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ON IMBEDDING NUMBERS OF DIFFERENTIABLE MANIFOLDS†

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§1. INTRODUCTION AND RESULTS

THE PAPER is concerned with certain numerical invariants which may be assigned to manifolds. By a manifold we shall always understand a differentiable manifold of class C^∞. All imbeddings and immersions will be differentiable of class C^∞; we shall write $M \subset R^{n+k}$ if M can be imbedded in R^{n+k} and $M \propto R^{n+k}$ if M can be immersed in R^{n+k}.

Definition 1.1. Let M be a closed n-manifold and let $N_k(M)$ be the least integer N such that there exists a covering of M by N open sets $U_1, \ldots, U_N$ for which $U_i \subset R^{n+k}$, $i = 1, \ldots, N$. We shall call $N_k(M)$ the *imbedding covering number* of M in codimension k.

Remark. $N_0(M)$ has the simple interpretation of being the least number of charts needed in order to define the differentiable structure of M.

Definition 1.2. Let M be a closed n-manifold and let $n_k(M)$ be the least integer N such that there exists a covering of M by N open sets $V_1, \ldots, V_N$ for which $V_i \propto R^{n+k}$, $i = 1, \ldots, N$. We shall call $n_k(M)$ the *immersion covering number* of M in codimension k.

In this paper we shall be concerned only with properties of $n_0(M)$ and $N_0(M)$. We shall prove

THEOREM 1.3. *Let M be an $[\frac{n}{2}]$-parallelizable closed n-manifold, $n \neq 4$. Then $N_0(M) = 2$ provided that one of the following conditions is satisfied:*

i) *n is odd;*

ii) *$n = 4s$ and the index $\tau(M) = 0$;*

iii) *$n = 4s + 2$ and the Arf–Kervaire invariant $c(M) = 0$ (for some framing of a neighborhood of the $2s + 1$-skeleton of M).*

Recall that a manifold M is k-parallelizable if the restriction of its tangent bundle to its k-skeleton is trivial.

Remarks. a) The vanishing of $\tau(M)$ in case ii) is necessary for $N_0(M) = 2$ (see Proposition 2.7); so is probably the vanishing of $c(M)$ in case iii) although the author has been able to prove it only for $n = 8s + 2$ and M simply connected with $w_2(M) = 0$‡ (see Proposition 2.9).

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‡ See footnote on next page.

b) In Theorem 1.3 we have $N_0(M) = 2$ under the same circumstances in which M is framed cobordant to a homotopy sphere (see [8]). The reason for this will become apparent in Section 5, containing the proof of 1.3, which is based on surgery.

COROLLARY 1.4. *Let M be a closed stably parallelizable n-manifold, $n \neq 4$, and $n \neq 8s+6$. Then $N_0(M) = 2$.*‡

For the proof of 1.4 it is enough to notice that under our assumptions the index $\tau(M)$ or the Arf–Kervaire invariant $c(M)$ vanish (for the latter see [4].)

The other results of this paper concern the imbedding covering numbers of real projective spaces P^n. Although the author has been unable to determine them completely, the lower and upper bounds given in the following theorem are relatively close and sometimes even coincide.

THEOREM 1.5. *Let $n = 2^q r - 1$, where r is odd. Then*

$$n_0(P^n) = N_0(P^n) = \max\{r, 2\} \quad \text{if} \quad q \leqq 3 \tag{1.1}$$

and

$$\max\left\{r, 2, \left[\frac{2^{q-1}r-1}{q+1}\right]+1\right\} \leqq n_0(P^n) \leqq N_0(P^n)$$
$$\leqq \max\left\{2, \left[\frac{n}{k(q)+1}\right]+1\right\} \quad \text{if} \quad q \geqq 4 \tag{1.2}$$

where

$$k(q) = \begin{cases} 2q & \text{if} \quad q \equiv 0(4), \\ 2q-1 & \text{if} \quad q \equiv 1, 2(4), \\ 2q+1 & \text{if} \quad q \equiv 3(4). \end{cases}$$

The proof of 1.5 will be given in Section 6. Section 2 contains the statements and proofs of some more or less elementary facts concerning imbedding and immersion covering numbers.

The author wishes to thank the referee for a number of valuable suggestions which helped to improve the exposition.

§2. BASIC PROPERTIES OF COVERING NUMBERS

PROPOSITION 2.1. $n_k(M) \leqq N_k(M) \leqq N_0(M)$.
The proof is trivial.

PROPOSITION 2.2. *$N_0(M) \leqq n+1$ or more generally $N_0(M) \leqq \left[\frac{n}{k}\right]+1$ if M is $(k-1)$-connected.*

‡ The author hopes that the restriction $n \neq 8s+6$ can be removed, at least partially, in view of some recent unpublished work of W. Browder.

This follows from the fact [17] that any closed $(k-1)$-connected n-manifold can be covered by $\left[\frac{n}{k}\right]+1$ balls. (See also Proposition 6.1.)

Let $p: E \to B$ be a fibre bundle over B. Following Švarc [14], we shall say that the genus of p is $\leqq m$ if B can be covered by m open sets over which the bundle is trivial. If f is the classifying map for the associated principal bundle, then genus $p = \text{cat } f$, where the category map $f: B \to Y$ is defined as the least cardinal number of a covering of B by open sets over which f is null-homotopic (see [2]).

PROPOSITION 2.3. *If M is a parallelizable closed manifold, then $n_0(M) = 2$; in all other cases $n_0(M)$ is equal to the genus of the tangent bundle of M.*

Proof. Use the Hirsch–Poenaru theorem ([6], [12]) according to which a non-closed n-manifold can be immersed in R^n if and only if it is parallelizable.

Let H be any multiplicative (ordinary or extraordinary) cohomology theory and let $\tilde{H}$ be the corresponding reduced theory, i.e. $\tilde{H}(X) = \tilde{H}(X, *)$, where $*$ is the base-point. Let $T(M)$ be the tangent bundle of M and let $f: M \to BO(n)$ be the classifying map for $T(M)$. Denote by

$$f^*: \tilde{H}(BO(n)) \to \tilde{H}(M)$$

the induced map in cohomology.

PROPOSITION 2.4. *Let $v_i \in \tilde{H}(BO(n))$, $i = 1, \ldots, s-1$ and suppose that the product $f^*v_1 \cup \ldots \cup f^*v_{s-1} \neq 0$. Then $n_0(M) \geqq s$.*

Proof. We apply 2.3 and the standard argument connecting cohomology length and category (see e.g. the proof of Proposition 1.10 in [2]). For CW-complexes it is valid for any reduced multiplicative cohomology theory.

COROLLARY 2.5. *Suppose $w_{i_1} \cup \ldots \cup w_{i_s} \neq 0$, where w_i, $i = 1, \ldots, n$ are the Stiefel–Whitney classes of M. Then $n_0(M) \geqq s+1$.*

COROLLARY 2.6. *Let $[T] \in \tilde{K}_R(M)$ correspond to the tangent bundle $T(M)$, i.e. $[T]$ is represented by $T(M) - \theta^n$, where θ^n is the trivial n-bundle. Then $[T]^s \neq 0$ implies $n_0(M) \geqq s+1$.*

Let M be a $2k$-dimensional oriented manifold. The intersection pairing

$$\langle\ ,\rangle : H_k(M, \Lambda) \otimes H_k(M, \Lambda) \to \Lambda \tag{2.1}$$

(where Λ is a commutative ring with unit) is symmetric if k is even and antisymmetric if k is odd.

Suppose that $k = 2s$ and that $\Lambda = Q$ (the rationals). The signature of the quadratic form over Q defined by (2.1) is called the index of M and is denoted by $\tau(M)$. Let r be the rank of the form and m the dimension of a maximal self-annihilating space; then $\tau(M) = r - 2m$.

PROPOSITION 2.7. *Let the dimension of the closed manifold M be a multiple of 4. Then $N_0(M) = 2$ implies $\tau(M) = 0$.*

Before proceeding to the proof of 2.7, we shall deduce from it

COROLLARY 2.8. *There exist closed manifolds M for which $n_0(M) \neq N_0(M)$.*

Proof of 2.8. Kervaire and Milnor [9] have constructed examples of closed manifolds M^{4s} which are almost parallelizable but have a nonzero index. By 2.7 such a manifold has $N_0(M) \geqq 3$; being almost parallelizable means that $M - pt$ is parallelizable so that by 2.3 $n_0(M) \leqq 2$.

Proof of 2.7. Let $M = U \cup V$, where U and V are open $4s$-manifolds, $U, V \subset R^{4s}$. By Lemma 2.11 below, we may assume that $M = A \cup B$, $A \subset U$, $B \subset V$, where A and B are compact $4s$-manifolds with common boundary $C = A \cap B$. Take the rationals as coefficient group and consider the subspace $X \subset H_{2s}(M)$ generated by the images of $H_{2s}(A)$ and $H_{2s}(B)$. Let r be the rank of the intersection quadratic form (r is nothing else than the $2s^{\text{th}}$ Betti number of M). In order to prove that $\tau(M) = 0$ it is enough to show that X is a self-annihilating subspace ($\langle X, X\rangle = 0$) of $H_{2s}(M)$ with $m = \dim X = \dfrac{r}{2}$.

a) $\langle X, X\rangle = 0$. If $x_1 \in \operatorname{Im} H_{2s}(A)$ and $x_2 \in \operatorname{Im} H_{2s}(B)$ then $\langle x_1, x_2\rangle = 0$ because we may represent x_1 and x_2 by cycles with disjoint carriers in $A - C$ and $B - C$. On the other hand if say $x, y \in \operatorname{Im} H_{2s}(A)$ then $\langle x, y\rangle = 0$ since $A \subset R^{4s}$ and the intersection number of any two cycles in R^{4s} is zero; similarly for $x, y \in \operatorname{Im} H_{2s}(B)$ we have $\langle x, y\rangle = 0$.

b) Let $Y = H_{2s}(M)$ and let X^*, Y^* be the dual vector spaces of X and Y. The intersection pairing induces the duality isomorphism $D : Y \approx Y^*$ such that $(Dy)(z) = \langle y, z\rangle$ $y, z \in Y$. In order to prove that $r = \dim Y = 2 \dim X$ it is enough to show the exactness of the sequence

$$0 \to X \xrightarrow{Di} Y^* \xrightarrow{i^*} X^* \to 0$$

where $i : X \to Y$ is the inclusion. The inclusion $\operatorname{Im} Di \subset \operatorname{Ker} i^*$ follows from a), while the inclusion $\operatorname{Ker} i^* \subset \operatorname{Im} Di$ is the consequence of the following remark:

c) Let $l : M \subset (M, B)$ and $e : (A,C) \to (M, B)$ be inclusions and let $y \in Y$ be such that $i^*Dy = 0$, i.e. $Dy(x) = \langle y, x\rangle = 0$ for every $x \in X$. Let $z = e_*^{-1}l_*(y)$; z is well defined since e_* is an excision isomorphism. For an arbitrary $\bar{x} \in H_{2s}(A)$ whose image in X is x we have $\langle z, \bar{x}\rangle = \langle z, x\rangle = 0$, which implies by Lefschetz duality that $z = 0$. Thus $l_*(y) = 0$, and by exactness of the homology sequence of the pair (M, B) and by the definition of X, we have $y \in \operatorname{Im} i$ whence $Dy \in \operatorname{Im} Di$.

Let us now consider the case of manifolds of dimension $8s + 2$. We shall this time take in (2.1) $\Lambda = Z_2$, so that the pairing is again symmetric. Let us first recall the definition of the Arf–Kervaire invariant as given by Brown [3]. There exists a secondary cohomology operation

$$\begin{aligned}\psi : H^{4s+1}(K, L) \cap \operatorname{Ker} Sq^{4s} \cap \operatorname{Ker} Sq^2Sq^{4s-1}\\ \to H^{8s+2}(K, L)/(\operatorname{Im} Sq^2 + \operatorname{Im} Sq^1)\end{aligned} \tag{2.2}$$

for any CW-pair (K, L). In the case of a simple connected closed $(8s + 2)$-manifold M admitting a spin structure (i.e. such that $w_2(M) = 0$). ψ is defined on all of H^{4s+1} and has no indeterminacy in H^{8s+2}. For any $u \in H_{4s+1}(M)$ define $c(u) = \psi(Du)[M]$, where Du is the dual cohomology class. We can always choose a symplectic basis of $H_{4s+1}(M)$, i.e. a basis $u_1, \ldots, u_m, v_1, \ldots, v_m$ such that $\langle u_i, v_j\rangle = \delta_{ij}$, $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0$. The Arf–Kervaire invariant $c(M)$ is defined as

$$c(M) = \sum_{i=1}^{m} c(u_i)c(v_i). \tag{2.3}$$

PROPOSITION 2.9. *Let M be an $(8s+2)$-dimensional simply connected closed spin manifold with $N_0(M) = 2$. Then $c(M) = 0$.*

The proof of 2.9 depends on Lemma 2.10 below. Let $A \subset R^{8q+2}$ be a compact $8s+2$-dimensional manifold with boundary C.

LEMMA 2.10. *Under the above assumptions, the operation*

$$Sq^{4s} : H^{4s+1}(A, C) \to H^{8s+1}(A, C)$$

is trivial.

Proof. Let $D = R^{8s+2}$-Int A. It follows from the Mayer–Vietoris sequence of the triad (R^{8s+2}, A, D) that the inclusion map induces a monomorphism $H^i(A) \to H^i(C)$ for all i and that therefore the co-boundary $\delta : H^{4s}(C) \to H^{4s+1}(A, C)$ is an epimorphism. Since $\delta Sq^{4s} = Sq^{4s}\delta$, it suffices to prove the triviality of

$$Sq^{4s} : H^{4s}(C) \to H^{8s}(C).$$

We may look upon C as imbedded in S^{8s+2}; let DC be a deformation retract of $S^{8s+2} - C$. It has been shown in [15] (see also [11] and [13], Ch. 3) that the action of Sq^{4s} on $H^{4s}(C)$ corresponds by Alexander duality to the action of a stable cohomological operation

$$\chi(Sq^{4s}) : H^1(DC) \to H^{4s+1}(DC).$$

But, it is well known that any element of even degree of the Steenrod algebra acts trivially on 1-dimensional classes (see for instance [13], Ch. 1, Lemma 2.4); going back to C by Alexander duality, we obtain the desired result.

Proof of 2.9. Let us assume that $M = A \cup B$, where A and B are $(8s+2)$-manifolds with common boundary $C = A \cap B$ (see Lemma 2.11 below) and $A \subset R^{8s+2}$ and $B \subset R^{8s+2}$. If we denote by X the subspace of $H_{4s+1}(M; Z_2)$ generated by the images of $H_{4s+1}(A)$ and $H_{4s+1}(B)$, the argument used to prove 2.7 (which is independent of the characteristic of the coefficient field), shows that X is a maximal selfannihilating subspace. A standard argument allows us to choose a symplectic basis $u_1, \ldots, u_m, v_1, \ldots, v_m$ of $H_{4s+1}(M)$ such that the elements $u_1, \ldots, u_m$ form some basis of X. Let us assume that $u \in \operatorname{Im} H_{4s+1}(A)$ (the case $u \in \operatorname{Im} H_{4s+1}(B)$ is similar). Then by duality $Du \in H^{4s+1}(M)$ lies in the image of $H^{4s+1}(M, B) \approx H^{4s+1}(A, C)$ i.e. $Du = l^*(w)$, $w \in H^{4s+1}(M, B)$. By Lemma 2.10, $Sq^{4s}w = 0$; on the other hand $Sq^2Sq^{4s-1}(w) = 0$, since the fundamental class of the imbedded manifold pair (A, C) is spherical. According to (2.2), $\psi(w) = 0$, again by the sphericity of the fundamental class. Since $\psi(w) = l^*\psi(Du)$, we have $c(u) = 0$. Thus ψ vanishes on all elements $u_1, \ldots, u_m$; this means by (2.3) that $c(M) = 0$.

LEMMA 2.11. *Let M be a closed n-manifold covered by two open subsets U_1 and U_2. Then there exist two compact manifolds $A \subset U_1, B \subset U_2$ with common boundary $C = A \cap B$ such that $M = A \cup B$.*

Proof. Let $K_1 = M - U_2$, $K_2 = M - U_1$; then $K_1 \cap K_2 = \varnothing$ and K_1 and K_2 are compact. Define a differentiable function $\lambda : M \to I$ (where I is the unit interval) with $\lambda(K_1) = 0$ and $\lambda(K_2) = 1$. According to Sard's theorem, there exists a regular value $a \in I$ of λ such that $0 < a < 1$. Then $A = \lambda^{-1}([0, a]) \subset U_1$, $B = \lambda^{-1}([a, 1]) \subset U_2$ and $C = \lambda^{-1}(a)$ have the required properties.

§3. GEOMETRIC LEMMAS

Throughout this section we shall use the terminology and some of the notation of [10], which will be our main reference.

Construct a Morse function $f : M \to R$ and a gradient like field ξ for f ([10], p. 20). Let us introduce the following notations; $A_a = f^{-1}((-\infty, a])$, $B_a = f^{-1}([a, \infty))$, $C_a = f^{-1}(a)$. We shall call f k-almost nice if f is self indexing, except that the values of f in critical points of index k may be any numbers between $k - 1$ and $k + 1$.

For a fixed $0 < i < n$ consider the "cobordism" $(A_b \cap B_a, C_a, C_b)$, $i - 1 < a < i < b < i + 1$. In $A_b \cap B_a$ the function f has only nondegenerated critical points $P_1, \ldots, P_m$ of index i and $f(P_j) = i$, $j = 1, \ldots, m$. The union of all trajectories of ξ in $A_b \cap B_a$ which start on C_a and end at P_j forms a differentiably imbedded i-disk; following [10] we shall call it the left-hand disk of P_j and denote it by $D_L^a(P_j)$, or, if no confusion can arise, by $D_L(P_j)$. Similarly, the union of all trajectories of ξ which end on C_b and which tend for $t \to -\infty$ to P_j forms the right-hand disk $D_R(P_j)$.

LEMMA 3.1. *Let a be a non-critical level such that A_a contains only points of index $\leqq i$. Then A_a has the homotopy type of an i-dimensional CW-complex.*

Proof. We apply the usual argument of Morse theory.

LEMMA 3.2. *Let A_a be as above and let K be a CW-complex of dimension $<n - i$. Then any map $\varphi : K \to A_a$ can be deformed* rel $\varphi^{-1}(C_a)$ *into a map $K \to C_a$.*

Proof. The union of all trajectories of ξ which end in critical points lying in A_a forms an i-dimensional complex $L \subset A_a$. Since $\dim K + i < n$, by a general position argument we can deform φ rel $\varphi^{-1}(C_a)$ into a map ψ such that $\psi(K) \cap L = \varnothing$. Through any point $x \in \psi(K)$ passes a unique trajectory of ξ, which ends in C_a and all we have to do is to push ψ into C_a along these trajectories.

Unless we specify the contrary the following assumptions will be made from now on: $n = 2k$, $p \leqq k - 1$, M^n is k-parallelizable and f is k-almost nice. If $k < b < k + 1$, this means, according to Lemma 3.1, that we may define a framing of the stable tangent bundle over A_b. By Lemma 3.2 any map $S^p \to M$ can be deformed into a map $\varphi_0 : S^p \to C_a$. where $a < b$ is a fixed non-critical level $k - 1 < a < k + 1$, and since $2p + 1 \leqq 2k - 1 = \dim C_a$, we may assume that φ_0 is an imbedding. By referring again to a general position argument, we may assume also that all trajectories of ξ which start or end in some neighborhood U

of $\varphi_0(S^p)$ can be extended up and down to levels $a+\mu$ and $a-\nu$, where $n-p-1<a+\mu<n-p$, $p<a-\nu<p+1$, $\mu, \nu>0$. Since $\varphi_0(S^p)$ has a stably trivial normal bundle in $C_a\subset A_b$, and $2p<n-1=\dim C_a$, $\varphi_0(S^p)$ has a trivial normal bundle in C_a. Let $q=n-p-1$; define an imbedding $\psi : S^p\times D^q\to C_a$, $\psi(S^p\times D^q)\subset U$ such that $\psi|S^p\times 0=\varphi_0$ and extend it to an imbedding $\varphi : S^p\times D^{q+1}\to M$ in the following way. Let $u=(u_0,\dots,u_p)$, $\bar v=(v_0,\dots,v_{q-1})$, $v=(v_0,\dots,v_q)$, $u\in D^p$, $\|u\|\leqq 1$, $\bar v\in D^q$, $\|\bar v\|\leqq 1$, $v\in D^{q+1}$, $\|v\|\leqq 1$. For $(u,v)\in S^p\times D^{q+1}$ define $\varphi(u,v)$ as the point P on the trajectory through $\psi(u,\bar v)$ lying on the level $\lambda(v_q)$, where λ is a C^∞ function $[-1,1]\to R$ such that $\lambda'(y)>0$, $p<\lambda(-1)<\lambda(0)=a<\lambda(1)<q+1$. Such a point is well defined, since for $|v_q|\leqq 1$, P lies between the levels p and $q+1$, where all trajectories passing through the neighborhood $U\supset\varphi_0(S^p)$ can be continued without meeting critical points. Moreover, the framing over A_b can be extended in a trivial way to $A_b\cup\varphi(S^p\times D^{q+1})$, which has the same homotopy type as A_b.

LEMMA 3.3. *In the above situation, with φ_0 fixed and λ fixed, the imbedding $\psi : S^p\times D^q\to C_a$ can be chosen in such a way that the manifold A^* obtained from $A_b\cup(S^p\times D^{q+1})$ by surgery along φ be framed.*

Proof. A^* is obtained from $((A_b\cup\varphi(S^p\times D^{q+1}))-\varphi_0(S^p\times 0))\cup(D^{p+1}\times S^q)$ via the identification $\varphi(u,tv)\sim(tu,v)$, $0<t\leqq 1$, so that a framing is defined in A^*- Image of $0\times S^q$; it is easy to see that the unique obstruction to the extension of the framing is given by an element $\chi_\varphi\in H^{p+1}(D^{p+1}\times S^q, 0\times S^q;\pi_p(SO))=\pi_p(SO)$. Furthermore, Kervaire and Milnor [8] have shown that if $\alpha : S^p\to SO_{q+1}$ and if $s_{q+1}:\pi_p(SO_{q+1})\to\pi_p(SO)$ is induced by inclusion, then the map $\varphi_\alpha : S^p\times D^{q+1}\to M$, defined by $\varphi_\alpha(u,v)=\varphi(u,v\,.\,\alpha(u))$ satisfies $\chi_{\varphi_\alpha}=\chi_\varphi+(s_{q+1})_*([\alpha])$, where $[\alpha]\in\pi_p(SO_{q+1})$ is the class of α. Under our assumptions, $p<q=n-p-1$ and therefore $s_{q,q+1}:\pi_p(SO_q)\to\pi_p(SO_{q+1})$ is an epimorphism, while s_{q+1} is an isomorphism. We may therefore choose $\beta : S^p\to SO_q$ such that $(s_{q+1}\circ s_{q,q+1})_*[\beta]=-\chi_\varphi$. Then, for the map φ_β defined with the help of ψ_β, and the fixed φ_0 and λ, where $\psi_\beta(u,\bar v\,.\,\beta(u))$, we have

$$\chi_{\varphi_\beta}=0$$

and the corresponding framing can be extended.

LEMMA 3.4. *Let ψ,φ be chosen as in Lemma 3.3 and let M^* be the manifold obtained from M be surgery along φ. There exists a Morse function f^* on M^*, which coincides with f on the complement of some neighborhood of $\varphi(S^p\times 0)$ and which has exactly two non-degenerate critical points, in addition to those of f. One of these critical points has index $p+1$ and the other has index q. Moreover,*

i) *if $p+1=q=k$, and if P and Q are the two additional critical points (of index k), then $f^*(P)=a-\varepsilon_1>k-1$, $f^*(Q)=a+\varepsilon_2<k+1$, $\varepsilon_1,\varepsilon_2>0$;*

ii) *if $p+1<q$, $f^*(P)=p+1$, $f^*(Q)=q=n-p-1$.*

Proof. M^* is obtained from the disjoint union $(M-\varphi(S^p\times 0))\cup(D^{p+1}\times S^q)$ via the identification $\varphi(u,tv)\sim(tu,v)$, $\|u\|=1$, $\|v\|=1$, $0<t\leqq 1$. Thus f induces under this identification a map $F:((D^{p+1}\times S^q)-(0\times S^q))\to R$, $F(u,v)=\lambda(\|u\|v_q)$. The problem reduces now to the definition of a new function $F^*(u,v)$, $(u,v)\in D^{p+1}\times S^q$, which coincides

with $F(u, v)$ on some neighborhood of the boundary $S^p \times S^q = \partial(D^{p+1} \times S^q)$ and which has exactly two critical points in the interior.

Let $\mu : [0, 1] \to [0, 1]$ be a function of class C^∞, such that

$$\mu(x) = x^2 + \tfrac{1}{2} \quad \text{for} \quad 0 \leqq x \leqq \tfrac{1}{2},$$

$$\mu(x) = x \quad \text{for} \quad \tfrac{7}{8} \leqq x \leqq 1,$$

$$\mu'(x) > 0 \quad \text{and} \quad \mu(x) > 0 \quad \text{for} \quad 0 < x < 1.$$

Define $F^*(u, v) = \lambda(\mu(\|u\|)v_q)$. Direct computation, by taking $(v_0, \ldots, v_{q-1})$ as local coordinates on S^p if $v_q < 0$ or $v_q > 0$, and v_q as one of the coordinates in the neighborhood of $v_q = 0$, shows the following:

a) $F^*(u, v)$ has as its only critical points $\|u\| = 0$, $v_q = \pm 1$; we shall denote by P the point $\|u\| = 0$, $v_q = -1$, and by Q the point $\|u\| = 0$, $v_q = +1$. The index of P is $p + 1$ and the index of Q is q.

b) For $p + 1 < q$ we choose the function λ so that $\lambda(-\frac{1}{2}) = p + 1$, $\lambda(\frac{1}{2}) = q$. Thus in this case $F^*(P) = p + 1$, $F^*(Q) = q$.

c) For $p + 1 = q = k$ we take $\lambda(-\frac{1}{2}) = a - \varepsilon_1 < a < \lambda(\frac{1}{2}) = a + \varepsilon_2$; then $k - 1 < F^*(P) < a < F^*(Q) < k + 1$.

The required function f^* is defined on $M^* = \pi((M - \varphi(S^p \times 0)) \cup D^{q+1} \times S^q)$, where π is the identification map, by setting $f^* = f\pi^{-1}$ on $\pi(M - \varphi(S^p \times D^{q+1}_{7/8}))$ and $f^* = F^* \pi^{-1}$ on $\pi(D^{p+1} \times S^q) - (D^{p+1}_{7/8} \times S^q))$; here $D_{7/8}$ is the ball of radius 7/8.

§4. ALGEBRAIC LEMMAS

We continue to assume here that $n = 2k$, $k > 2$, that M^n is k-parallelizable, and that a k-almost nice function f and a gradient like field are defined on M, the notations being the same.

Let $0 < a_0 < 1 < \ldots < i < a_i < i + 1 < \ldots < n = a_n$ and let us use the notations $A_i = A_{a_i}$, $C_i = C_{a_i}$, $B_i = B_{a_i}$, $W_i = A_i \cap B_{i-1}$. We assume that all critical points of index k lie between C_{k-1} and C_k. Let $X_i = H_i(W_i, C_{i-1})$; X_i is a free abelian group generated by the oriented left-hand disks of the critical points of index i; the composition

$$H_i(W_i, C_{i-1}) \to H_i(A_i, A_{i-1}) \to H_{i-1}(A_{i-1}, A_{i-2}) \leftrightarrow H_{i-1}(W_{i-1}, C_{i-2})$$

defines a boundary operator $\partial : X_i \to X_{i-1}$. The homology of the chain-complex (X, ∂) is isomorphic to $H_*(M; Z)$ [10]. Similarly, the right-hand disks generate a chain-complex $(\bar{X}, \bar{\partial})$, also yielding the homology of M, and the intersection between left-hand and right-hand disks defines an orthogonal pairing

$$\langle , \rangle : X_i \otimes \bar{X}_{n-i} \to Z.$$

With respect to this pairing $\bar{\partial}$ is the adjoint of ∂ [10].

Remark. Given any chain complex X we can always add to it an elementary chain complex with two generators $x \in X_i$ and $x \in X_{i+1}$ such that $\partial y = x$, and that the resulting chain complex X' has the following property

(α_i) given any $\gamma \in H_i(X')$, there is a representative $c' \in \gamma$, $c' \in X_i'$, which is indivisible.

Indeed, if $c \in X$ represents γ, $c = c' + x \sim c$ is indivisible. We shall assume henceforth that both (X, ∂) and $(\bar{X}, \bar{\partial})$ have property (α_{k-1}). This is easily achieved by adding pairs of non-essential critical points of index $k - 1$ and k to f and $-f$ [see e.g. [10, §8].†

Choose $a_{k-1} < t < k + 1$ and an imbedding of a sphere $\varphi_0 : S^{k-1} \to C_t$, and perform the framed surgery along φ_0 described in 3.3, by modifying f and ξ as shown in 3.4. Let f^* and ξ^* be the new function and field.

LEMMA 4.1. *If the homology class γ is represented by φ_0, one can further modify f^* and ξ^* in the neighborhood of the additional critical points P and Q, such that the new function and field (also denoted by f^* and ξ^*) satisfy the following conditions:*

i) $f^*(P) = t - \varepsilon_1 > k - 1, f^*(Q) = t + \varepsilon_2 < k + 1.$

ii) *If (X^*, ∂^*) is the new chain-complex, $X_i^* = X_i$ for $i \neq k$ and $X_k = X_k + F + G$ where F and G are infinite cyclic groups with generators $a = D_L(P)$ and $b = D_L(Q)$;*

iii) $\partial^*|X_i^* = \partial|X_i$ *for* $i \neq k, k + 1$; $\partial^*|X_k = \partial|X_k$;

iv) *The class of $\partial^* a$ in $H_{k-1}(X, \partial) = H_{k-1}(M)$ is γ;*

v) $\partial^* b = 0$ *and there exists* $h \in X_{k+1}^* = X_{k+1}$, *such that* $b - \partial^* h \in X_k$.

Proof. According to 3.4, $X_k^* = X_k + K$, where K is the free abelian group generated by the left-hand disks of the new critical points P and Q. Since γ is killed by surgery in M^*, we have for some $a \in K$, $\partial^* a = c$, where $c \in X_{k-1}$ represents γ in (X, ∂). In the dual complex $(\bar{X}^*, \bar{\partial}^*)$; $\bar{X}_k^* = \bar{X}_k + \bar{K}$, where $\bar{K}$ is generated by the right-hand disks of P and Q. Here we also have

$$\bar{c} = \bar{\partial}^* \bar{b} \tag{4.1}$$

where $\bar{b} \in K$ and $\bar{c}$ represents γ. According to (α_{k-1}) we can assume that both $\bar{c}$ and c are indivisible; if not we may add to a and $\bar{b}$ some elements of X_k or $\bar{X}_k$ as in the proof of the basis theorem. This will change the representatives of c and $\bar{c}$ so that they become indivisible. Since $\bar{c}$ is indivisible, there exists $h \in X_{k+1}$ such that $\langle h, \bar{c}\rangle = \langle h, \bar{\partial}^* \bar{b}\rangle = \langle \partial^* h^*, \bar{b}\rangle = 1$. Let $\partial^* h = x + b$ where $x \in X_k$, $b \in K$. Then $\langle x, \bar{b}\rangle = 0$ and we have $\langle b, \bar{b}\rangle = 1$. I claim that $\{a, b\}$ is a basis for K with the required properties.

First, it is clear that $x + b = \partial^* h$ means that $x = \partial h$, since the incidence numbers between the $(k + 1)$-disks and the k-disks are not affected by surgery, so that the component of $\partial^* h$ in X is exactly ∂h. Therefore, $\partial^* x = \partial x = 0$, which implies that $\partial^* b = 0$. If now $\mu a + \nu b = 0$, $\partial^*(\mu a + \nu b) = 0$, whence $\mu c = 0$ and $\mu = 0$ and $\nu = 0$. Next, if $\mu a + \nu b = \eta d$, $d \in K$ and μ and ν are relatively prime, we have on one hand, by applying ∂^*, $\mu c = \eta \partial^* d$, whence $\eta | \mu$, and on the other hand $\eta\langle d, \bar{b}\rangle = \langle \mu a + \nu b, \bar{b}\rangle = \mu\langle a, \bar{b}\rangle + \nu$, whence $\eta | \nu$, so that $\eta = 1$. This means that $\{a, b\}$ generate a subgroup $L \subset K$ of rank 2 such that K/L is free, i.e. $L = K$.

Now, as in the proof of the basis theorem [10], we modify the function f^* and the gradient like field in the image of $D^k \times S^{n-k}$ in M^*, so that the left-hand disks of P and Q

† The author is indebted for the above remark to W. Browder.

represent the basis a and b of K. This can be done without affecting the values $f^*(P) = t - \varepsilon_1, f^*(Q) = t + \varepsilon_2$. The basis $\{a, b\}$ satisfies all the requirements of 4.1.

Let $\bar{a} = D_R(P)$, $\bar{b} = D_R(Q)$, where we are in the conditions of Lemma 4.1. Then

LEMMA 4.2. *$\bar{\partial}^*\bar{a} = 0$, $\bar{\partial}^*\bar{b} = \bar{c}$, where the class of $\bar{c}$ in $H_{k-1}(\bar{X}, \bar{\partial}) = H_{k-1}(M)$ represents γ. Moreover there exists $\bar{h}$ in $\bar{X}_{k+1} = \bar{X}_{k+1}$ such that $\bar{a} - \bar{\partial}^*\bar{h} \in \bar{X}_k$.*

Thus the roles of $\bar{a}$ and $\bar{b}$ are reversed.

Proof. We have $\langle a, \bar{a}\rangle = \langle b, \bar{b}\rangle = 1$, $\langle a, \bar{b}\rangle = \langle b, \bar{a}\rangle = 0$. If $\bar{\partial}^*\bar{a} \neq 0$, there exists $z \in X_{k+1}$ such that $\langle z, \partial^*\bar{a}\rangle = \langle \partial^* z, \bar{a}\rangle = q \neq 0$. This implies that $\partial^* z = qa + pb + y$, $y \in X_k$, $y = \partial z$. Applying ∂^* again, and noticing that $\partial^* y = \partial y$ and $\partial^* b = 0$ we obtain

$$0 = qc + \partial y = qc + \partial\partial z = qc$$

which is a contradiction, showing that $\bar{\partial}^*\bar{a} = 0$.

On the other hand, for some $\mu\bar{a} + \nu\bar{b}$ we have $\bar{\partial}^*(\mu\bar{a} + \nu\bar{b}) = \bar{c}$ where $\bar{c}$ represents γ in $(\bar{X}, \bar{\partial})$. Since $\partial^* a = 0$, it follows immediately that $\nu\bar{\partial}^* b = \bar{c}$, whence $\bar{\partial}^*\bar{b} = \bar{c}$ because $\bar{c}$ is indivisible.

The existence of $\bar{h}$ is proved exactly as that of h in 4.1.

Let M^* be a $(k-1)$-connected manifold obtained from M by framed surgery. We may assume that the last stage of the surgery is realized by killing the generators of a direct sum decomposition of $H_{k-1}(M)$ in a minimal number of cyclic groups. Moreover, if f^*, ξ^* are obtained from the original f, ξ by successively applying 3.3 and 3.4, then $X_k^* = X_k + F + G$, where F is generated by a_i and G is generated by b_i. Each element a_i, b_i and their duals satisfy the conditions of 4.1 and 4.2.

LEMMA 4.3. *Let $u \in H_k(M^*) = H_k(X^*, \partial^*) = H_k(\bar{X}^*, \bar{\partial}^*)$. We can choose representatives $x^* \in X_k^*$, $\bar{y}^* \in \bar{X}_k^*$ of u such that*

$$x^* = x + \sum \mu_i a_i, \qquad x \in X_k,$$
$$\bar{y}^* = \bar{y} + \sum \bar{\nu}_i \bar{b}_i, \qquad \bar{y} \in \bar{X}_k.$$

Proof. In view of 4.1 and 4.2 it is enough to prove the first of the two statements; the proof of the other is similar.

In general we have

$$x^* = x + \sum \mu_i a_i + \sum \nu_i b_i;$$

however, in view of 4.1, v) the cycles b_i may be successively replaced by homologous cycles not containing b_i.

LEMMA 4.4. *If $z = \sum \mu_i a_i$ is a cycle, then $z = 0$.*

Proof. Let $\mu_i \neq 0$ for some i, whence $\langle z, \bar{a}_i\rangle \neq 0$. By 4.3, $\bar{a}_i \sim \bar{y}^*$, where $\bar{y}^* = \bar{y} + \sum \nu_i \bar{b}_i$, $\bar{y} \in \bar{X}_k$ and $\langle z, \bar{y}^*\rangle = 0$, which is a contradiction.

Let $u_1, \ldots, u_m, v_1, \ldots, v_m$ be a basis for $H_k(M^*)$. Let $x_i^* \in X_k^*$ be representatives for u_i, $i = 1, \ldots, m$, chosen in accordance with 4.3.

$$x_i^* = x_i + \sum \mu_{ij} a_j, \; x_i \in X_k. \tag{4.2}$$

LEMMA 4.5. *The elements $x_1, \ldots, x_m$ can be extended to a basis of X_k.*

Proof. It is enough to show that if the g.c.d. of a system of numbers $\lambda_1, \ldots, \lambda_m$ is 1, then $x = \lambda_1 x_1 + \cdots + \lambda_m x_m$ is not divisible in X_k.

Let $x^* = \lambda_1 x_1^* + \cdots + \lambda_m x_m^*$; $x^* \in X_k^*$ is a representative cycle of $u = \lambda_1 u_1 + \cdots + \lambda_m u_m$. Since u is indivisible and of infinite order, there exists $v \in H_k(M^*)$ such that $\langle u, v\rangle = 1$. Let $\bar{y}^* = \bar{y} + \sum \bar{v}_i \bar{b}_i$ represent v according to 4.3, where $\bar{y} \in \bar{X}_k$. We have $\langle u, v\rangle = \langle x^*, \bar{y}^*\rangle = \langle x, \bar{y}\rangle = 1$ since $\langle a_k, \bar{b}_j\rangle = \langle x, \bar{a}_j\rangle = \langle a_j, \bar{y}\rangle = 0$. This means that x cannot be divisible.

§5. PROOF OF 1.3

Let us suppose that either k is even and the index of M is zero, or the Arf–Kervaire invariant of M is zero. Start with a nice (selfindexing) function f on M. In both cases we can perform framed surgery on M, as described in the previous two sections such that the resulting $(k-1)$-connected manifold M^* possesses a symplectic basis $u_1, \ldots, u_m, v_1, \ldots, v_m$. In the zero Arf–Kervaire invariant case we may assume that $c(u_1) = \ldots = c(u_m) = 0$. Let x_i^* be representatives of u_i of the form (4.2). Apply 4.5 and the basis theorem [10] to the original manifold M and the original nice function f and modify f, ξ in the neighborhood of the level k, so that $x_1, \ldots, x_m$ are a part of a basis of X_k and are represented by left-hand disks of critical points $R_1, \ldots, R_m$. By changing slightly f, we may assume that $f(R_i) = t - \varepsilon_1 > k - 1$ and that any other critical point of index k of f lies on the level $t + \varepsilon_2 < k + 1$.

Let A_t and B_t have the meaning of the beginning of Section 3 with respect to the new function f.

LEMMA 5.1. *A_t and B_t can be imbedded in R^n.*

Before proving 5.1 we shall prove a few additional lemmas.

Let f^*, ξ^* be obtained from f, ξ according to 3.3 and 3.4 by performing again the surgery which makes M^* $(k-1)$-connected, in the same order as before and on imbeddings of $S^p \times D^{n-p}$ isotopic to the original 'ones with $\varphi(S^p \times D) \subset C_t$, so that M^* is $(k-1)$-connected.

LEMMA 5.2. *$A_t \subset A_t^*$, $B_t \subset B_t^*$, where A_t^*, B_t^* are related to f^*, ξ^* in the same way in which A_t and B_t are related to f, ξ.*

Proof. According to 3.4, each elementary surgery introduces one additional critical point below the level t, i.e. a handle is attached to A_t. Similarly a handle is attached to B_t.

Let $C_t^* = A_t^* \cap B_t^*$ i.e. $C_t^* = f^{*-1}(t)$.

LEMMA 5.3. i) *The inclusion $A_t^* \to M^*$ induces a monomorphism $H_k(A_t^*) \to H_k(M^*)$; its image is generated by the elements $u_1, \ldots, u_m$.*

ii) *The inclusion $C_t^* \to A_t^*$ induces an epimorphism $H_k(C_t^*) \to H_k(A_t^*)$.*

Proof. i) Let (Y, ∂^*) be the chain-complex generated by the left-hand disks of f^*, ξ^* lying in A_t^*. Clearly $Y_i^* = X_i^*$ for $i < k$, $Y_i^* = 0$ for $i > k$ a d Y_k^* is generated by

$x_1, \ldots, x_m, a_1, \ldots, a_q$. Therefore the group of cycles of Y_k^* coincides with $H_k(A_t^*)$ and by 4.3 it contains $x_1^*, \ldots, x_m^*$, which are mapped by inclusion on $u_1, \ldots, u_m$. If $x^* = \sum \lambda_i x_i + \sum \mu_i a_i$ is a cycle, then by 4.4., $z = x^* - \sum \lambda_i x_i^*$ vanishes, so that $x^* = \sum \lambda_i x_i^*$.

ii) Since $H_k(A_t^*)$ has no torsion, by Lefschetz duality the intersection pairing between $H_k(A_t^*)$ and $H_k(A_t^*, C_t^*)$ is orthogonal. It is enough therefore to show that the intersection number of any two cycles in A_t^* is zero, which is immediate since $\langle u_i, u_j \rangle = 0$ for all i and j. Indeed, this implies that $H_k(A_t^*) \to H_k(A_t^*, C_t^*)$ is trivial, which immediately yields ii).

LEMMA 5.4. *C_t^* is simply connected and one may find elements $z_1, \ldots, z_m \in H_k(C_t^*)$ whose images are $u_1, \ldots, u_m$ and which are represented by spherical cycles.*

Proof. The first assertion is a ready consequence of 3.2; the second follows from the next two commutative diagrams of exact sequences

$$\begin{array}{ccccccc} & & \pi_k(A_t^*) & \to & \pi_k(M^*) & \to & \pi_k(M^*, A_t^*) \\ & & \downarrow & & \downarrow \approx & & \downarrow \approx \\ 0 & \to & H_k(A_t^*) & \to & H_k(M^*) & \to & H_k(M^*, A_t^*) \end{array} \tag{5.1}$$

$$\begin{array}{ccccc} \pi_k(C_t^*) & \to & \pi_k(A_t^*) & \to & \pi_k(A_t^*, C_t^*) \\ \downarrow & & \downarrow & & \downarrow \approx \\ H_k(C_t^*) & \to & H_k(A_t^*) & \to & H_k(A_t^*, C_t^*) \end{array} \tag{5.2}$$

By 3.2, the pairs (M^*, A_t^*) and (A_t^*, C_t^*) are $(k-1)$-connected so that the last vertical arrows in (5.1) and (5.2) are relative Hurewicz isomorphisms. The second vertical arrow in (5.1) is an absolute Hurewicz isomorphism. Since u_i is the image of $x_i^* \in H_k(A_t^*)$, its image in $H_k(M^*, A_t^*)$ is trivial, so that its representative $\eta_i \in \pi_k(M^*)$ is null-homotopic in $\pi_k(M^*, A_t^*)$, which means that η_i is the image of some $\bar{\eta}_i \in \pi_k(A_t^*)$. The latter has to be mapped onto x_i^* since by 5.3. i) the lower left arrow is a monomorphism. In (5.2) the image of $\bar{\eta}_i$ in $\pi_k(A_t^*, C_t^*)$ is zero (since the image of x_i^* in $H_k(A_t^*\ \ C_t^*)$ is zero by 5.3 ii). Therefore $\bar{\eta}_i$ comes from $\bar{\bar{\eta}}_i \in \pi_k(C_t^*)$; we may take as z_i the image of $\bar{\bar{\eta}}_i$ in $H_k(C_t^*)$.

Proof of 5.1. According to 5.2 it suffices to prove that $A_t^* \subset R^n$. Let W be a tubular neighborhood of C_t^* in M^*; the complement $M^* - W$ is diffeomorphic to the disjoint union of A_t^* and B_t^*. Therefore our goal will be attained if we succeed to perform surgery in W, so that to transform M^* into a homotopy sphere $\sum$. Then $A_t^*, B_t^* \subset \sum - pt$ which is diffeomorphic to a ball.

According to 5.4, $u_1, \ldots, u_m$ are represented by spherical cycles in W. Since $A_t^* \approx A_t^* \cup W$, W can be framed. Moreover all intersections between u_i and u_j are zero, and $c(u_1) = \ldots = c(u_m) = 0$; therefore $u_1, \ldots, u_m$ are represented by imbedded spheres in W on which surgery can be done. All we have to do is to kill these spheres by surgery as in [8]. The resulting manifold is $\sum$.

Theorem 1.3 ii) and iii) follow directly from 5.1. 1.3i) admits a similar and much easier proof, but it also follows from Proposition 6.1 in the next section.

§6. PROOF OF 1.5

We shall first prove the following proposition, which is of some independent interest.

PROPOSITION 6.1. Let M be a k-parallelizable closed n-manifold, $k < \frac{n}{2}$. Then $N_0(M) \leqq p + 1$ where $p = \left[\frac{n}{k+1}\right]$.

We recall that a *regular neighborhood* N of a subcomplex K of a (combinatorial) manifold M is a subcomplex of some subdivision of M, which is also a manifold and which collapses to K. A *smooth regular neighborhood* of K in a differentiable manifold M is a regular neighborhood of K in some smooth triangulation of M, which is a smooth submanifold of M [7].

The following Lemma is known [5].

LEMMA A. Let M be a combinatorial n-manifold. For any $k \geqq 0$ there exists a subdivision of M and $p+1$ subcomplexes $K_i \subset M$, $\dim K_i \leqq k$, $i = 0, \ldots, p = \left[\frac{n}{k+1}\right]$, such that regular neighborhoods $N(K_i)$ of K_i cover M.

Remark. Lemma A is actually proved in [5] in the more general case when M is an arbitrary n-complex.

Proof of 6.1. Apply Lemma A to a smooth triangulation of M. Let $\bar{U}_i$ be smooth regular neighborhoods of K_i, $U_i \supset K_i$, $i = 0, \ldots, p$. According to Theorem 1 of [7] such neighborhoods exist and it follows from the proof of that theorem that we may assume that $\bar{U}_0, \ldots, \bar{U}_p$ form a covering of M. Since $\bar{U}_i$ collapses to a k-dimensional complex, $\bar{U}_i$ is parallelizable and thus by the Hirsch–Poenaru theorem [6], [12], there exist immersions $\theta_i : \bar{U}_i \to R^n$. Since $k < \frac{n}{2}$ we may apply [16, Th. 2(e)] and assume that $\theta_i | K_i$ are imbeddings. Then θ_i are imbeddings on some smooth regular neighborhoods $\bar{V}_i \supset K_i$, $\bar{V}_i \subset U_i$, which again by [7, Th. 1] are diffeomomorphic to $\bar{U}_i$. Thus $M = U_0 \cup \ldots \cup U_p$ and $\bar{U}_i \subset R^n$. This completes the proof.

We shall now recall the results of Adams [1, §7.4] concerning the reduced real K-ring $\tilde{K}_R(P^n)$.

Let ξ be the reduced stable class of the canonical line-bundle over P^n and let $f = f(n)$ be the number of all natural numbers $\leqq n$ congruent to 0, 1, 2 or 4 mod 8. Additively $\tilde{K}_R(P^n)$ is a cyclic group of order 2^f generated by ξ; multiplicatively $\xi^2 = -2\xi$ so that $\xi^{f+1} = 0$.

LEMMA 6.2. *Let $k(q)$ be the largest integer k such that $f(k) \leqq q$. Then*

$$k(q) = \begin{cases} 2q & \text{if } q \equiv 0(4), \\ 2q - 1 & \text{if } q \equiv 1, 2(4), \\ 2q + 1 & \text{if } q \equiv 3(4). \end{cases}$$

Proof. It follows from the definition of $f(k)$ that

$$f(k) = \begin{cases} 4s & \text{if } \; k = 8s, \\ 4s+1 & \text{if } \; k = 8s+1, \\ 4s+2 & \text{if } \; k = 8s+2, \quad \text{or} \quad k = 8s+3, \\ 4s+3 & \text{if } \; k = 8s+4, \quad \text{or} \quad k = 8s+5,\ k = 8s+6,\ k = 8s+7. \end{cases} \tag{6.2}$$

If $q = 4s$, the largest k such that $f(k) \leqq q$ is $8s$ i.e. $2q$. If $q = 4s + 1$, the largest k such that $f(k) \leqq q$ is $8s + 1 = 2q - 1$. The other values of $k(q)$ can be similarly read of from (5.1).

Let $n + 1 = 2^q r$ where r is odd.

LEMMA 6.3. *Let $s(n)$ be the largest integer s such that $qs + s - 1 < f(n)$. Then*

$$s(n) = \left[\frac{2^{q-1}r - 1}{q+1}\right] \tag{6.3}$$

provided $q \geqq 3$.

Proof. Since $n = 2^q r - 1$ and $q \geqq 3$, the last line of (6.2) implies that $f(n) = \dfrac{2^q r - 2}{2} = 2^{q-1}r - 1$. Therefore we have to solve the inequality

$$qs + s - 1 < 2^{q-1}r - 1$$

i.e. $(q+1)s < 2^{q-1}r$ and $s < \dfrac{2^{q-1}r}{q+1}$. It is clear that the largest solution is given by (6.3).

Proof of 1.5. The first non-vanishing Stiefel–Whitney class of P^n is w_{2^q} and $(w_{2^q})^{r-1} \neq 0$. Therefore 2.5 implies that $N_0(P^n) \geqq n_0(P^n) \geqq r$. On the other hand, the reduced stable class of the tangent bundle of P^n is $\tau = (n+1)\xi$; hence according to [1, §7.4], $\tau | P^k = 0$ if $k \leqq k(q)$. Since P^k is the k-skeleton of P^n, this means that P^n is $k(q)$-parallelizable. If $k(q) \geqq \dfrac{n}{2}$, 1.3 implies that $N_0(P^n) = 2$; if however $k(q) < \dfrac{n}{2}$, 6.1 implies that $N_0(P^n) \leqq \left[\dfrac{n}{k(q)+1}\right] + 1$. If $q \leqq 3$, $\left[\dfrac{n}{k(q)+1}\right] + 1 = r$ so that we obtain (1.1). If $q \geqq 3$, let s be the largest number such that $\tau^s \neq 0$. Since $\tau = (n+1)\xi = 2^q r\xi$, $\tau^s = 2^{qs} r^s \xi^s = \pm 2^{qs+s-1} r^s \xi$ and $\tau^s \neq 0$ if $qs + s - 1 < f(n)$. According to 6.3, the value of s is given by (6.3), which by 2.6 yields the first inequality of (1.2).

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