is rather close to the unit matrix. This picture seems to suggest two quite different origins for the respective mixing matrices: quark masses strongly dominating the mixing patterns, whereas geometrical factors found from symmetries shaping the leptonic mixing, with only a weak intervention from the lepton masses [34,35].

Fermion masses, on the other hand, are also as puzzling as the mixing matrices: the top quark mass is by far the largest among the charged fermions, there are six orders of magnitude separating the top quark from the electron mass, six orders of magnitude separating the largest neutrino mass from the electron mass (assuming a neutrino mass scale of 0.1 eV). There are three orders of magnitude between the masses of the up-type quarks, whereas two orders of magnitudes in the down-quark sector. Top and bottom quark are separated by two orders of magnitude — the lightest charged lepton and the heaviest quark by again six orders of magnitude. Within each (charged) fermion species ($f = u, d, e$), the masses follow a hierarchy $m_{f,3} \gg m_{f,2} \gg m_{f,1}$,

$$m_u : m_c : m_t \approx 10^{-6} : 10^{-3} : 1, \qquad m_d : m_s : m_b \approx 10^{-4} : 10^{-2} : 1,$$
$$m_e : m_\mu : m_\tau \approx 10^{-4} : 10^{-2} : 1, \tag{1}$$

while the two squared mass differences measured from neutrino oscillations obey a much weaker hierarchy,

$$\Delta m_{21}^2 : \Delta m_{31(32)}^2 \approx 10^{-2} : 1. \tag{2}$$

Quark masses plus mixing parameters sum up to ten arbitrary physical parameters in the SM. Consideration of neutrino masses, whether Dirac or Majorana, adds at least ten more parameters to the count. Two more complex phases and a possibly arbitrary number of masses for sterile neutrinos appear in the more general cases including Majorana neutrinos [36]. The SM *per se* seems to lack a course of action on how to relate the mixing matrix elements to the corresponding fermion masses.

The first realization of a mixing angle in terms of the masses is commonly assigned to Gatto et al. [1] which is referred to as the Gatto–Sartori–Tonin relation. This relation is an expression of the Cabibbo angle commonly written as

$$\theta_{12}^q \approx \sqrt{\frac{m_d}{m_s}}, \tag{3}$$

where originally, the authors of [1] found a similar relation in terms of light meson masses from the demand of weak self-masses being free from quadratic divergences. In a footnote, they break it down to an elementary discussion in a "naive quark model" and state

$$\tan^2\theta = \frac{m_n - m_p}{m_p} = \frac{m_n}{m_\lambda}, \tag{4}$$

where m_n, m_p, and m_λ are the old notations of down-, up-, and strange-quark masses (moreover, the second equal sign was misleadingly written as a minus sign). The first work referring to [1] as origin of "$\tan\theta = m_n/m_\lambda$" was [3] (even though with a typo in the abstract). For small angles, $\tan\theta \approx \theta$ and we are at Eq. (3). Since $\sqrt{m_d/m_s}$ is an astonishingly good approximation for the Cabibbo angle, we will show in the course of this paper how to rearrive at this expression in a formal way of parametrizing mixing matrices in terms of invariants.

The work of [1] was followed by derivations of the same formula focused on the derivation in a more model-building related approach using left-right symmetric scenarios [1–5,12,37]. In the same decade, a model independent approach was initiated where mass matrices with different null matrix elements ("texture zeros") were considered [38–43]; similar relations were then found for other mixing angles. Subsequently, horizontal or family discrete symmetries were used in order to relate the three families in a non-trivial fashion [11,13,14,44–48]. In their initial stage, though, the experimental uncertainty in the mixing angles and fermion masses was still too large as to build a stable model consistent with the unstable phenomenology. This approach was vigorously resurrected in the last decade when precision measurements for neutrino oscillations started [35,49,50]. Relations between the neutrino mixing angles and lepton mass hierarchies were found [51,52] where the values for the three neutrino masses are compatible with what follows from our method, though θ_{13} was predicted too low (only about 3°). Nevertheless, up to now, no complex mass matrix with a well-motivated constrained set of parameters has been found to entirely and successfully postdict the Cabibbo–Kobayashi–Maskawa (CKM) quark mixing matrix or to predict the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix in the lepton sector. In this work, we do not focus on a specific model predicting mixing angles, but give explicit relations following from a model independent treatment based on the observation of the two strong hierarchies $m_3 \gg m_2 \gg m_1$ in the charged fermion masses. Moreover, we dare to apply the same fomulae to the neutrino mixing and derive the PMNS angles with astonishingly good agreement.

This paper is organized in the following way: first, we start discussing the generic treatment of mixing matrices following from hierarchical mass matrices in Section 2, where we focus on the mathematical derivation of relations among fermion mass ratios and mixing angles. This result

gets applied to the phenomenological data in Section 3. Finally, we conclude. In Appendices A, B, C, D, we review the current status of input data, give a brief statement about the applicability of the method elaborated in this work, comment on the hierarchical structure of the mass matrices as a consequence of hierarchical masses and minimal flavor symmetry breaking, and provide the explicit, approximative formulae that gave the results of Section 3.

2. Mass and mixing matrices

Let us extend the SM by three right-handed neutrinos to have a more symmetric treatment of the problem in the quark and lepton sector. Dirac neutrinos alone still leave the question open why the Yukawa couplings for neutrinos are so much smaller than for the charged fermions. Nonetheless, in the description of fermion mixings in terms of fermion masses this assumption does not play a role and later we take an effective neutrino mass matrix without the need to specify whether neutrinos are Dirac or Majorana. The most general, renormalizable and gauge invariant construction of fermion mass matrices follows from the Yukawa Lagrangian

$$-\mathcal{L}_Y = \sum_{f=d,e} \mathcal{Y}_f^{ij} \overline{\psi}_{fL,i} \Phi \psi_{fR,j} + \sum_{f=u,v} \mathcal{Y}_f^{ij} \overline{\psi}_{fL,i} (i\sigma_2 \Phi^*) \psi_{fR,j} + \text{H.c.}, \tag{5}$$

where $i, j = 1, 2, 3$ are family indices and summation over them is implicitly understood. The generic fermion fields are denoted as ψ_f, where the left-handed fermions are grouped into $\text{SU}(2)_L$ doublets and the right-handed ones are the usual singlets. The Higgs doublet is given by $\Phi = (\phi^+, \phi^0)$ whereas its nonvanishing vacuum expectation value $v = \langle \phi^0 \rangle = 174$ GeV. The spontaneous breakdown of electroweak symmetry gives rise to four Dirac mass matrices of the form

$$\mathcal{M}_f = v \mathcal{Y}_f. \tag{6}$$

These mass matrices are 3×3 complex arbitrary matrices; each of them is diagonalized by a biunitary transformation

$$D_f = L^f \mathcal{M}_f R^{f\dagger}, \tag{7}$$

where D_f is a diagonal matrix with real and positive entries while L^f and R^f are two unitary matrices acting in family space on left- and right-handed fermions of type f respectively. Both transformations, L^f and R^f, correspond to the unitary matrices appearing in the *Singular Value Decomposition* of $\mathcal{M}_f$. These unitary matrices transform the sets of three left- or three right-handed fermion fields each from the interaction basis to the physical mass basis

$$\psi'_{f,L} = L^f \psi_{f,L} \quad \text{and} \quad \psi'_{f,R} = R^f \psi_{f,R}. \tag{8}$$

The mass eigenstates are therewith ψ'_f. In return, the diagonal weak charged current interactions are no longer diagonal, and mix different fermion families. This occurs as a consequence of the mismatch between the two different left unitary matrices acting inside the same fermion sector which results in the observable mixing matrices in the charged current interactions

$$V_{\text{CKM}} = L^u L^{d\dagger} \quad \text{and} \quad U_{\text{PMNS}} = L^e L^{v\dagger}. \tag{9}$$

2.1. The double mass hierarchy pattern (DMHP)

The singular values of the diagonal matrix D_f in Eq. (7) are to be identified with the measured fermion masses (see Appendix A). An interesting and not yet exploited fact is that the observed hierarchies in the masses (singular values) can be used to approximate the original mass matrices by lower-rank matrices as stated in the Schmidt–Mirsky approximation theorem [53–56].[1]

The left and right unitary matrices, L^f and R^f are decomposed into the left and right singular vectors, $l_{f,i}$ and $r_{f,i}$ ($i = 1, 2, 3$), and built up as $L^{f\dagger} = [l_{f,1}, l_{f,2}, l_{f,3}]$ and $R^{f\dagger} = [r_{f,1}, r_{f,2}, r_{f,3}]$. Each pair of singular vectors correspond to the singular value $m_{f,i}$. For square matrices when all three singular values can be ordered as $m_{f,3} > m_{f,2} > m_{f,1} \geq 0$, the decomposition is unique up to a shared complex phase for each pair of singular vectors.[2]

The number of non-zero singular values equals the rank of the mass matrix $\mathcal{M}_f$. The mass matrix can be written in terms of its singular values with the respective left and right singular vectors as a sum of rank-one matrices,

$$\mathcal{M}_f = \left[\left(l_{f,1} \frac{m_{f,1}}{m_{f,2}} r_{f,1}^\dagger + l_{f,2} r_{f,2}^\dagger \right) \frac{m_{f,2}}{m_{f,3}} + l_{f,3} r_{f,3}^\dagger \right] m_{f,3}. \tag{10}$$

Any hierarchy among the singular values is of major interest to us as it leads to a lower-rank approximation $\mathcal{M}_f^r$ ($r = \text{rank}[\mathcal{M}_f^r] < 3$). The lower-rank approximation is the closest matrix of the given rank to the original matrix, where "close" has to be specified (see Appendix B). We obtain it by keeping the largest singular values and setting the smaller ones equal to zero. The lower-rank matrices are unique if and only if all the kept singular values are larger than those set to zero.

Because of $m_{f,3} \gg m_{f,2} \gg m_{f,1}$, Eq. (10) provides a powerful way to appreciate the double hierarchy of its singular values and the emerging relation to its rank by the use of Schmidt–Mirsky's approximation theorem. As both types of quarks and charged lepton masses satisfy those two hierarchies, we conclude, that their mass matrices can be safely approximated as either matrices of rank one or rank two, depending on how strong their double mass hierarchy pattern (DMHP) is.

As illustrated in Eq. (10), this expression points also to the fact that the fermion mass ratios $m_{f,1}/m_{f,2}$ and $m_{f,2}/m_{f,3}$ play the dominant role in determining the structure of the mass matrix whereas $m_{f,3}$ sets the overall mass scale. Only those two ratios will be necessary in the determination of the mixing parameters, since the overall mass scale can be factored out. For later use, we abbreviate $\hat{m}_{f,1} = m_{f,1}/m_{f,3}$ and $\hat{m}_{f,2} = m_{f,2}/m_{f,3}$. In the following, the hat (ˆ) denotes the division by the largest mass $m_{f,3}$.

2.1.1. The four mass ratios parametrization

The fact, that only two mass ratios for each fermion species are independent parameters, gives four independent mass ratios in each sector (quarks and leptons). An important remark at this point is, that also four parameters are needed to fully parametrize the mixing. This observation shall be used to build up the mixing matrix. In the standard parametrization, those four values are three angles and one phase — additional phases are to be rotated away by redefinition of the

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[1] It is often wrongly called the Eckart–Young–Mirsky or simply Eckart–Young theorem, see [57] for an early history on the Singular Value Decomposition.

[2] In the case of degeneracy among some of the singular values, there is no longer a unique Singular Value Decomposition for $\mathcal{M}_f$. This matters in the discussion of degenerate neutrino masses.

fermion fields. The case of Majorana neutrinos does not allow to rotate away the phases for the neutrinos, so two "Majorana phases" are left. In the following, we will leave aside the issue of Majorana phases and only discuss the Dirac phases. We shall show that it is possible to use the four mass ratios of each fermion sector to entirely parametrize the mixing without introduction of new parameters.

It is interesting to note, that a complete parametrization of the fermion mixing in terms of the fermion mass ratios only works in the two- and three-family case. To completely parametrize the mixing matrix, for $n > 1$ families, we need $(n-1)^2$ mixing parameters. On the other hand, $n-1$ mass ratios are independent for each fermion species. Therefore, only when the number of mass ratios in the corresponding fermion sector is equal to or larger than the number of mixing parameters, $2(n-1) \geq (n-1)^2$, this parametrization will be possible. In general, this only works out for two or three families.

2.2. The lower-rank approximations

Let us investigate the effect of neglecting the first generation masses. From now on we will work with the singular values normalized by the largest one. In the $\hat{m}_{f,1} \to 0$ limit, the application of Schmidt–Mirsky's approximation theorem to the mass matrices is consistent with the rank-two approximation. As we are neglecting all contributions $\mathcal{O}(\hat{m}_{f,1})$ we shall take into account all corrections of the same order later on to get a more precise result and reduce the error stemming from this approximation.

The rank-two mass matrices are then given by

$$\hat{\mathcal{M}}_f^{r=2} = \left[l_{f,2} \hat{m}_{f,2} r_{f,2}^\dagger + l_{f,3} r_{f,3}^\dagger \right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hat{m}_{22}^f & \hat{m}_{23}^f \\ 0 & \hat{m}_{32}^f & \hat{m}_{33}^f \end{pmatrix}. \tag{11}$$

In general, all the matrix elements should be different from zero. However, it is crucial to establish a connection between a lower-rank approximation and its origin to the Yukawa interactions. That is, $\hat{m}_{f,1} = 0$ is equivalent to decoupling the first fermion family from the Higgs field, $Y_{1j}^f = 0 = Y_{j1}^f$. Effectively, thus, we are left with a 2×2 mass matrix. In the 1–1 sector, in contrast, a phase freedom corresponding to U(1) rotations for the left- and right-handed fields is left, where the second and third generation share one common phase.

Up to now, we have only used the hierarchy $m_{f,2} \gg m_{f,1}$ to decouple the first generation masses. According to the lower-rank approximation theorem, the rank-two approximation differs in every element from the full-rank matrix, whereas its norm, for any chosen one, only changes slightly. The DMHP furthermore shows $m_{f,3} \gg m_{f,2}$ which can be exploited to further approximate the initial mass matrix by a rank-one matrix,

$$\hat{\mathcal{M}}_f^{r=1} = l_{f,3} r_{f,3}^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

Successively reducing the rank of the mass matrices helps to simplify the parametrization without loosing track of the parameters. It is, however, not necessary to work in the very crude rank-one approximation, but sufficient to consider as a starting point the rank-two approximation.

Eq. (12) reveals a left-over U(2) rotation in the 1–2 plane and one common U(1) factor for the third generation. We want to emphasize that the described picture of lower-rank approximations follows what is discussed in the literature as minimally broken flavor symmetry [22,58,59]. In the

limit of vanishing Yukawa couplings, the SM exhibits a $[U(3)]^5$ global flavor symmetry ($[U(3)]^6$ if right-handed neutrinos are considered). Each individual U(3) flavor symmetry gets gradually broken

$$U(3) \xrightarrow{M_3} U(2) \xrightarrow{M_2} U(1) \xrightarrow{M_1} \text{nothing},$$

with $M_3 > M_2 > M_1$ which simultaneously occurs in the up- and down sector and trivial U(1)s are left out for readability. After the first symmetry breaking step at M_3, one global phase freedom is left for the third generation that is combined to a global U(1) for the second and third after the following symmetry breaking. There is one residual U(1) symmetry left for all fermions in each sector at the end which is either baryon or lepton number. It is not only safe to work with $M_3 \gg M_2$ — where we are at the U(2) flavor symmetries of [22,58,59], but even $M_2 \gg M_1$ which allows to work with the rank-two approximation at a sufficiently low scale and perform the final symmetry breaking step at say the electroweak scale.

U(2) symmetric Yukawa couplings give a well-motivated and frequently used setup to study flavor physics in supersymmetric [60,61] and unified [22] theories and are still a viable tool to discuss recent results in flavor physics [62,63]. Application to lepton flavor physics was also considered [64–67], recently also in the context of $[U(3)]^5$ breaking [68]. The implication of U(2) flavor symmetries which can be used in a weaker symmetry assignment [69], is the arrangement of the first two families into one doublet whereas the third family transforms as a singlet under the flavor symmetry. This assignment can be achieved with the minimal discrete symmetry S_3 [70–74] that was applied to neutrinos [75] as well as quarks [28].

The important point in the discussion of fermion mixings in terms of fermion masses via lower-rank approximations is, that we implicitly assume the maximal $[U(3)]^6$ flavor symmetry broken with each symmetry breaking step *occurring simultaneously* for each subgroup $[U(3)]^6 = U(3)_Q \times U(3)_u \times U(3)_d \times U(3)_L \times U(3)_e \times U(3)_\nu$.

2.2.1. Order of independent rotations

To parametrize the three-fold mixing, we follow the commonly used three successive rotations depending on one angle and one phase each. The order of these transformations needs to follow the consecutive breakdown of the initial U(3) symmetry as implied by the hierarchy in the masses. Therefore,

$$L^f = L_{12}^f\big(\theta_{12}^f, \delta_{12}^f\big) L_{13}^f\big(\theta_{13}^f, \delta_{13}^f\big) L_{23}^f\big(\theta_{23}^f, \delta_{23}^f\big), \tag{13}$$

where each individual rotation is parametrized by one angle θ_{ij}^f and one phase δ_{ij}^f.[3]

Note that this set of rotations diagonalize the mass matrices for each fermion type. The resulting mixing matrices are the product of all the individual rotations

$$V_{\text{CKM}} = L^u L^{d\dagger} = L_{12}^u L_{13}^u L_{23}^u \big(L_{23}^d\big)^\dagger \big(L_{13}^d\big)^\dagger \big(L_{12}^d\big)^\dagger$$

and

$$U_{\text{PMNS}} = L^e L^{\nu\dagger} = L_{12}^e L_{13}^e L_{23}^e \big(L_{23}^\nu\big)^\dagger \big(L_{13}^\nu\big)^\dagger \big(L_{12}^\nu\big)^\dagger.$$

By convention, up- and down-type rotations are exchanged for leptons.

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[3] Later, when reparametrizing the individual rotations in terms of the masses we will see that some of these six mixing parameters are unphysical while the rest can be expressed solely by two mass ratios.

2.3. The effective 2×2 mass matrix

It is instructive to first study the two-family limit in the rank-two approximation following from $\hat{m}_{f,1} \ll 1$. The second hierarchy $m_{f,2} \ll m_{f,3}$ implies a 2×2 mass matrix of the form

$$\hat{\mathbf{m}}^f = \begin{pmatrix} \hat{m}_{ss}^f & \hat{m}_{sl}^f \\ \hat{m}_{ls}^f & \hat{m}_{ll}^f \end{pmatrix}, \tag{14}$$

with hierarchical elements $|\hat{m}_{ll}^f|^2 \gg |\hat{m}_{sl}^f|^2$, $|\hat{m}_{ls}^f|^2 \gg |\hat{m}_{ss}^f|^2$ and where we are now generically treating two fermion families whose singular values obey the hierarchy, $\sigma_l \gg \sigma_s$. In general, the matrix elements are complex numbers. The labeling s and l refers to the corresponding smaller and larger singular value, respectively. It can be shown that the order of magnitude of $\hat{m}_{ss}^f$ is about $\mathcal{O}(|\hat{m}_{sl}^f|^2)$ (see Appendix C). In the following, we work with the approximation $\hat{m}_{ss}^f = 0$.

Unlike most considerations, we take the outcome of the DMHP and minimal flavor symmetry breaking to set the magnitudes of the off-diagonals equal — the phases are not constrained, such that

$$|\hat{m}_{sl}^f| = |\hat{m}_{ls}^f| \quad \text{not} \quad \hat{m}_{sl}^f = (\hat{m}_{ls}^f)^*,$$

as implied by the requirement of an Hermitian mass matrix. We only need normal mass matrices.[4] In both cases (normal and Hermitian), the left and right Hermitian products are diagonalized by the same unitary transformation. For a normal mass matrix, however, the phases can be arranged in a way that the off-diagonal magnitudes do not have to be the same. We only constrain the matrix of absolute values to be symmetric, whereas the phases can be arbitrary:

$$\hat{\mathbf{m}}^f = \begin{pmatrix} 0 & |\hat{m}_{sl}^f| e^{i\delta_{sl}^f} \\ |\hat{m}_{sl}^f| e^{i\delta_{ls}^f} & \hat{m}_{ll}^f \end{pmatrix}. \tag{15}$$

As a self-consistency check, it is important to verify that the required hierarchy in all the mass matrix elements of the full-rank scenario actually is respected when expressing the matrix elements in terms of the masses (singular values).

2.3.1. Reparametrization in terms of the singular values

Due to our lack of knowledge of right-handed flavor mixing, the relevant object that determines our phenomenology is the Hermitian product $\mathbf{n}^f = \mathbf{m}^f (\mathbf{m}^f)^\dagger$, which exhibits two invariants: $\operatorname{tr} \mathbf{n}^f = \sigma_s^{f2} + \sigma_l^{f2}$ and $\det \mathbf{n}^f = \sigma_s^{f2} \sigma_l^{f2}$. The small and large singular value are denoted by σ_s^f and σ_l^f, respectively. Through means of the two invariants, we find

$$|\hat{m}_{sl}^f| = \sqrt{\hat{\sigma}_{sl}^f}, \quad \text{and} \quad |\hat{m}_{ll}^f| = 1 - \hat{\sigma}_{sl}^f, \tag{16}$$

where we have expressed for a generic treatment the normalized ratio of the small singular value over the large one as $\hat{\sigma}_{sl}^f \equiv \sigma_s^f / \sigma_l^f$.

This reparametrization nicely shows the result of the Schmidt–Mirsky approximation theorem: on the one hand, $|\hat{m}_{ll}^f|^2 \gg |\hat{m}_{sl}^f|^2$, while on the other hand, $|\hat{m}_{ll}^f| = 1$ is the only non-vanishing matrix element in the limit $\hat{\sigma}_s^f \to 0$.

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[4] A matrix is normal if the left and right Hermitian products are the same: $\mathbf{mm}^\dagger = \mathbf{m}^\dagger \mathbf{m}$.

The left unitary transformation corresponding to the diagonalization of this matrix is given by

$$L_{sl}^f(\hat{\sigma}_{sl}^f, \delta_{sl}^f) = \frac{1}{\sqrt{1 + \hat{\sigma}_{sl}^f}} \begin{pmatrix} 1 & e^{-i\delta_{sl}^f}\sqrt{\hat{\sigma}_{sl}^f} \\ -e^{i\delta_{sl}^f}\sqrt{\hat{\sigma}_{sl}^f} & 1 \end{pmatrix}. \tag{17}$$

This result has been already discussed previously by many authors [8,9,18,26]. The mixing angle can be obtained from $\tan\theta_{sl}^f = \sqrt{\hat{\sigma}_{sl}^f}$.[5] Note that this relation indeed is the Gatto–Sartori–Tonin result, see Eq. (3).

2.3.2. The two-family mixing matrix

Eq. (17) diagonalizes the mass matrix of one fermion type. In the weak charged current, an a-type fermion ($a = u, e$) meets a b-type fermion ($b = d, \nu$), so we need two such diagonalizations to describe fermion mixing in the charged current interactions. Anyway, two unitary 2×2 rotations do not commute, and the new mixing parameters are not just the sum or difference of the former ones: $\theta_{sl} \neq \theta_{sl}^a \pm \theta_{sl}^b$ and $\delta \neq \delta_{sl}^a \pm \delta_{sl}^b$. Explicitly,

$$V_{sl} = L_{sl}^a L_{sl}^{b\dagger} = \text{diag}(1, e^{-i\delta_{sl}^a}) \begin{pmatrix} \sqrt{1-\lambda^2}e^{-i\delta_0} & \lambda e^{-i\delta} \\ -\lambda e^{i\delta} & \sqrt{1-\lambda^2}e^{i\delta_0} \end{pmatrix} \text{diag}(1, e^{i\delta_{sl}^a}), \tag{18}$$

where we factored out the phase δ_{sl}^a. This choice is completely arbitrary, the same is true for δ_{sl}^b. The relevant phases *inside* the matrix only depend on the *difference*. The mixing can then be obtained in the following way

$$\lambda = \sin\theta_{sl} = \sqrt{\frac{\hat{\sigma}_{sl}^a + \hat{\sigma}_{sl}^b - 2\sqrt{\hat{\sigma}_{sl}^a \hat{\sigma}_{sl}^b}\cos(\delta_{sl}^a - \delta_{sl}^b)}{(1 + \hat{\sigma}_{sl}^a)(1 + \hat{\sigma}_{sl}^b)}}, \tag{19}$$

$$\tan\delta = \frac{\hat{\sigma}_{sl}^b \sin(\delta_{sl}^a - \delta_{sl}^b)}{\hat{\sigma}_{sl}^a - \hat{\sigma}_{sl}^b \cos(\delta_{sl}^a - \delta_{sl}^b)}, \tag{20}$$

$$\tan\delta_0 = \frac{\hat{\sigma}_{sl}^a \hat{\sigma}_{sl}^b \sin(\delta_{sl}^a - \delta_{sl}^b)}{1 + \hat{\sigma}_{sl}^a \hat{\sigma}_{sl}^b \cos(\delta_{sl}^a - \delta_{sl}^b)}. \tag{21}$$

The functional dependence on the two initial complex phases is found to be only their difference. From the hierarchies $\hat{\sigma}_{sl}^x = \sigma_s^x/\sigma_l^x \ll 1$ (for $x = a, b$) follow the new phases to be approximately given by $\tan\delta \approx -\tan(\delta_{sl}^a - \delta_{sl}^b)$ and $\tan\delta_0 \approx 0$. For the full-rank scenario, however, this simple conclusion cannot be drawn — it actually holds for the "initial" 2–3 rotation, but not anymore when subsequent rotations are added.

2.3.3. Comment on the complex phases

In general, the complex phases of the initial mass matrix elements are not constrained to a particular value. The employed matrix invariants only restrict the moduli of the matrix elements, the phases are unconstrained. There is nevertheless an ambiguity in those phases that is not necessary to set up a full parametrization of fermion mixing in the SM. The standard parametrization uses three successive rotations with $\theta_{ij} \in [0, \frac{\pi}{2}]$ and one complex phase $\delta_{CP} \in [0, 2\pi)$. These

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[5] Another solution can be found, that behaves wrongly in the limit $\hat{\sigma}_{sl}^f \to 0$ and gives maximal mixing $\tan\theta_{sl}^f \to \infty$ instead of zero mixing.

four parameters are sufficient to describe both mixing and CP violation in each fermion sector (unless we want to include a description of Majorana phases for neutrinos). In contrast, we have four mass ratios — and the freedom to put either real or purely imaginary matrix elements. This last choice can be achieved by restricting all phases to be either maximal CP violating ($\pi/2$ or $3\pi/2$) or CP conserving (0 or π). Interestingly, at the end, there is no freedom in phase choices at all and we find that only the 1–2 phase is allowed to be maximally CP violating, which indeed follows from a symmetry argument.

2.4. The full-rank picture

Working in the lower-rank approximations, we are neglecting the first generation mass ($\hat{m}_{f,1} = 0$) in the 2–3 rotation and the second generation mass ($\hat{m}_{f,2} = 0$) while performing the 1–3 rotation. The last transformation that appears in Eq. (13) acting in the 1–2 plane needs no approximation. It affects only the upper left 2×2 submatrix and is an exact diagonalization. In all cases, the mass matrices are of the form (14), where the elements are properly distributed over the 3×3 matrix elements. All residual matrix elements are zero. The same holds for the arising rotation matrices that are 3×3 generalizations of Eq. (17).

Working in the leading order approximations shows a subtle inconsistency: neglecting $\mathcal{O}(\hat{m}_{f,2})$ terms in the 1–3 rotation means actually ignoring a large effect, because $\mathcal{O}(\hat{m}_{f,1}) = \mathcal{O}(\hat{m}_{f,2}^2)$. Moreover, to include $\mathcal{O}(\hat{m}_{f,1})$ contributions in the 1–3 rotation following the initial rotation in the 2–3 plane, we first have to consider contributions of the same order that were missing in the initial rotation. Therefore, we briefly discuss how to consistently include corrections of missing pieces to improve the result.

2.4.1. Inclusion of corrections

We include the corrections as correcting (small) rotations. This procedure is crucial in view of the symmetry breaking chain from an enhanced flavor symmetry, as $[U(3)]^3$ (corresponding to a rank-zero mass matrix), down to the least symmetry left over. Since each breaking step is done by a small parameter, we do not disturb much by adding perturbations. Moreover, by repeatedly applying rotations, this guarantees from the very beginning normalized eigenvectors, and furthermore, an inclusion of formally higher order terms in perturbation theory. This can be seen from the following example of two real rotations, where $\hat{\epsilon} \lesssim \hat{\sigma}_{sl}^{f}$[6]:

$$\left(L_{sl}^{f}\right)^{(p=1)} = \left(L_{sl}^{f}\right)^{(1)}(\pm\hat{\epsilon})\left(L_{sl}^{f}\right)^{(0)}\left(\hat{\sigma}_{sl}^{f}\right) = \begin{pmatrix} \cos(\theta_{sl}^{f})^{(p=1)} & \sin(\theta_{sl}^{f})^{(p=1)} \\ -\sin(\theta_{sl}^{f})^{(p=1)} & \cos(\theta_{sl}^{f})^{(p=1)} \end{pmatrix}, \tag{22}$$

and the new angle is given by

$$\sin(\theta_{sl}^{f})^{(p=1)} = \frac{\sqrt{\hat{\sigma}_{sl}^{f}} \pm \sqrt{\hat{\epsilon}}}{\sqrt{(1 + \hat{\sigma}_{sl}^{f})(1 + \hat{\epsilon})}}. \tag{23}$$

For real rotations, the requirement $\hat{\epsilon} \lesssim \hat{\sigma}_{sl}^{f}$ is irrelevant, because O(2) rotations commute. Therefore, there is also no need to specify any order in the addition of correcting rotations in each $i{-}j$ plane.

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[6] The two signs reflect the freedom of choice for a clockwise or counterclockwise correcting rotation.

Inverting this procedure shows that it is equivalent to add the perturbation term

$$-\sqrt{\hat{\epsilon}}\Big[1 + (\hat{\sigma}_{sl}^f)^2 - 2\hat{\sigma}_{sl}^f + \sqrt{\hat{\epsilon}}\hat{\sigma}_{sl}^f(\hat{\sigma}_{sl}^f - 1)\Big](1 - \hat{\epsilon}) \tag{24}$$

to the off diagonal matrix elements s–l and l–s.

Continuing this, an arbitrary number of correcting rotations could be added in each 2×2 rotation:

$$\sin\big(\theta_{sl}^f\big)^{(p=n)} = \frac{\sum_{j=0}^{n}(-1)^{\delta_j}\sqrt{\hat{a}_j} + \mathcal{O}([\sqrt{\hat{a}_i\hat{a}_j\hat{a}_k}]_{i\neq j\neq k})}{\sqrt{(1+\hat{a}_0)(1+\hat{a}_1)(1+\hat{a}_2)\cdots(1+\hat{a}_n)}}, \tag{25}$$

where we have denoted $\hat{a}_0 \equiv \hat{\sigma}_{sl}^f$ and $\hat{a}_{i>0}$ for the parameters of the following rotations. Each $(-1)^{\delta_i}$ is the orientation of the i-th rotation, which is either clockwise or counterclockwise (plus or minus). We neglect in Eq. (25) all trilinear and higher products of $\hat{a}_i$, where no $\hat{a}_i^2$ and no even products appear. Let us emphasize here, nevertheless, that these correcting rotations do not follow the traditional procedure of perturbation theory where we could naively think that the following new correcting rotation is a power of the previous one. Inclusion of new correcting rotations requires a careful treatment. We have found to be sufficient to include two correcting rotations to the mixing matrix parametrization which are the contributions $\mathcal{O}(\hat{m}_{f,1})$, $\mathcal{O}(\hat{m}_{f,2}^2)$, and $\mathcal{O}(\hat{m}_{f,1} \cdot \hat{m}_{f,2})$ which are of the same order as the neglected terms in each case.

2.4.2. First rotation: the 2–3 sector

Starting from the rank-two approximation, we loose track of all $\sqrt{\hat{m}_{f,1}}$ contributions in the mass matrix. However, all correcting rotations have to be consistent with the initial approximation ($\hat{m}_{f,1} \to 0$) and, moreover, all "higher order" contributions ($\sim\hat{m}_{f,2}^2$, $\sim\hat{m}_{f,1}^2$) are already covered as can be seen from (24). We therefore conclude, that all reasonable rotations in the 2–3 plane can be expressed as

$$\big(L_{23}^f\big)^{(p=2)} = \big(L_{23}^f\big)^{(2)}(\hat{m}_{f,1}\cdot\hat{m}_{f,2})\big(L_{23}^f\big)^{(1)}(\hat{m}_{f,1})\big(L_{23}^f\big)^{(0)}(\hat{m}_{f,2}). \tag{26}$$

Additionally, in principle, there is a freedom in the choice of the complex phase, which can be boiled down to the two different sign choices.

2.4.3. Second rotation: the 1–3 sector

What follows is the same procedure in the 1–3 sector after the 2–3 rotations have been done. In this case, the $p = 2$ leading correcting rotations are

$$\big(L_{13}^f\big)^{(p=2)} = \big(L_{13}^f\big)^{(2)}(\hat{m}_{f,1}\cdot\hat{m}_{f,2})\big(L_{13}^f\big)^{(1)}(\hat{m}_{f,2}^2)\big(L_{13}^f\big)^{(0)}(\hat{m}_{f,1}). \tag{27}$$

2.4.4. Last rotation: the 1–2 sector

No approximation is left anymore, therefore the exact rotation is expressed as

$$L_{12}^f = L_{12}^f\bigg(\frac{\hat{m}_{f,1}}{\hat{m}_{f,2}}, \delta_{12}^f\bigg), \tag{28}$$

where we now explicitly put the phase δ_{12}^f. This occurrence is very clear from the rank evolution: in the rank-one approximation, there is the freedom of a U(2) rotation left in the 1–2 block. The initial 2–3 and 1–3 rotations can always be taken real, the only possible phase then sits in the 1–2 rotation.

The necessity of correcting rotations is very apparent from the flavor symmetry breaking chain: First, in the rank-two approximation we have

$$
\begin{pmatrix} 0 & 0 & 0 \\ 0 & X & X \\ 0 & X & X \end{pmatrix} \xrightarrow{L^{(0)}_{23}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{pmatrix}.
$$

After performing the symmetry breaking step to the full-rank matrix, we get contributions in all matrix elements not larger than $\mathcal{O}(\sqrt{\hat{m}_{f,1}})$ — also in off-diagonal components that were already rotated away:

$$
\begin{pmatrix} * & * & * \\ * & X & * \\ * & * & X \end{pmatrix}.
$$

So, we indeed have to consider higher order corrections to the initial rotation. The correcting rotations also do not spoil the required hierarchy. After the successive 2–3 and 1–3 rotations there is a contribution shuffled into the 1–1 entry which is $\sim s^2_{13} m_{33} \sim \mathcal{O}(\hat{m}_{f,1})$ and therefore of higher order compared to $\mathcal{O}(\hat{m}_1/\hat{m}_2)$, the original 1–1 element.

3. Applying the DMHP to phenomenology

By building up the mixing matrices following the procedure of the previous section, there appears the impression of an arbitrariness in the choice of complex phases. This arbitrariness can be attenuated taking into account some well motivated considerations. First, complex phases appear pairwise in the up- and down-type fermion sectors. We therefore have the freedom to keep track of them in only one sector and set all phases in the other one equal to zero. The charged current mixing matrix is therefore constructed in the following way:

$$
V_{\text{CKM}} = L^u L^{d\dagger},
$$

$$
L^u = L^u_{12}\left(\frac{m_u}{m_c}\right) L^u_{13}\left(\frac{m_u m_c}{m_t^2}\right) L^u_{13}\left(\frac{m_c^2}{m_t^2}\right) L^u_{13}\left(\frac{m_u}{m_t}\right) L^u_{23}\left(\frac{m_u m_c}{m_t^2}\right) L^u_{23}\left(\frac{m_u}{m_t}\right) L^u_{23}\left(\frac{m_c}{m_t}\right),
$$

$$
(29)
$$

$$
L^{d\dagger} = \left(L^d_{23}\right)^\dagger\left(\frac{m_s}{m_b}, \delta^{(0)}_{23}\right)\left(L^d_{23}\right)^\dagger\left(\frac{m_d}{m_b}, \delta^{(1)}_{23}\right)\left(L^d_{23}\right)^\dagger\left(\frac{m_d m_s}{m_b^2}, \delta^{(2)}_{23}\right)\left(L^d_{13}\right)^\dagger\left(\frac{m_d}{m_b}, \delta^{(0)}_{13}\right)
$$

$$
\times \left(L^d_{13}\right)^\dagger\left(\frac{m_s^2}{m_b^2}, \delta^{(1)}_{13}\right)\left(L^d_{13}\right)^\dagger\left(\frac{m_d m_s}{m_b^2}, \delta^{(2)}_{13}\right)\left(L^d_{12}\right)^\dagger\left(\frac{m_d}{m_s}, \delta_{12}\right),
\qquad (30)
$$

$$
U_{\text{PMNS}} = L^e L^{\nu\dagger},
$$

$$
L^e = L^e_{12}\left(\frac{m_e}{m_\mu}\right) L^e_{13}\left(\frac{m_\mu^2}{m_\tau^2}\right) L^e_{13}\left(\frac{m_e m_\mu}{m_\tau^2}\right) L^e_{13}\left(\frac{m_e}{m_\tau}\right)
$$

$$
\times L^e_{23}\left(\frac{m_e m_\mu}{m_\tau^2}\right) L^e_{23}\left(\frac{m_e}{m_\tau}\right) L^e_{23}\left(\frac{m_\mu}{m_\tau}\right),
\qquad (31)
$$

$$
L^{\nu\dagger} = \left(L^\nu_{23}\right)^\dagger\left(\frac{m_{\nu2}}{m_{\nu3}}, \delta^{(0)}_{23}\right)\left(L^\nu_{23}\right)^\dagger\left(\frac{m_{\nu1}}{m_{\nu3}}, \delta^{(1)}_{23}\right)\left(L^\nu_{23}\right)^\dagger\left(\frac{m_{\nu1}m_{\nu2}}{m_{\nu3}^2}, \delta^{(2)}_{23}\right)\left(L^\nu_{13}\right)^\dagger\left(\frac{m_{\nu1}}{m_{\nu3}}, \delta^{(0)}_{13}\right)
$$

$$
\times \left(L^\nu_{13}\right)^\dagger\left(\frac{m_{\nu2}^2}{m_{\nu3}^2}, \delta^{(1)}_{13}\right)\left(L^\nu_{13}\right)^\dagger\left(\frac{m_{\nu1}m_{\nu2}}{m_{\nu3}^2}, \delta^{(2)}_{13}\right)\left(L^\nu_{12}\right)^\dagger\left(\frac{m_{\nu1}}{m_{\nu2}}, \delta_{12}\right).
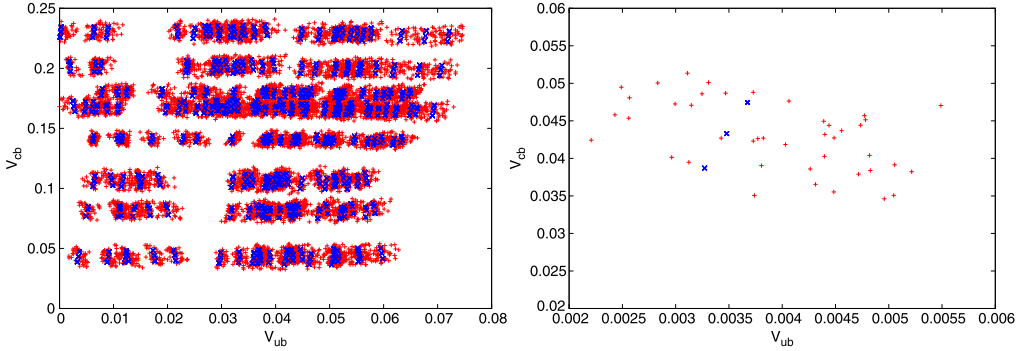\qquad (32)
$$

Fig. 1. Distribution of allowed values in the V_{ub}–V_{cb} plane. The small red points show allowed regions where the masses were varied in their 1σ regimes, the blue crosses show the values coming from the central values of the masses. Right: zoom into the phenomenological viable region. There are only three distinct phase choices leading to both small values for V_{ub} and V_{cb}. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
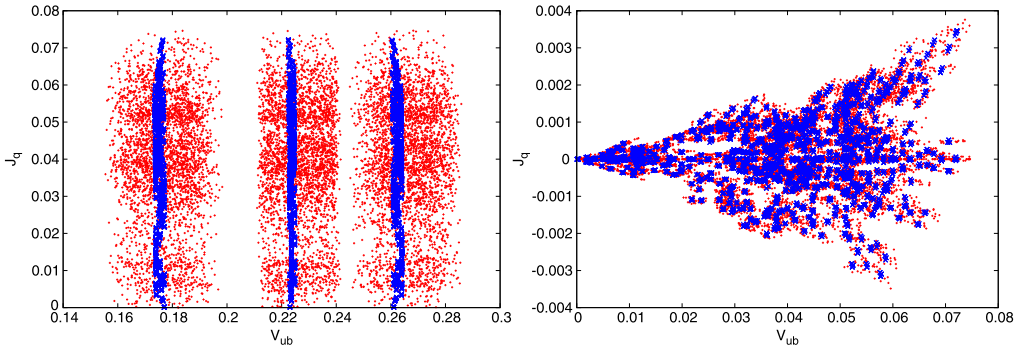
Fig. 2. The left plot shows the regions for the Cabibbo angle (more exactly V_{us} — clearly the solution $\delta_{12} = \frac{\pi}{2}$ is favored (which corresponds to the stripe in the center). On the right side, the rephasing invariant J_q is shown against V_{ub}. Color code as in Fig. 1: red dots are points with masses varied in the 1σ regimes, blue crosses are the central values. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The method itself is not quite arbitrary at all. For the CKM mixing it gives well-separated regions that have to be entered with a specific choice for the phases (see Fig. 1). Since both quark masses as well as CKM mixing matrix entries are rather well measured, this observations allows us to set the phases. We find only one distinct choice. Moreover, we make a *minimal* choice: on the one hand, we allow CP phases to be either maximally CP violating or CP conserving. On the other hand, we find, that the only maximally CP violating phase has to be in the 1–2 rotation of the down-type quarks or neutrinos, respectively. This can be seen from Fig. 2 where the three bands correspond to a phase $\delta_{12} = 0, \frac{\pi}{2}$ and π (Table 1).

The previously derived subsequent rotations only depend on four mass ratios in each fermion sector and have to be faced with phenomenological data. As input values we are using the quark and lepton masses only (see Appendix A) and then give a prediction for the neutrino masses to be in agreement with observations of neutrino mixing in this setup.

Table 1
The choice of phases in Eqs. (30) and (32) leading to the mixing matrices shown in (33) and (38).

	δ_{12}	$\delta_{13}^{(0)}$	$\delta_{13}^{(1)}$	$\delta_{13}^{(2)}$	$\delta_{23}^{(0)}$	$\delta_{23}^{(1)}$	$\delta_{23}^{(2)}$
CKM	$\frac{\pi}{2}$	0	π	π	0	π	π
PMNS	$\frac{\pi}{2}$	0	π	π	π	π	0

3.1. Minimal or maximal CP violation

The nature of the complex phases and its impact in the mixing matrix elements needs further investigation. Giving a solution to this problem is, however, outside the scope of this work. We shall use our observation to distribute the CP violating phase properly and leave the origin of CP violation for later work.

A final comment can be done, though, that guarantees the uniqueness of the parametrization. In Fig. 1, we show the maximally allowed ranges for the mixing matrix elements V_{ub} and V_{cb}. The amount of data points was constructed choosing the quark masses from their 1σ regimes and randomly taking every phase in the final parametrization from the set $\{0, \frac{\pi}{2}, \pi\}$. It is sufficient to constrain oneself to this set which gives the minimal and maximal allowed amount of CP violation [76] — and connected to that minimal and maximal mixing. The latter can be seen from Eq. (19) for the two-generation sub-case: the phase difference $\delta_{sl}^{a} - \delta_{sl}^{b}$ controls the magnitude of the mixing angle between minimal ($\delta_{sl}^{a} - \delta_{sl}^{b} = 0$) and maximal ($\pi$) mixing.

The fact, that only *one* combination of phases survives, is astonishing: note that all possible combinations in Eq. (30) are generically $3^7 = 2187$ choosing from $\{0, \frac{\pi}{2}, \pi\}$. Still, after taking $\delta_{12} = \frac{\pi}{2}$ and constraining the remnant phases to be either zero or π, 64 combinations are left. It is therefore not *a priori* clear that the mass ratios alone give the right mixing. The functional dependence on the mass ratios, however, is unique once the phases are set. We therefore use this description to determine the position of the maximal CP phase, where in contrast the other phases give relative minus signs. The maximal CP violating phase in the neutrino 1–2 mixing is somewhat different to what was found in connection with maximal atmospheric mixing [77].

3.2. Projected values of V_{CKM}^{th} and J_q

Consideration of all the aforementioned prescriptions gives the following numbers for the magnitude of the mixing matrix elements (see Appendix D for the explicit formulae of the mixing angles and the Jarlskog invariant),

$$\left|V_{CKM}^{th}\right| = \begin{pmatrix} 0.974^{+0.004}_{-0.003} & 0.225^{+0.016}_{-0.011} & 0.0031^{+0.0018}_{-0.0015} \\ 0.225^{+0.016}_{-0.011} & 0.974^{+0.004}_{-0.003} & 0.039^{+0.005}_{-0.004} \\ 0.0087^{+0.0010}_{-0.0008} & 0.038^{+0.004}_{-0.004} & 0.9992^{+0.0002}_{-0.0001} \end{pmatrix} \tag{33}$$

and the following amount of CP violation as measured by the Jarlskog invariant,

$$J_q = \mathrm{Im}\left(V_{us} V_{cb} V_{ub}^* V_{cs}^*\right) = \left(2.6^{+1.3}_{-1.0}\right) \times 10^{-5}, \tag{34}$$

where all quantities here are seen to be in quite good agreement within the errors compared to the global fit result given by the PDG 2014 [78] (see Appendix A for present knowledge on masses and mixings). Note that generically, the amount of CP violation is much larger (Fig. 2) and a small value of V_{ub} is connected to a small J_q, as expected.

3.3. Lepton sector

Quark masses show a very strong hierarchy. Charged lepton masses also do. Neutrinos, though, do not do. Is it really viable to apply the DMHP also to lepton mixing? Leptonic mixing angles are large, this observation may hint to a different mechanism. However, mass ratios for neutrinos are also large. The parametrization of fermion mixing in terms of mass ratios allows to also cope with large mixings by large mass ratios. Nevertheless, we have to include a solid examination of the errors in this approximation and see whether the same procedure as for quarks is viable also for leptons.

Are neutrino masses hierarchical? Neither the quasidegenerate solution nor the strong hierarchy are excluded yet. A hierarchical mass spectrum in any case predicts a very light lightest neutrino (it still can be exactly massless — in this case we would only have a rank-two mass matrix), where degenerate masses are likely to be tested in the near future.

The power of the mixing parametrization in terms of mass ratios lies in its invertibility: the formulae give us a unique description of the missing mass ratio once the mixing angle is measured. The pattern of neutrino masses brings us into the comfortable situation of nearly disentangling the 1–2 from the 2–3 mixing, because $\Delta m_{21}^2/\Delta m_{31}^2 \ll 1$. Additionally, the 1–2 mixing angle has the smallest error in the global fit.

3.3.1. Predicted neutrino masses

We do not focus on a specific model behind the theory of neutrino masses. It is sufficient to consider an effective neutrino mass matrix irrespective of the UV completion behind. To embed our description into a theory of neutrino flavor, it definitely matters if neutrinos are Dirac or Majorana. The size of the masses, however, allows to neglect RG running in any case. Therefore, we also ignore the nature of the neutrino mass operator. Since we take the magnitudes of the Dirac masses symmetric for quarks, the only difference would be the off-diagonal phase. Having this similarity in mind, the 1–2 approximation for neutrinos follows directly from Eq. (18) and the determining equation for the missing mass ratio from Eq. (19) with obvious relabellings:

$$|U_{e2}| \approx \sqrt{\frac{\hat{m}_{e\mu} + \hat{m}_{\nu 12} - 2\sqrt{\hat{m}_{e\mu}\hat{m}_{\nu 12}}\cos(\delta_{12}^e - \delta_{12}^\nu)}{(1 + \hat{m}_{e\mu})(1 + \hat{m}_{\nu 12})}}, \tag{35}$$

where the mass ratios are $\hat{m}_{e\mu} = m_e/m_\mu$ and $\hat{m}_{\nu 12} = m_{\nu 1}/m_{\nu 2}$. The three individual neutrino masses[7] are obtained via the mass squared differences:

$$m_{\nu 2} = \sqrt{\Delta m_{21}^2/\left(1 - \hat{m}_{\nu 12}^2\right)},$$
$$m_{\nu 1} = \sqrt{m_{\nu 2}^2 - \Delta m_{21}^2},$$
$$m_{\nu 3} = \sqrt{\Delta m_{31}^2 - \Delta m_{21}^2 + m_{\nu 2}^2}. \tag{36}$$

In Eq. (35), there appears the phase difference $\delta_{12}^e - \delta_{12}^\nu$. Although a twofold rotation shows no CP violation, this phase has to be considered because it appears last in the order of successive rotations. Moreover, we observed a maximal CP phase in the quark 1–2 sector. Albeit there

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[7] We are implicitly assuming normal ordering. Inverted ordering is excluded by construction because it is not hierarchical in the minimal flavor symmetry breaking chain.
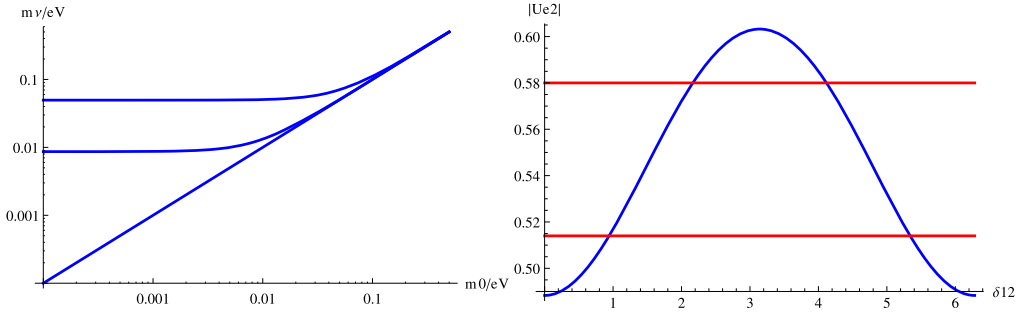
Fig. 3. Left: Evaluation of the three neutrino masses with the lightest mass (m_0, in eV). In the regime $m_0 < 0.1$ eV the assumption of a hierarchical pattern is indeed viable. Note also, that the ratio $m_{\nu 2}/m_{\nu 3}$ basically does not change with decreasing $m_0 = m_{\nu 1}$. Right: The value of $|U_{e2}|$ in dependency from δ_{12}^ν — the experimentally allowed 3σ region (indicated by the horizontal red lines) is compatible with the choice $\delta_{12}^\nu = \frac{\pi}{2}$, while not with $\delta_{12}^\nu = 0$ or π. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

is no connection between quark and lepton mixing at this stage, we shall keep the assignment $\delta_{12}^e - \delta_{12}^\nu = \frac{\pi}{2}$ (for the reasoning why, see Fig. 3) and get

$$\hat{m}_{\nu 1} = \frac{|U_{e2}|^2 (1 + \hat{m}_e) - \hat{m}_e}{1 - |U_{e2}|^2 (1 + \hat{m}_e)} = 0.41 \ldots 0.45 \tag{37}$$

using $\hat{m}_e = 0.00474$ and $|U_{e2}| = \sin\theta_{12} = 0.54 \ldots 0.56$. The masses are calculated as

$$m_{\nu 1} = (0.0041 \pm 0.0015) \text{ eV},$$
$$m_{\nu 2} = (0.0096 \pm 0.0005) \text{ eV},$$
$$m_{\nu 3} = (0.050 \pm 0.001) \text{ eV}.$$

The errors were propagated from the Δm^2 and added linearly to be more conservative. Within 3σ, the lightest neutrino can be massless. This prediction, however, will significantly improve with the improved errors on Δm_{21}^2.

The minimally and maximally allowed neutrino masses (corresponding to $\delta_{12}^e - \delta_{12}^\nu = 0, \pi$) are very close:

min (in eV)	max (in eV)
$m_{\nu 1} = 0.0029 \pm 0.0017$	$m_{\nu 1} = 0.0062 \pm 0.0017$
$m_{\nu 2} = 0.0091 \pm 0.0003$	$m_{\nu 2} = 0.011 \pm 0.001$
$m_{\nu 3} = 0.050 \pm 0.001$	$m_{\nu 3} = 0.050 \pm 0.001$

In any case, the lightest neutrino is much lighter than 0.01 eV.

3.3.2. U_{PMNS}^{th} as implied by the four leptonic mass ratios

Albeit the hierarchy is not as strong as for quarks and charged neutrinos, we dare to use the same description and show that indeed large mass ratios in the four mass ratio parametrization also lead to large mixing angles. The applicability of the whole method depends on hierarchical masses. In Appendix B we give a simple criterion parameter to check whether the lower-rank approximations are good approximations. Indeed, the deviation from unity is only a few percent. Therefore, we safely use the previous described procedure.

With the predicted neutrino masses (which only know about $|U_{e2}|$) and the knowledge of the charged fermion mass ratios, the leptonic mixing matrix exhibits the following numerical values

$$\left|U_{\text{PMNS}}^{\text{th}}\right| = \begin{pmatrix} 0.83^{+0.04}_{-0.05} & 0.54^{+0.06}_{-0.09} & 0.14 \pm 0.03 \\ 0.38^{+0.04}_{-0.06} & 0.57^{+0.03}_{-0.04} & 0.73 \pm 0.02 \\ 0.41^{+0.04}_{-0.06} & 0.61^{+0.03}_{-0.04} & 0.67 \pm 0.02 \end{pmatrix}, \tag{38}$$

whereas the implied amount of CP violation is displayed as

$$J_\ell = \text{Im}\left(U_{e2}U_{\mu 3}U_{e3}^* U_{\mu 2}^*\right) = 0.031^{+0.006}_{-0.007}. \tag{39}$$

We remark an astonishingly good agreement with the measured values (see Appendix A) and observe a close-to-maximal CP violation in the lepton sector! ($\delta_{\text{CP}} = 70°$ from the central values: $J_\ell = J_\ell^{\text{max}} \sin \delta_{\text{CP}}$, the error on J_ℓ^{max} is nevertheless compatible with maximal CP violation, $\delta_{\text{CP}} = 90°$.)

3.4. About precision

The goal of the presented work is not to be a precision analysis of quark and lepton mixing. The projected values of the mixing matrices are rather a rough-and-ready estimate compatible though very well with experimental data. We wanted to show that the knowledge of fermion masses is sufficient to describe their mixing accepting a hierarchical nature.

The errors that are presented in Eqs. (33) and (38) follow from the uncertainties in the masses. Better precision in the determination of quark masses leads to better discrimination in future whether the described procedure is valid. The estimates are not too bad, nevertheless, we ignored radiative corrections to the mixing matrices and constrain ourselves on a tree-level discussion. One-loop corrections to the masses or Yukawa couplings would be suppressed by factors $Y_{ij}Y_{jk}Y_{kl}/(16\pi^2)$ and are therefore in the range of the errors for the masses. Renormalization group running of the parameters is also negligible: quark mixing angles do basically not run. The running of fermion mixing parameters depends on a factor $(m_i + m_j)/(m_i - m_j)$ which is small for the hierarchical spectra. Especially neutrino masses and mixings run only slightly in the scenario which is under consideration in this work.

4. Conclusions

We investigated the long-standing question of understanding the functional description of the mixing matrices in terms of the fermion masses. The pure phenomenological observation of strong hierarchies among the charged fermion masses $m_{f,3} \gg m_{f,2} \gg m_{f,1}$ guides the way to a parametrization of fermion mixings in terms of mass ratios without further assumptions. By solely exploiting the mathematical properties of the mass matrices, namely their Singular Value Decomposition, and making use of the double mass hierarchy pattern (DMHP), we have shown that four mass ratios in each fermion sector and a maximal CP violating phase in the 1–2 rotation are sufficient to reproduce the numerical quantities of the fermionic mixing matrices. Hierarchical masses guarantee a unique decomposition into singular vectors up to a complex phase shared by the respective pair of singular vectors of a singular value. This uniqueness theorem dissolves the common ambiguities found in the literature originated in the freedom of weak bases. Schmidt–Mirsky's approximation theorem has been used to approximate the hierarchical mass matrices by lower-rank matrices that are the closest one to the given full-rank

matrix. The connection of each lower-rank approximation to the nature of the Yukawa interactions, $m_{f,i} = 0 \rightarrow Y_{ij}^f = 0 = Y_{ji}^f$, helps to simplify the reparametrization of the mass matrix without loosing track of the parameters. This connection is established via the minimal breaking of maximal flavor symmetry $[U(3)]^3 \rightarrow [U(2)]^3 \rightarrow [U(1)]^3 \rightarrow U(1)_F$ in each fermion sector, where the remnant $U(1)_F$ symmetry is either baryon or lepton number. The approximation, however, neglects sizeable terms in the mass matrices that have been consistently added by use of correcting rotations. The arbitrariness of complex phases is reduced by requiring them to be either maximally CP violating ($\pi/2$) or CP conserving (π). This assumption is motivated by the fact that the four mass ratios should be enough to serve as mixing parameters in the unitary 3×3 mixing matrix.

We found a remarkably good agreement of the projected magnitudes of both the CKM and PMNS matrix elements and reproduce the Jarlskog invariant of the quark sector quite well. The strength of this description in terms of mass ratios lies in its invertibility. In the leptonic sector, we have calculated the neutrino mass spectrum following from the inversion of the formulae in the 1–2 mixing sector and the measured mass squared differences. The lightest neutrino has a mass well below 0.01 eV, while the largest neutrino mass lies around 0.05 eV. We therefore conclude that, if also in the neutrino sector the mixing is determined by the mass ratios without any further contribution, the electron neutrino mass escapes its nearby measurement from tritium decay. Moreover, we give a prediction for the leptonic CP phase close to maximal, $\delta_{CP}^\nu \approx 90°$.

Hence, contrary to the common expectation, leptonic mixing angles are found to be determined solely by the four leptonic mass ratios: $m_e/m_\mu, m_\mu/m_\tau, m_{\nu 1}/m_{\nu 2}$, and $m_{\nu 2}/m_{\nu 3}$ without any relation to the geometrical factors observed in most flavor models. Notwithstanding, we see a great power of the described method in the application to flavor model building: once a model gives hierarchical masses, the mixing follows from this hierarchy. In contrast, our approach gives viable patterns and textures for mass matrices in terms of the singular values (fermion masses). We explicitly leave the question of a model behind open. Likewise, the origin of CP violation stays unexplained, though our observation about the distribution of CP phases gives an important starting point.

Acknowledgements

Appendix A. State of the art in the fermion masses and mixing matrices

In this section, we collect the current knowledge about fermion mixing data and specify the input values we use in the following for the masses.

Table A.1

The quark masses are run to the Z boson mass scale by virtue of the RunDec package [80]. The mass inputs correspond to the experimental measured values while the outputs, evaluated at the Z pole, include the resummation of higher order corrections from QCD by the RG running. RunDec takes properly into account the decoupling of heavy quarks below their scale. All masses are given in GeV.

Input	Output
$m_u(2\,\text{GeV}) = 0.0023^{+0.0007}_{-0.0005}$	$m_u(M_Z) = 0.0013^{+0.0004}_{-0.0003}$
$m_d(2\,\text{GeV}) = 0.0048^{+0.0005}_{-0.0003}$	$m_d(M_Z) = 0.0028^{+0.0003}_{-0.0002}$
$m_s(2\,\text{GeV}) = 0.095 \pm 0.005$	$m_s(M_Z) = 0.055 \pm 0.003$
$m_c(m_c) = 1.275 \pm 0.025$	$m_c(M_Z) = 0.622 \pm 0.012$
$m_b(m_b) = 4.18 \pm 0.03$	$m_b(M_Z) = 2.85 \pm 0.02$
$m_t(\text{OS}) = 173.07 \pm 1.24$	$m_t(M_Z) = 172.16^{+1.47}_{-1.46}$

For all numerical evaluations made in this work, we stick to the updated values of the quark mixing matrix [78],

$$|V_{\text{CKM}}| = \begin{pmatrix} 0.97427 \pm 0.00014 & 0.22536 \pm 0.00061 & 0.00355 \pm 0.00015 \\ 0.22522 \pm 0.00061 & 0.97343 \pm 0.00015 & 0.0414 \pm 0.0012 \\ 0.00886^{+0.00033}_{-0.00032} & 0.0405^{+0.0011}_{-0.0012} & 0.99914 \pm 0.00005 \end{pmatrix}, \qquad (A.1)$$

with the Jarlskog invariant equal to $J_q = (3.06^{+0.21}_{-0.20}) \times 10^{-5}$. In the standard parametrization by the Particle Data Group (PDG), the central values give the following mixing angles,

$$\theta_{12}^q \approx 13.3°, \qquad \theta_{13}^q \approx 0.2°, \qquad \theta_{23}^q \approx 2.4°. \qquad (A.2)$$

The most recent update on the 3σ allowed ranges of the elements of the PMNS mixing matrix are given by [79]

$$|U_{\text{PMNS}}| = \begin{pmatrix} 0.801 \to 0.845 & 0.514 \to 0.580 & 0.137 \to 0.158 \\ 0.225 \to 0.517 & 0.441 \to 0.699 & 0.614 \to 0.793 \\ 0.246 \to 0.529 & 0.464 \to 0.713 & 0.590 \to 0.776 \end{pmatrix}. \qquad (A.3)$$

Where the best fit points of the mixing angles are

$$\theta_{12}^\ell = 33.48°, \qquad \theta_{13}^\ell = 8.50°, \qquad \theta_{23}^\ell = 42.3°. \qquad (A.4)$$

The maximal value of the leptonic Jarlskog invariant is given by $J_\ell^{\text{max}} = 0.033 \pm 0.010$ and different from zero at more than 3σ — still, the proper J_ℓ has first to be multiplied by $\sin\delta_{\text{CP}}$ and is supposed to be smaller.

The study of the mixing matrices in terms of the masses is done at the scale of the Z boson mass. The input values for the numerical calculations are obtained using the experimental values of the quark masses as given by the PDG Review 2014 [78] and running them to the scale of the Z boson determining the electroweak scale. We include highest precision running in QCD by the virtue of the RunDec package [80]. For completeness, we show the input values and their uncertainties as well as the resulting outputs in Table A.1.

The reported measured on-shell values in MeV for the charged lepton masses are

$$m_e = 0.510998928, \qquad m_\mu = 105.6583715, \qquad m_\tau = 1776.82 \pm 0.16, \qquad (A.5)$$

Table A.2
Charged fermions mass ratios at the M_Z scale.

f	$m_{f,1}/m_{f,2}$	$m_{f,1}/m_{f,3}$	$m_{f,2}/m_{f,3}$
u	$0.0021^{+0.0007}_{-0.0005}$	$(7.6^{+2.4}_{-1.8}) \times 10^{-6}$	0.0036 ± 0.0001
d	$0.051^{+0.009}_{-0.006}$	$(9.8^{+1.1}_{-0.7}) \times 10^{-4}$	0.019 ± 0.0012
e	0.00474	0.000279	0.0588

where we have neglected the tiny experimental errors in the first two generation masses. The recent changes of this values affect only the few last digits. Therefore, we safely trust the results of [81] for their values at the Z scale (in MeV):

$$m_e(M_Z) = 0.486570161, \qquad m_\mu(M_Z) = 102.7181359,$$
$$m_\tau(M_Z) = 1746.24^{+0.20}_{-0.19}. \tag{A.6}$$

The nine mass ratios are of essential use in the evaluation of the analytic formulae to describe fermion mixing. We show our input values determined from Table A.1 and Eq. (A.6) in Table A.2.

In the case of neutrinos, only two squared mass differences have been measured whose values are taken from [79],

$$\text{NO:} \quad \Delta m_{31}^2 = +2.457 \pm 0.002 \times 10^{-3} \text{ eV}^2,$$
$$\text{IO:} \quad \Delta m_{32}^2 = -2.448 \pm 0.047 \times 10^{-3} \text{ eV}^2, \qquad \Delta m_{21}^2 = 7.50^{+0.19}_{-0.17} \times 10^{-5} \text{ eV}^2, \tag{A.7}$$

where NO and IO stand for normal and inverted ordering, respectively.

Still, the most recent direct bound on the neutrino mass scale stems from tritium beta decay experiments: $m(\nu_e) \lesssim 2$ eV at 95% C.L. [82]. The KATRIN experiment is going to improve this bound by one order of magnitude [83].

Appendix B. Applicability of the method

The Schmidt–Mirsky theorem relates the validity of the lower-rank approximation to a measure of being close to the full-rank matrix. This measure has to be a scalar parameter and can be any norm. In the original formulation, the Frobenius norm was used, which is also the most natural choice since it is the square root over the sum of squared singular values and directly related to one of the invariants of the mass matrix

$$\|\mathcal{M}_f\|_{\mathbf{F}} = \sqrt{\sum_{i=1,2,3} m_{f,i}^2}. \tag{B.1}$$

The use of this norm serves as a way to define a criterion which allows us to distinguish when the hierarchy is strong enough as to safely make an approximation. In this regard, we define the parameter x_f^r as

$$x_f^r \equiv \frac{\sqrt{(r-1)m_{f,2}^2 + m_{f,3}^2}}{\|\mathcal{M}_f\|_{\mathbf{F}}} = \sqrt{\frac{(r-1)m_{f,2}^2 + m_{f,3}^2}{m_{f,1}^2 + m_{f,2}^2 + m_{f,3}^2}}, \tag{B.2}$$

where $r = \text{rank}[\mathcal{M}_f^r] \in \{1, 2\}$. The approximation becomes better the closer x_f^r is to one and is exact in the $x_f^r \to 1$ limit. Eq. (B.2) is actually the ratio of the lower-rank approximated mass

Table B.1

Values of the criterion parameter $x_f^r \equiv \sqrt{[(r-1)m_{f,2}^2 + m_{f,3}^2]/(m_{f,1}^2 + m_{f,2}^2 + m_{f,3}^2)}$, for the different cases of the fermion masses, where x_f^r provides a measure of the applicability of the method. The fact that all cases here are sufficiently close to one guarantees the safe use of the lowest-rank approximations. Even for neutrinos, x_ν^2 is close to one, where we exploit the prediciton for neutrino masses from Section 3.3.

x_f^r	u	d	e	ν
$r = 1$	0.999993	0.999816	0.998274	0.978894
$r = 2$	0.999999	0.999999	0.999999	0.996773

matrix norm with the original norm. Hence, x_f^r is a measure of the applicability of the method. Table B.1 shows the different values obtained of x_f^r for the several charged fermion masses. The values in the rank-one approximation, $r = 1$, for all practical purposes equal to one, though for both charged and neutral leptons deviate in the per mill and percent regime, respectively. From here we can already understand why the quark mixing matrix is so close to the unit matrix which is the trivial mixing matrix in the rank-one approximation. In a similar manner, the very mild hierarchy for neutrinos leads to a stronger deviation from the rank-one approximation and therefore larger mixing angles.

Appendix C. Hierarchical mass matrices

We show how to derive the hierarchical structure of the mass matrices by the use of the lower-rank approximation theorem and the principle of minimal flavor violation. Let us consider the two-flavor case and the mass matrix

$$\mathbf{m} = \begin{pmatrix} m_{ss} & m_{sl} \\ m_{ls} & m_{ll} \end{pmatrix}, \tag{C.1}$$

with the two singular values σ_s and σ_l respecting the hierarchy $\sigma_s \ll \sigma_l$.

We decompose the mass matrix in terms of the Singular Value decomposition

$$\mathbf{L}\mathbf{m}\mathbf{R}^\dagger = \mathrm{diag}(\sigma_s, \sigma_l), \tag{C.2}$$

where the left and right unitary matrices diagonalize the Hermitian products

$$\mathbf{L}\mathbf{m}\mathbf{m}^\dagger\mathbf{L}^\dagger = \begin{pmatrix} \sigma_s^2 & 0 \\ 0 & \sigma_l^2 \end{pmatrix} = \mathbf{R}\mathbf{m}^\dagger\mathbf{m}\mathbf{R}^\dagger. \tag{C.3}$$

Each Hermitian product can be expressed as a sum of rank-one matrices with the components of $\mathbf{L}$ and $\mathbf{R}$,

$$\mathbf{m}\mathbf{m}^\dagger = \sigma_s^2 \begin{pmatrix} |L_{11}|^2 & L_{11}L_{21}^* \\ L_{11}^*L_{21} & |L_{21}|^2 \end{pmatrix} + \sigma_l^2 \begin{pmatrix} |L_{12}|^2 & L_{12}L_{22}^* \\ L_{12}^*L_{22} & |L_{22}|^2 \end{pmatrix} \tag{C.4}$$

and

$$\mathbf{m}^\dagger\mathbf{m} = \sigma_s^2 \begin{pmatrix} |R_{11}|^2 & R_{11}R_{21}^* \\ R_{11}^*R_{21} & |R_{21}|^2 \end{pmatrix} + \sigma_l^2 \begin{pmatrix} |R_{12}|^2 & R_{12}R_{22}^* \\ R_{12}^*R_{22} & |R_{22}|^2 \end{pmatrix}. \tag{C.5}$$

Due to our lack of knowledge of right-handed flavor mixing, the relevant object that determines our phenomenology is the left Hermitian product, $\mathbf{m}\mathbf{m}^\dagger$.

C.1. Applying Schmidt–Mirsky's approximation theorem

Consider the rank-one approximation in Eq. (C.4) by $\hat{\sigma} = \sigma_s/\sigma_l = 0$ normalized with respect to the larger singular value

$$\hat{\mathbf{m}}^{r=1}\left(\hat{\mathbf{m}}^{r=1}\right)^{\dagger} = \begin{pmatrix} |L_{12}|^2 & L_{12}L_{22}^* \\ L_{12}^*L_{22} & |L_{22}|^2 \end{pmatrix}. \tag{C.6}$$

The components of the left unitary matrix depend on $\hat{\sigma}$. In the limit $\hat{\sigma} \to 0$, there is trivial mixing and the rank-one left Hermitian product is

$$\hat{\mathbf{m}}^{r=1}\left(\hat{\mathbf{m}}^{r=1}\right)^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{C.7}$$

A small breaking of the $[\mathrm{U}(1)]^2$ symmetry for the massless fermions implies only a small deviation from the trivial mixing:

$$|\mathbf{L}| \sim \begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}. \tag{C.8}$$

The mixing angle is related to the parameter of symmetry breaking $\hat{\sigma}$ and it is an easy exercise to derive $\theta \sim \sqrt{\hat{\sigma}}$ from Eq. (C.3).

We then get an estimate on the magnitudes of each element in Eq. (C.4)

$$|\hat{\mathbf{m}}\hat{\mathbf{m}}^{\dagger}| \sim \begin{pmatrix} \mathcal{O}(\theta^2) & \mathcal{O}(\theta) \\ \mathcal{O}(\theta) & 1 + \mathcal{O}(\theta^2) \end{pmatrix}. \tag{C.9}$$

The explicit form of the mass matrix $\mathbf{m}$ stays unknown as long as we have no information about $\mathbf{R}$. However, the minimal breaking of the maximal flavor symmetry applies to both chiralities simultaneously and the argument from above is the same for the right Hermitian product. We therefore know that $\mathbf{L}$ and $\mathbf{R}$ have the same moduli and get the hierarchical structure of $\mathbf{m}$:

$$\hat{\mathbf{m}} = \begin{pmatrix} m_{ss} & m_{sl} \\ m_{ls} & m_{ll} \end{pmatrix} \sim \begin{pmatrix} \mathcal{O}(\theta^2) & \mathcal{O}(\theta) \\ \mathcal{O}(\theta) & 1 + \mathcal{O}(\theta^2) \end{pmatrix}, \tag{C.10}$$

with $|m_{sl}| = |m_{ls}|$ as a natural consequence of hierarchical masses and minimal flavor symmetry breaking. The hierarchical structure for the mass matrix and its Hermitian product is the same. Hence, due to the strong hierarchy in the masses we can neglect the role of $|m_{ss}|^2 \sim \theta^4$ in (C.9) working with the leading order contributions in θ and assume $m_{ss} = 0$ as done in Eq. (15). This gives corrections to the Gatto–Satori–Tonin relation, $\tan\theta = \sqrt{\sigma_s/\sigma_l} = \sqrt{\hat{\sigma}}$, which are $\mathcal{O}(\theta^3) = \mathcal{O}(\hat{\sigma}\sqrt{\hat{\sigma}})$ and therefore neglected.

Appendix D. Explicit approximate formulae for the mixing angles and the Jarlskog invariant

The explicit formulae for the distinct mixing matrix elements in terms of the mass ratios is rather lengthy. We opt then, to show only the three mixing angles, used in the standard parametrization, with the corresponding Jarlskog invariant. This allows to express the mixing angles in terms of three moduli of the mixing matrix

$$\sin\theta_{23}^{f=q,\ell} = \frac{|V_{23}^{f=q,\ell}|}{\sqrt{1 - |V_{13}^{f=q,\ell}|^2}}, \qquad \sin\theta_{12}^{f=q,\ell} = \frac{|V_{12}^{f=q,\ell}|}{\sqrt{1 - |V_{13}^{f=q,\ell}|^2}},$$

$$\sin\theta_{13}^{f=q,\ell} = |V_{13}^{f=q,\ell}|. \tag{D.1}$$

In the four mass ratios parametrization it is more natural to give not the formulae of the mixing angles in terms of the masses but rather of the aforementioned moduli

$$\left|V_{12}^{f=q,\ell}\right| \approx \sqrt{\frac{\hat{m}_{12}^a + \hat{m}_{12}^b}{(1 + \hat{m}_{12}^a)(1 + \hat{m}_{12}^b)}}, \tag{D.2}$$

$$\left|V_{23}^{f=q,\ell}\right| \approx \mp \frac{\sqrt{\hat{m}_{13}^a} + \sqrt{\hat{m}_{13}^b} + \sqrt{\hat{m}_{23}^a} \mp \sqrt{\hat{m}_{23}^b} + \sqrt{\hat{m}_{13}^a \hat{m}_{23}^a} \pm \sqrt{\hat{m}_{13}^b \hat{m}_{23}^b}}{\sqrt{(1 + \hat{m}_{13}^a)(1 + \hat{m}_{13}^b)(1 + \hat{m}_{23}^a)(1 + \hat{m}_{23}^b)(1 + \hat{m}_{13}^a \hat{m}_{23}^a)(1 + \hat{m}_{13}^b \hat{m}_{23}^b)}}, \tag{D.3}$$

$$\left|V_{13}^{f=q,\ell}\right| \approx \mp |V_{23}^{f=q,\ell}| \sqrt{\frac{\hat{m}_{12}^a}{1 + \hat{m}_{12}^a}}$$

$$+ \frac{\sqrt{\hat{m}_{13}^a} - \sqrt{\hat{m}_{13}^b} + \sqrt{\hat{m}_{13}^a \hat{m}_{23}^b} + \sqrt{\hat{m}_{13}^b \hat{m}_{23}^b} + \hat{m}_{23}^a + \hat{m}_{23}^b}{\sqrt{(1 + \hat{m}_{13}^a)(1 + \hat{m}_{13}^a \hat{m}_{23}^a)(1 + (\hat{m}_{23}^a)^2)(1 + \hat{m}_{12}^a)(1 + \hat{m}_{13}^b)(1 + \hat{m}_{13}^b \hat{m}_{23}^b)(1 + (\hat{m}_{23}^b)^2)}}, \tag{D.4}$$

where we have denoted $\hat{m}_{ij}^{a(b)} = m_i^{a(b)}/m_j^{a(b)}$, the upper and lower signs in Eq. (D.3) correspond to q and ℓ, respectively. The two fermion species of each sector are $a = u, e$ and $b = d, \nu$.

The Jarlskog invariant is given by,

$$J_{f=q,\ell} \approx \cos\theta_{12}^b \sin\theta_{12}^b \sin\theta_{23}^{f=q,\ell} \left(\sin\theta_{12}^a \sin\theta_{23}^{f=q,\ell} + \sin\theta_{13}^a - \sin\theta_{13}^b\right), \tag{D.5}$$

where

$$\sin\theta_{12}^{a(b)} = \sqrt{\frac{\hat{m}_{12}^{a(b)}}{1 + \hat{m}_{12}^{a(b)}}} \quad \text{and}$$

$$\sin\theta_{13}^{a(b)} \approx \frac{\pm\sqrt{\hat{m}_{13}^{a(b)}} + \sqrt{\hat{m}_{13}^{a(b)} \hat{m}_{23}^{a(b)}} + \hat{m}_{23}^{a(b)}}{\sqrt{(1 + \hat{m}_{13}^{a(b)})(1 + \hat{m}_{13}^{a(b)} \hat{m}_{23}^{a(b)})(1 + (\hat{m}_{23}^{a(b)})^2)}}. \tag{D.6}$$

The approximate relations here given differ from the complete one in $\sim 1\%$ order.

References

[1] R. Gatto, G. Sartori, M. Tonin, Weak selfmasses, Cabibbo angle, and broken SU(2) × SU(2), Phys. Lett. B 28 (1968) 128–130, http://dx.doi.org/10.1016/0370-2693(68)90150-0.

[2] N. Cabibbo, L. Maiani, Dynamical interrelation of weak, electromagnetic and strong interactions and the value of theta, Phys. Lett. B 28 (1968) 131–135, http://dx.doi.org/10.1016/0370-2693(68)90151-2.

[3] K. Tanaka, P. Tarjanne, Cabibbo angle and selfconsistency condition, Phys. Rev. Lett. 23 (1969) 1137–1139, http://dx.doi.org/10.1103/PhysRevLett.23.1137.

[4] R. Oakes, SU(2) × SU(2) breaking and the Cabibbo angle, Phys. Lett. B 29 (1969) 683–685, http://dx.doi.org/10.1016/0370-2693(69)90110-5.

[5] H. Genz, J. Katz, L. Ram Mohan, S. Tatur, Chiral symmetry breaking and the Cabibbo angle, Phys. Rev. D 6 (1972) 3259–3265, http://dx.doi.org/10.1103/PhysRevD.6.3259.

[6] H. Pagels, Vacuum stability and the Cabibbo angle, Phys. Rev. D 11 (1975) 1213, http://dx.doi.org/10.1103/PhysRevD.11.1213.

[7] A. Ebrahim, Chiral SU(4) × SU(4) symmetry breaking and the Cabibbo angle, Phys. Lett. B 69 (1977) 229–230, http://dx.doi.org/10.1016/0370-2693(77)90650-5.

[8] H. Fritzsch, Calculating the Cabibbo angle, Phys. Lett. B 70 (1977) 436, http://dx.doi.org/10.1016/0370-2693(77)90408-7.

[9] S. Weinberg, The problem of mass, Trans. N. Y. Acad. Sci. 38 (1977) 185–201, http://dx.doi.org/10.1111/j.2164-0947.1977.tb02958.x.

[10] A. De Rujula, H. Georgi, S. Glashow, A theory of flavor mixing, Ann. Phys. 109 (1977) 258, http://dx.doi.org/10.1016/0003-4916(77)90172-5.

[11] F. Wilczek, A. Zee, Discrete flavor symmetries and a formula for the Cabibbo angle, Phys. Lett. B 70 (1977) 418, http://dx.doi.org/10.1016/0370-2693(77)90403-8.

[12] R.N. Mohapatra, G. Senjanovic, Cabibbo angle, CP violation and quark masses, Phys. Lett. B 73 (1978) 176, http://dx.doi.org/10.1016/0370-2693(78)90829-8.

[13] F. Wilczek, A. Zee, Horizontal interaction and weak mixing angles, Phys. Rev. Lett. 42 (1979) 421, http://dx.doi.org/10.1103/PhysRevLett.42.421.

[14] D. Wyler, The Cabibbo angle in the SU(2)$_L$ × U(1) gauge theories, Phys. Rev. D 19 (1979) 330, http://dx.doi.org/10.1103/PhysRevD.19.330.

[15] H. Fritzsch, Hierarchical chiral symmetries and the quark mass matrix, Phys. Lett. B 184 (1987) 391, http://dx.doi.org/10.1016/0370-2693(87)90186-9.

[16] P.H. Frampton, Y. Okada, Simplified symmetric quark mass matrices and flavor mixing, Mod. Phys. Lett. A 6 (1991) 2169–2172, http://dx.doi.org/10.1142/S0217732391002360.

[17] J.L. Rosner, M.P. Worah, Models of the quark mixing matrix, Phys. Rev. D 46 (1992) 1131–1140, http://dx.doi.org/10.1103/PhysRevD.46.1131.

[18] S. Raby, Introduction to theories of fermion masses, arXiv:hep-ph/9501349, 1995, 0126–180.

[19] T. Ito, M. Tanimoto, The unitarity triangle and quark mass matrices on the NNI basis, Phys. Rev. D 55 (1997) 1509–1514, http://dx.doi.org/10.1103/PhysRevD.55.1509, arXiv:hep-ph/9603393.

[20] Z.-z. Xing, Implications of the quark mass hierarchy on flavor mixings, J. Phys. G 23 (1997) 1563–1578, http://dx.doi.org/10.1088/0954-3899/23/11/006, arXiv:hep-ph/9609204.

[21] S.-S. Xue, Quark masses and mixing angles, Phys. Lett. B 398 (1997) 177–186, http://dx.doi.org/10.1016/S0370-2693(97)00213-X, arXiv:hep-ph/9610508.

[22] R. Barbieri, L.J. Hall, S. Raby, A. Romanino, Unified theories with U(2) flavor symmetry, Nucl. Phys. B 493 (1997) 3–26, http://dx.doi.org/10.1016/S0550-3213(97)00134-X, arXiv:hep-ph/9610449.

[23] D. Falcone, F. Tramontano, Relation between quark masses and weak mixings, Phys. Rev. D 59 (1999) 017302, http://dx.doi.org/10.1103/PhysRevD.59.017302, arXiv:hep-ph/9806496.

[24] A. Mondragon, E. Rodriguez-Jauregui, The breaking of the flavor permutational symmetry: mass textures and the CKM matrix, Phys. Rev. D 59 (1999) 093009, http://dx.doi.org/10.1103/PhysRevD.59.093009, arXiv:hep-ph/9807214.

[25] A. Mondragon, E. Rodriguez-Jauregui, The CP violating phase δ_{13} and the quark mixing angles θ_{13}, θ_{23} and θ_{12} from flavor permutational symmetry breaking, Phys. Rev. D 61 (2000) 113002, http://dx.doi.org/10.1103/PhysRevD.61.113002, arXiv:hep-ph/9906429.

[26] H. Fritzsch, Z.-z. Xing, Mass and flavor mixing schemes of quarks and leptons, Prog. Part. Nucl. Phys. 45 (2000) 1–81, http://dx.doi.org/10.1016/S0146-6410(00)00102-2, arXiv:hep-ph/9912358.

[27] G. Branco, D. Emmanuel-Costa, C. Simoes, Nearest-neighbour interaction from an Abelian symmetry and deviations from hermiticity, Phys. Lett. B 690 (2010) 62–67, http://dx.doi.org/10.1016/j.physletb.2010.05.009, arXiv:1001.5065.

[28] F. González Canales, A. Mondragón, M. Mondragón, U.J. Saldaña Salazar, L. Velasco-Sevilla, Quark sector of S3 models: classification and comparison with experimental data, Phys. Rev. D 88 (2013) 096004, http://dx.doi.org/10.1103/PhysRevD.88.096004, arXiv:1304.6644.

[29] L.J. Hall, A. Rasin, On the generality of certain predictions for quark mixing, Phys. Lett. B 315 (1993) 164–169, http://dx.doi.org/10.1016/0370-2693(93)90175-H, arXiv:hep-ph/9303303.

[30] A. Rasin, Diagonalization of quark mass matrices and the Cabibbo–Kobayashi–Maskawa matrix, arXiv:hep-ph/9708216.

[31] A. Rasin, Hierarchical quark mass matrices, Phys. Rev. D 58 (1998) 096012, http://dx.doi.org/10.1103/PhysRevD.58.096012, arXiv:hep-ph/9802356.

[32] H. Fritzsch, Z.-z. Xing, The light quark sector, CP violation, and the unitarity triangle, Nucl. Phys. B 556 (1999) 49–75, http://dx.doi.org/10.1016/S0550-3213(99)00337-5, arXiv:hep-ph/9904286.

[33] Z.-z. Xing, Model-independent access to the structure of quark flavor mixing, Phys. Rev. D 86 (2012) 113006, http://dx.doi.org/10.1103/PhysRevD.86.113006, arXiv:1211.3890.

[34] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, et al., Non-Abelian discrete symmetries in particle physics, Prog. Theor. Phys. Suppl. 183 (2010) 1–163, http://dx.doi.org/10.1143/PTPS.183.1, arXiv:1003.3552.

[35] S.F. King, C. Luhn, Neutrino mass and mixing with discrete symmetry, Rep. Prog. Phys. 76 (2013) 056201, http://dx.doi.org/10.1088/0034-4885/76/5/056201, arXiv:1301.1340.

[36] J. Schechter, J. Valle, Neutrino masses in SU(2) × U(1) theories, Phys. Rev. D 22 (1980) 2227, http://dx.doi.org/10.1103/PhysRevD.22.2227.

[37] H. Fritzsch, Quark masses and flavor mixing, Nucl. Phys. B 155 (1979) 189, http://dx.doi.org/10.1016/0550-3213(79)90362-6.

[38] H. Fritzsch, Weak interaction mixing in the six-quark theory, Phys. Lett. B 73 (1978) 317–322, http://dx.doi.org/10.1016/0370-2693(78)90524-5.

[39] P. Ramond, R. Roberts, G.G. Ross, Stitching the Yukawa quilt, Nucl. Phys. B 406 (1993) 19–42, http://dx.doi.org/10.1016/0550-3213(93)90159-M, arXiv:hep-ph/9303320.

[40] G. Branco, D. Emmanuel-Costa, R. Gonzalez Felipe, Texture zeros and weak basis transformations, Phys. Lett. B 477 (2000) 147–155, http://dx.doi.org/10.1016/S0370-2693(00)00193-3, arXiv:hep-ph/9911418.

[41] R. Roberts, A. Romanino, G.G. Ross, L. Velasco-Sevilla, Precision test of a fermion mass texture, Nucl. Phys. B 615 (2001) 358–384, http://dx.doi.org/10.1016/S0550-3213(01)00408-4, arXiv:hep-ph/0104088.

[42] H. Fritzsch, Z.-z. Xing, Four zero texture of Hermitian quark mass matrices and current experimental tests, Phys. Lett. B 555 (2003) 63–70, http://dx.doi.org/10.1016/S0370-2693(03)00048-0, arXiv:hep-ph/0212195.

[43] M. Gupta, G. Ahuja, Flavor mixings and textures of the fermion mass matrices, Int. J. Mod. Phys. A 27 (2012) 1230033, http://dx.doi.org/10.1142/S0217751X12300335, arXiv:1302.4823.

[44] S. Pakvasa, H. Sugawara, CP violation in six quark model, Phys. Rev. D 14 (1976) 305, http://dx.doi.org/10.1103/PhysRevD.14.305.

[45] S. Pakvasa, H. Sugawara, Discrete symmetry and Cabibbo angle, Phys. Lett. B 73 (1978) 61, http://dx.doi.org/10.1016/0370-2693(78)90172-7.

[46] E. Derman, Flavor unification, τ decay and b decay within the six quark six lepton Weinberg–Salam model, Phys. Rev. D 19 (1979) 317–329, http://dx.doi.org/10.1103/PhysRevD.19.317.

[47] Y. Yamanaka, H. Sugawara, S. Pakvasa, Permutation symmetries and the fermion mass matrix, Phys. Rev. D 25 (1982) 1895, http://dx.doi.org/10.1103/PhysRevD.29.2135.

[48] R. Yahalom, Horizontal permutation symmetry, fermion masses and pseudogoldstone bosons in $SU(2)_L$ × U(1), Phys. Rev. D 29 (1984) 536, http://dx.doi.org/10.1103/PhysRevD.29.536.

[49] G. Altarelli, Status of neutrino mass and mixing, Int. J. Mod. Phys. A 29 (2014) 1444002, http://dx.doi.org/10.1142/S0217751X14440023, arXiv:1404.3859.

[50] S.F. King, A. Merle, S. Morisi, Y. Shimizu, M. Tanimoto, Neutrino mass and mixing: from theory to experiment, New J. Phys. 16 (2014) 045018, http://dx.doi.org/10.1088/1367-2630/16/4/045018, arXiv:1402.4271.

[51] H. Fritzsch, Z.-z. Xing, Lepton mass hierarchy and neutrino mixing, Phys. Lett. B 634 (2006) 514–519, http://dx.doi.org/10.1016/j.physletb.2006.02.028, arXiv:hep-ph/0601104.

[52] H. Fritzsch, Z.-z. Xing, Relating the neutrino mixing angles to a lepton mass hierarchy, Phys. Lett. B 682 (2009) 220–224, http://dx.doi.org/10.1016/j.physletb.2009.11.018, arXiv:0911.1857.

[53] E. Schmidt, Zur theorie der linearen und nichtlinearen integralgleichungen – I. Teil: entwicklung willkürlicher funktionen nach systemen vorgeschriebener, Math. Ann. 63 (4) (1907) 433–476.

[54] G. Eckart, G. Young, The approximation of one matrix by another of lower rank, Psychometrika 1 (1936) 211–218.

[55] L. Mirsky, Symmetric gauge functions and unitarily invariant norms, Q. J. Math. 11 (1960) 50–59.

[56] G. Golub, A. Hoffman, G. Stewart, A generalization of the Eckart–Young–Mirsky matrix approximation theorem, Linear Algebra Appl. 88–89 (1987) 317–327.

[57] G. Stewart, On the early history of the singular value decomposition, SIAM Rev. 35 (4) (1993) 551–566.

[58] R. Barbieri, G. Dvali, L.J. Hall, Predictions from a U(2) flavor symmetry in supersymmetric theories, Phys. Lett. B 377 (1996) 76–82, http://dx.doi.org/10.1016/0370-2693(96)00318-8, arXiv:hep-ph/9512388.

[59] R. Barbieri, L.J. Hall, A. Romanino, Consequences of a U(2) flavor symmetry, Phys. Lett. B 401 (1997) 47–53, http://dx.doi.org/10.1016/S0370-2693(97)00372-9, arXiv:hep-ph/9702315.

[60] A. Crivellin, U. Nierste, Supersymmetric renormalisation of the CKM matrix and new constraints on the squark mass matrices, Phys. Rev. D 79 (2009) 035018, http://dx.doi.org/10.1103/PhysRevD.79.035018, arXiv:0810.1613.

[61] A. Crivellin, L. Hofer, U. Nierste, D. Scherer, Phenomenological consequences of radiative flavor violation in the MSSM, Phys. Rev. D 84 (2011) 035030, http://dx.doi.org/10.1103/PhysRevD.84.035030, arXiv:1105.2818.

[62] R. Barbieri, D. Buttazzo, F. Sala, D.M. Straub, Flavour physics from an approximate $U(2)^3$ symmetry, J. High Energy Phys. 1207 (2012) 181, http://dx.doi.org/10.1007/JHEP07(2012)181, arXiv:1203.4218.

[63] A.J. Buras, J. Girrbach, On the correlations between flavour observables in minimal $U(2)^3$ models, J. High Energy Phys. 1301 (2013) 007, http://dx.doi.org/10.1007/JHEP01(2013)007, arXiv:1206.3878.

[64] C.D. Carone, L.J. Hall, Neutrino physics from a U(2) flavor symmetry, Phys. Rev. D 56 (1997) 4198–4206, http://dx.doi.org/10.1103/PhysRevD.56.4198, arXiv:hep-ph/9702430.

[65] M. Tanimoto, Neutrino mass matrix with U(2) flavor symmetry and its test in neutrino oscillation experiments, Phys. Rev. D 57 (1998) 1983–1986, http://dx.doi.org/10.1103/PhysRevD.57.1983, arXiv:hep-ph/9706497.

[66] L.J. Hall, N. Weiner, U(2) and maximal mixing of muon-neutrino, Phys. Rev. D 60 (1999) 033005, http://dx.doi.org/10.1103/PhysRevD.60.033005, arXiv:hep-ph/9811299.

[67] R. Barbieri, P. Creminelli, A. Romanino, Neutrino mixings from a U(2) flavor symmetry, Nucl. Phys. B 559 (1999) 17–26, http://dx.doi.org/10.1016/S0550-3213(99)00425-3, arXiv:hep-ph/9903460.

[68] G. Blankenburg, G. Isidori, J. Jones-Perez, Neutrino masses and LFV from minimal breaking of $U(3)^5$ and $U(2)^5$ flavor symmetries, Eur. Phys. J. C 72 (2012) 2126, http://dx.doi.org/10.1140/epjc/s10052-012-2126-7, arXiv:1204.0688.

[69] A. Aranda, C.D. Carone, R.F. Lebed, U(2) flavor physics without U(2) symmetry, Phys. Lett. B 474 (2000) 170–176, http://dx.doi.org/10.1016/S0370-2693(99)01497-5, arXiv:hep-ph/9910392.

[70] L.J. Hall, H. Murayama, A geometry of the generations, Phys. Rev. Lett. 75 (1995) 3985–3988, http://dx.doi.org/10.1103/PhysRevLett.75.3985, arXiv:hep-ph/9508296.

[71] J. Kubo, A. Mondragon, M. Mondragon, E. Rodriguez-Jauregui, The flavor symmetry, Prog. Theor. Phys. 109 (2003) 795–807, http://dx.doi.org/10.1143/PTP.109.795, arXiv:hep-ph/0302196.

[72] S. Morisi, S(3) family permutation symmetry and quark masses: a model independent, arXiv:hep-ph/0604106.

[73] F. Feruglio, Y. Lin, Fermion mass hierarchies and flavour mixing from a minimal discrete symmetry, Nucl. Phys. B 800 (2008) 77–93, http://dx.doi.org/10.1016/j.nuclphysb.2008.02.008, arXiv:0712.1528.

[74] T. Teshima, Y. Okumura, Quark/lepton mass and mixing in S_3 invariant model and CP-violation of neutrino, Phys. Rev. D 84 (2011) 016003, http://dx.doi.org/10.1103/PhysRevD.84.016003, arXiv:1103.6127.

[75] R. Jora, J. Schechter, M. Naeem Shahid, Perturbed S(3) neutrinos, Phys. Rev. D 80 (2009) 093007, http://dx.doi.org/10.1103/PhysRevD.82.079902, arXiv:0909.4414.

[76] I. Masina, C.A. Savoy, Real and imaginary elements of fermion mass matrices, Nucl. Phys. B 755 (2006) 1–20, http://dx.doi.org/10.1016/j.nuclphysb.2006.07.021, arXiv:hep-ph/0603101.

[77] I. Masina, A maximal atmospheric mixing from a maximal CP violating phase, Phys. Lett. B 633 (2006) 134–140, http://dx.doi.org/10.1016/j.physletb.2005.10.097, arXiv:hep-ph/0508031.

[78] K. Olive, et al., Review of particle physics, Chin. Phys. C 38 (2014) 090001, http://dx.doi.org/10.1088/1674-1137/38/9/090001.

[79] M. Gonzalez-Garcia, M. Maltoni, T. Schwetz, Updated fit to three neutrino mixing: status of leptonic CP, arXiv:1409.5439.

[80] K. Chetyrkin, J.H. Kuhn, M. Steinhauser, RunDec: a Mathematica package for running and decoupling of the strong coupling and quark masses, Comput. Phys. Commun. 133 (2000) 43–65, http://dx.doi.org/10.1016/S0010-4655(00)00155-7, arXiv:hep-ph/0004189.

[81] Z.-z. Xing, H. Zhang, S. Zhou, Updated values of running quark and lepton masses, Phys. Rev. D 77 (2008) 113016, http://dx.doi.org/10.1103/PhysRevD.77.113016, arXiv:0712.1419.

[82] V. Aseev, et al., An upper limit on electron antineutrino mass from Troitsk experiment, Phys. Rev. D 84 (2011) 112003, http://dx.doi.org/10.1103/PhysRevD.84.112003, arXiv:1108.5034.

[83] A. Osipowicz, et al., KATRIN: a next generation tritium beta decay experiment with sub-eV sensitivity for the electron neutrino mass, Letter of intent. arXiv:hep-ex/0109033.