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Identifying codes and locating–dominating sets on paths and cycles[☆]

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ABSTRACT

Let $G = (V, E)$ be a graph and let $r \geq 1$ be an integer. For a set $D \subseteq V$, define $N_r[x] = \{y \in V : d(x, y) \leq r\}$ and $D_r(x) = N_r[x] \cap D$, where $d(x, y)$ denotes the number of edges in any shortest path between x and y. D is known as an r-identifying code (r-locating-dominating set, respectively), if for all vertices $x \in V$ ($x \in V \setminus D$, respectively), $D_r(x)$ are all nonempty and different. Roberts and Roberts [D.L. Roberts, F.S. Roberts, Locating sensors in paths and cycles: the case of 2-identifying codes, European Journal of Combinatorics 29 (2008) 72–82] provided complete results for the paths and cycles when $r = 2$. In this paper, we provide results for a remaining open case in cycles and complete results in paths for r-identifying codes; we also give complete results for 2-locating-dominating sets in cycles, which completes the results of Bertrand et al. [N. Bertrand, I. Charon, O. Hudry, A. Lobstein, Identifying and locating–dominating codes on chains and cycles, European Journal of Combinatorics 25 (2004) 969–987].

1. Introduction

We investigate the well-known identifying codes problem which originated, for instance, from fault diagnosis in multiprocessor systems. The purpose of fault diagnosis is to test the multiprocessor system and locate faulty processors. A multiprocessor system can be modeled as an undirected graph $G = (V, E)$, where V is the set of processors, E is the set of links in the system. Specific detectors are executed on certain selected processors to carry out diagnosis. The selection of these processors is done by generating the code D that allows for unique identification of faulty processors. Every processor corresponding to a codeword vertex tests itself and the processors that are in some areas. Hence, an optimal code (with minimum number of codewords) minimizes the amount of overhead required to implement fault diagnosis.

More precisely, let $G = (V, E)$ be an undirected graph and let $r \geq 1$ be an integer. Assume that D is a subset of V at which we place detectors. We define $N_r[x] = \{y \in V : d(x, y) \leq r\}$ and $D_r(x) = N_r[x] \cap D$, where $d(x, y)$ denotes the number of edges in any shortest path between x and y. In this sense, $D_r[x]$ is the set of all detectors that can detect an defect at x. We say that D is an *r-identifying code* (*r-IC*) *in* G if $D_r(x) \neq \emptyset$ for every vertex $x \in V$ and $D_r(x) \neq D_r(y)$ whenever $x \neq y$. In an r-IC, the set of detectors activated by an defect provides a unique signature that allows us to determine where the defect took place. We denote the minimum cardinality of an r-identifying code D of G by $M_r^I(G)$. Note that not all graphs admit an r-identifying code. A necessary and sufficient condition to admit an r-identifying code is that for any pair of distinct vertices x and y we have $N_r[x] \neq N_r[y]$.

A closely related concept is defined as follows. If for all vertices $x \in V \setminus D$, $D_r(x)$ are all not empty and different, then we say that D is an *r-locating–dominating set* or *r-LD set* for short. The smallest d such that there is an r-LD set of size d is

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denoted by $M_r^{LD}(G)$. The concept was introduced (for $r = 1$) by Slater [22], motivated by nuclear power plant safety. It can be used for fault detection in a distributed system. We also note that r-LD sets always exist, since the entire vertex set of a graph is an r-LD set.

Identifying codes were introduced in [18], locating–dominating sets in [12]. The literature about r-identifying codes and r-locating-dominating sets has become quite extensive. There are now numerous papers dealing with identifying codes and locating–dominating sets (see for instance [19] for an up-to-date bibliography). The problems of finding optimal r-ICs or r-LDs in a graph are NP-hard (see [6,8,10–12]). On the other hand, many special graphs have been investigated (see for instance, [2–5,7,9,16,17,21]). In this paper we are interested in studying r-IC for cycles and paths and 2-LD sets for cycles. This subject was already investigated in [1,14,20,22,23]. Let P_n (C_n, respectively) be a path (cycle, respectively) of n vertices. For $r = 1$, the exact values of $M_1^I(P_n)$ and $M_1^I(C_n)$ for even cycles was given by [1]; Gravier et al. [14] gave the exact values of $M_1^I(C_n)$ for odd cycles. Its analogue for 1-LD sets was given by Slater [22]. For $r = 2$, a complete solution for $M_2^I(C_n)$ and $M_2^I(P_n)$ was provided in [20]. Bertrand et al. [1] provided complete results about $M_2^{LD}(P_n)$ and gave the exact value of $M_2^{LD}(C_n)$ for $n = 6k$. For all $r \geq 1$, partial results about r-ICs for paths and cycles can be found in [1,14,23]. Some results on r-LD sets in paths are given by Honkala [15].

The structure of this paper is the following. Motivated by the method in [20], we give all values of $M_r^I(C_n)$ for an odd cycle C_n with $2r + 5 \leq n \leq 3r + 1$ in Section 2. In Section 3 we provide complete results for r-identifying codes in paths. In Section 4 we find the values of $M_2^{LD}(C_n)$. The paper is concluded in Section 5.

2. r-identifying codes for cycle C_n

In the following, we assume that the vertices of C_n are labeled consecutively as $x_1, x_2, \dots, x_n$. When we are dealing with a cycle, we also use addition and subtraction modulo n, so that, for example, x_{5n+2} means x_2. It is obvious that $M_r^I(C_n)$ is undefined for $n \leq 2r + 1$. Hence, we assume that $n \geq 2r + 2$. The case n even is solved in [1]. For the case n odd, what remains unknown in [14,23] is the subcase $2r + 5 \leq n \leq 3r + 1$. Here, we discuss this open case.

Lemma 1. *Suppose graph G has maximum degree 2, $y_1, y_2, \dots, y_{2r+2}$ is a path in G, and D is an r-IC for G. Then it is impossible to have $y_1 \notin D$ and $y_{2r+2} \notin D$.*

Proof. If $y_1 \notin D$ and $y_{2r+2} \notin D$, then $D_r(y_{r+1}) = D_r(y_{r+2})$. $\square$

Lemma 2. *If $2r + 5 \leq n \leq 3r + 1$, let $n = 2r + 1 + q$ ($4 \leq q \leq r$), D is an r-IC for C_n if and only if*

(1) *$x_i \in D$ or $x_{i+q} \in D$ for all $i \in \{1, \dots, n\}$;*
(2) *there is at most one set $\{x_{i+1}, x_{i+2}, \dots, x_{i+q}\}$ such that none of which is in D.*

Proof. ($\Rightarrow$:) Suppose to the contrary that $x_i \notin D$ and $x_{i+q} \notin D$. Since $N_r[x_{i-r-1}] = V \setminus \{x_i, x_{i+1}, \dots, x_{i+q-1}\}$ and $N_r[x_{i-r}] = V \setminus \{x_{i+1}, x_{i+2}, \dots, x_{i+q}\}$, this leads to the equality $D_r(x_{i-r-1}) = D_r(x_{i-r})$, a contradiction. If there exist two distinct sets $\{x_{i+1}, x_{i+2}, \dots, x_{i+q}\}$ and $\{x_{j+1}, x_{j+2}, \dots, x_{j+q}\}$ such that none of which is in D, then $D_r(x_{i-r}) = D_r(x_{j-r})$, which follows from $N_r[x_{i-r}] = V \setminus \{x_{i+1}, x_{i+2}, \dots, x_{i+q}\}$ and $N_r[x_{j-r}] = V \setminus \{x_{j+1}, x_{j+2}, \dots, x_{j+q}\}$.

($\Leftarrow$:) For every vertex $x_i \in V$, $N_r[x_i] = \{x_{i-r}, x_{i-r+1}, \dots, x_{i+r}\}$. As $q \leq r$, both x_i and x_{i+q} are in $N_r[x_i]$, and by condition (1), we can conclude that $D_r(x_i) \neq \emptyset$. For distinct vertices x_i and x_j, without loss of generality, we assume that $i < j$ and the distance from x_i to x_j in a clockwise direction around the cycle is no larger than in a counterclockwise direction. If $i+1 \leq j \leq i+q$, we have $x_{i+r+1} \in N_r[x_j] \setminus N_r[x_i]$ and $x_{i+r+q+1} \in N_r[x_i] \setminus N_r[x_j]$. Hence, by condition (1), $D_r(x_i) \neq D_r(x_j)$. Assume that $j > i+q$. Let $A = \{x_{i+r+1}, x_{i+r+2}, \dots, x_{i+r+q}\}$ and $B = \{x_{j+r+1}, x_{j+r+2}, \dots, x_{j+r+q}\}$. Then $N_r[x_i] = V \setminus A$ and $N_r[x_j] = V \setminus B$. Since $j > i+q$ and the distance from x_i to x_j in a clockwise direction around the cycle is no larger than in a counterclockwise direction, it implies that $A \cap B = \emptyset$. By condition (2), either $A \cap D \neq \emptyset$ or $B \cap D \neq \emptyset$ holds. Without loss of generality, we assume that $A \cap D \neq \emptyset$ and $x_{i+r+t} \in D$ for some $t \in \{1, 2, \dots, q\}$. Then $x_{i+r+t} \notin D_r(x_i)$, but $x_{i+r+t} \in D_r(x_j)$ as $A \cap B = \emptyset$. Hence, we have that $D_r(x_i) \neq D_r(x_j)$. $\square$

Theorem 3. *For the cycle C_{2k+1} with $2r + 5 \leq 2k + 1 \leq 3r + 1$, let $2k + 1 = 2r + 1 + q = \ell q + m$ ($4 \leq q \leq r$), where $\ell \geq 3$ is an integer and $m \in \{0, 1, \dots, q - 1\}$, then*

(1) *$M_r^I(C_{2k+1}) = k + 2$ if ℓ is odd, $m = q - 1$, $2k + 1 \geq 5q$ or ℓ is even, $m = 1$;*
(2) *$M_r^I(C_{2k+1}) = \gcd(q, 2k + 1)\lceil \frac{2k+1}{2\gcd(q,2k+1)} \rceil$ otherwise.*

Proof. Since $2k + 1 = 2r + 1 + q = \ell q + m$, then q is even, and hence m is odd. Let D be an r-IC for C_{2k+1}, by Lemma 2, it must satisfy $2k + 1$ constraints: $x_i \in D$ or $x_{i+q} \in D$ for $i = 1, 2, \dots, 2k + 1$. For notational convenience, we abbreviate x_i by i and $x_i \in D$ or $x_j \in D$ by $i \vee j$ in the constraints. Choose $i \in \{1, \dots, q\}$ and consider the following stream of constraints, which we call *stream i*:

$$i \vee i + q \vee i + 2q \vee \cdots \vee i + g_i q \vee h_i,$$

where $i + g_i q \leq 2k + 1 < i + (g_i + 1)q \equiv h_i \pmod{2k+1}$ and $h_i \in \{1, 2, \dots, q\}$. For stream i, it represents $g_i + 1$ constraints of type $j \vee k$: $i \vee i + q, \dots, i + g_i q \vee h_i$, which must be fulfilled.

Then stream 1 leads into stream h_1, which leads into stream h_{h_1}, and so on, end with last $h_i = 1$. When $\gcd(q, 2k+1) \neq 1$, it gives us a full stream, denoted by *full stream* 1, which contains $\frac{2k+1}{\gcd(q,2k+1)}$ constraints and $\frac{2k+1}{\gcd(q,2k+1)} + 1$ vertices (where two 1s). Similarly, we can get *full stream* 2, ..., *full stream* $\gcd(q, 2k+1)$. There are $\gcd(q, 2k+1)$ full streams and each full stream contains $\frac{2k+1}{\gcd(q,2k+1)}$ constraints and $\frac{2k+1}{\gcd(q,2k+1)} + 1$ vertices. To satisfy all constraints, we need at least $\frac{2k+1}{2\gcd(q,2k+1)}$ vertices from each full stream to be put into D. It follows that $|D| \geq \gcd(q, 2k+1)\lceil \frac{2k+1}{2\gcd(q,2k+1)} \rceil$.

Let us denote f_i^j the jth vertex of full stream i for $1 \leq i \leq \gcd(q, 2k+1)$ and $1 \leq j \leq \frac{2k+1}{\gcd(q,2k+1)} + 1$. Since $2k+1$ is odd, each full stream has even number of vertices. Let $D = \{f_i^j : 1 \leq i \leq \gcd(q, 2k+1), 1 \leq j \leq \frac{2k+1}{\gcd(q,2k+1)} + 1, i + j \text{ is odd}\}$. It is easy to see that D satisfies conditions of Lemma 2. Hence, D is an r-IC with $|D| = \gcd(q, 2k+1)\lceil \frac{2k+1}{2\gcd(q,2k+1)} \rceil$ and we are now in case (2) of theorem. For example, if $k = 10$ and $r = 7$, then $q = 6$ and $\gcd(6, 21) = 3$. Full stream 1 is given by $1 \vee 7 \vee 13 \vee 19 \vee 4 \vee 10 \vee 16 \vee 1$; Full stream 2 is given by $2 \vee 8 \vee 14 \vee 20 \vee 5 \vee 11 \vee 17 \vee 2$; Full stream 3 is given by $3 \vee 9 \vee 15 \vee 21 \vee 6 \vee 12 \vee 18 \vee 3$. Then the set $\{7, 19, 10, 1, 2, 14, 5, 17, 9, 21, 12, 3\}$ is an r-IC for C_{21}.

We now turn to the case $\gcd(q, 2k+1) = 1$. $\gcd(q, 2k+1) = 1$ implies that there is only one full stream, which contains $2k+1$ constraints. We discuss it as the following three cases, according to the values of m.

Case 1. $m = q - 1$, $2k + 1 = \ell q + q - 1$.

In this case, the full stream consists of stream 1, stream 2, ..., stream q in turn. Suppose that D is an r-IC for C_{2k+1} with $|D| = k + 1$. Since there are $2k+1$ constraints, there must be exactly one constraint where both vertices are in D, and all other constraints have exactly one of their vertices in D. Without loss of generality, we take 1 and $1 + q$ in D, then the rest of the membership of D is forced upon us. If l is even, the membership of D is just the following vertices:

- from stream i: use vertices $i + zq$, $i + z$ is even, $i \in \{1, 2, \dots, q\}$.

  It is easy to check that there are no q consecutive vertices none of which is in D. By Lemma 2, D is an r-IC with $|D| = k + 1$ we are in case (2) of theorem.

  If ℓ is odd, the membership of D is just the following vertices:
- from stream i: use vertices $i + zq$, z is odd, $i \in \{1, 2, \dots, q\}$.

  When $2k+1 < 5q$, i.e., $2k+1 = 4q-1$, D satisfies conditions of Lemma 2, and hence D is also an r-IC with $|D| = k+1$. We are still in case (2) of theorem. However, when $2k + 1 \geq 5q$, condition (2) of Lemma 2 is violated, there exist two sets $\{1 + 2q, 2 + 2q, \dots, q + 2q\}$ and $\{1 + 4q, 2 + 4q, \dots, q + 4q\}$ such that none of which is in D. Then D is not an r-IC. So, we conclude that $M_r^I(C_{2k+1}) \geq k+2$. Now we construct an r- IC with $k+2$ vertices as follows and we are now in case (1) of theorem.
- From stream i: use vertices $i + zq$, z is odd and $i \neq \frac{q}{2} + 1$;
- from stream $\frac{q}{2} + 1$: use vertices $\frac{q}{2} + 1 + zq$, z is even;
- add the vertex $\frac{q}{2} + 1 + lq$.

Case 2. $m = 1$, $2k + 1 = lq + 1$.

In this case, the full stream consists of stream q, stream $q - 1$, ..., stream 1 in turn. We can also prove that $M_r^I(C_{2k+1}) = k + 1$ if ℓ is odd and $M_r^I(C_{2k+1}) \geq k+2$ if ℓ is even. The proof is analogous with case 1 and is omitted in here. If ℓ is even, we can construct an r-IC with $k+2$ vertices as follows:

- from stream i: use vertices $i + zq$, z is odd and $i \neq \frac{q}{2} + 1$;
- from stream $\frac{q}{2} + 1$: use vertices $\frac{q}{2} + 1 + zq$, z is even.

Case 3. $1 < m < q - 1$, $2k + 1 = lq + m$.

Let D denote an r-IC for C_{2k+1} with $k+1$ vertices. When ℓ is odd, without loss of generality, we take q and $2q$ in D, the rest of the membership of D is forced upon us, and condition (1) of Lemma 2 holds. Next, we prove that there are no q consecutive vertices none of which should be in D. i.e., D satisfies a stronger property than condition (2) of Lemma 2. Suppose to the contrary that $i+pq, i+1+pq, \dots, q+pq, 1+(p+1)q, 2+(p+1)q, \dots, (i-1)+(p+1)q \notin D$ for some $i \in \{1, 2, \dots, q\}$ and $p \in \{0, 1, \dots, \ell\}$. By the selection of D, we know that $q \in D$ and $q + zq \in D$ for all odd $z \leq \ell$. So, p is even.

Since $q - m + zq \in D$ for all even $z \leq \ell - 1$, then $q - m \leq i - 1$. If $q - m > m$, then stream $q - m$ leads into stream $q - 2m$, and $q - 2m + zq \in D$ for all odd $z \leq \ell$. Thus, $q - 2m \geq i$. It contradicts that $q - m \leq i - 1$. Hence, $q - m \leq m$. Then stream $q - m$ leads into stream $2q - 2m$, and $2q - 2m + zq \in D$ for all even $z \leq \ell$. Therefore, $2q - 2m \leq i - 1$. Similarly, we have $2q - 2m \leq m$ and stream $2q - 2m$ leads into stream $3q - 3m$. Let t_0 be the minimum integer such that $t_0(q-m) > max\{i-1, m\}$. We have that: stream $q \Rightarrow$ stream $q-m \Rightarrow$ stream $2(q-m) \Rightarrow \cdots \Rightarrow$ stream $(t_0-1)(q-m) \Rightarrow$ stream $t_0(q - m)$. By the selection of t_0, we know that $t(q - m) + zq \in D$ for all $1 \leq t \leq t_0$ and all even $z \leq \ell$. Hence, $t_0(q - m) \leq i - 1$ and it implies that $t_0(q - m) > m$. Therefore, stream $t_0(q - m)$ leads into stream $t_0(q - m) - m$ and $t_0(q - m) - m + zq \in D$ for all odd $z \leq \ell$. So, we know that $t_0(q - m) - m \geq i$. It contradicts that $t_0(q - m) \leq i - 1$. So, there are no q consecutive vertices none of which is in D, and hence D is an r-IC with $k + 1$ vertices.

When ℓ is even, without loss of generality, we take 1 and $1 + q$ in D, the rest of the membership of D is forced upon us. The remainder proof is analogous and is omitted here. □

3. r-identifying codes for path P_n

We turn now to the path P_n. We assume that the vertices of P_n are labeled consecutively as $x_1, x_2, \ldots, x_n$. First it is easy to see that $M_r^I(P_n)$ is undefined if and only if $n \le 2r$. In the following, we assume that $n \ge 2r+1$.

Lemma 4. *If D is an r-IC for P_n, then $x_{r+2}, x_{r+3}, \ldots, x_{2r+1} \in D$ and $x_{n-r-1}, x_{n-r-2}, \ldots, x_{n-2r} \in D$.*

Proof. For $i = 1, 2, \ldots, r$, $D_r(x_i) \ne D_r(x_{i+1})$ implies that $x_{i+r+1} \in D$, and $D_r(x_{n-i}) \ne D_r(x_{n-i+1})$ implies that $x_{n-r-i} \in D$. □

Lemma 5. *D is an r-IC for P_n if and only if the following conditions hold:*

(1) *there are no $2r+2$ consecutive vertices with the first and last not in D;*
(2) *there are no $2r+1$ consecutive vertices none of which is in D;*
(3) *$\{x_{r+2}, x_{r+3}, \ldots, x_{2r+1}\} \subseteq D$ and $\{x_{n-r-1}, x_{n-r-2}, \ldots, x_{n-2r}\} \subseteq D$.*
(4) *$\{x_1, x_2, \ldots, x_{r+1}\} \cap D \ne \emptyset$ and $\{x_n, x_{n-1}, \ldots, x_{n-r}\} \cap D \ne \emptyset$.*

Proof. ($\Rightarrow$:) Necessity of (1) follows from Lemma 1, and necessity of (2) follows from $D_r(x) \ne \emptyset$ for every vertex $x \in V$. Necessity of (3) follows from Lemma 4, and necessity of (4) follows from $D_r(x_1) \ne \emptyset$ and $D_r(x_n) \ne \emptyset$.

($\Leftarrow$:) By conditions (2), (3) and (4), $D_r(x) \ne \emptyset$ for every vertex $x \in V$. Consider x_i and x_j, without loss of generality, we assume that $i < j$. If $i+1 \le j \le i+2r+1$ and $i > r$, by condition (1), either $x_{i-r} \in D$ or $x_{i+r+1} \in D$ holds, and hence $D_r(x_i) \ne D_r(x_j)$. If $i+1 \le j \le i+2r+1$ and $i \le r$, by condition (3), we have $x_{i+r+1} \in D$, and hence $D_r(x_i) \ne D_r(x_j)$. If $j > i+2r+1$ and $i > r$, by condition (2), $\{x_{i-r}, x_{i-r+1}, \ldots, x_{i+r}\} \cap D \ne \emptyset$, so $D_r(x_i) \ne D_r(x_j)$. If $j > i+2r+1$ and $i \le r$, by condition (4), $\{x_1, x_2, \ldots, x_{r+1}\} \cap D \ne \emptyset$, so $D_r(x_i) \ne D_r(x_j)$. □

Lemma 5 allows us to proceed for a path much as we did with a cycle. Constraint streams are again the focus of our argument. Similarly, we use i as an abbreviation for vertex x_i and we modify the definition of constraint stream i as follows:

$$i \vee i+(2r+1) \vee i+2(2r+1) \vee \cdots \vee i+g_i(2r+1),$$

where $i + g_i(2r+1) \le n$ and $1 \le i \le 2r+1$.

The following theorem gives all results for $M_r^I(P_n)$.

Theorem 6. *Let $n = (2r+1)p+q$, $p \ge 1$, $q \in \{0, 1, \ldots, 2r\}$.*

(1) *If $q = 0$, then $M_r^I(P_n) = \frac{(2r+1)p}{2} + 1$ if p is even; $M_r^I(P_n) = \frac{(2r+1)(p-1)}{2} + 2r$ if p is odd.*
(2) *If $1 \le q \le r+1$, then $M_r^I(P_n) = \frac{(2r+1)p}{2} + q$ if p is even; $M_r^I(P_n) = \frac{(2r+1)(p-1)}{2} + 2r + 1$ if p is odd.*
(3) *If $r+2 \le q \le 2r$, then $M_r^I(P_n) = \frac{(2r+1)p}{2} + q - 1$ if p is even; $M_r^I(P_n) = \frac{(2r+1)(p-1)}{2} + 2r + 1$ if p is odd.*

Proof. Let D be an r-IC for P_n. We first discuss the case $q = 0$.

(1) If $q = 0$, then $r+2, r+3, \ldots, 2r+1, 1+(p-1)(2r+1), 2+(p-1)(2r+1), \ldots, r+(p-1)(2r+1) \in D$, which follows from condition (3) of Lemma 5. For $i \in \{1, 2, \ldots, 2r+1\}$, the constraint stream i is given as follows: $i \vee i+(2r+1) \vee \cdots \vee i+(p-1)(2r+1)$. To satisfy condition (4) of Lemma 5, there are four possible cases:

(1A) $r+1 \in D$ and $r+1+(p-1)(2r+1) \in D$;
(1B) $i \in D$ for some $i \in \{1, 2, \ldots, r\}$ and $r+1+(p-1)(2r+1) \in D$;
(1C) $r+1 \in D$ and $j+(p-1)(2r+1) \in D$ for some $j \in \{r+2, r+3, \ldots, 2r+1\}$;
(1D) $i \in D$ for some $i \in \{1, 2, \ldots, r\}$ and $j+(p-1)(2r+1) \in D$ for some $j \in \{r+2, r+3, \ldots, 2r+1\}$.

First consider the case (1A). For each stream i ($i \in \{1, 2, \ldots, r\}$), we have already taken $i+(p-1)(2r+1)$ into D, satisfying the last constraint, and there are $p-2$ remaining constraints. So, we need to take at least $\lceil \frac{p-2}{2} \rceil$ vertices from each stream i ($i \in \{1, 2, \ldots, r\}$) into D to satisfy the remaining constraints. Turn to stream $r+1$, since $r+1$ and $r+1+(p-1)(2r+1)$ are already put in D, satisfying the first and last constraints in stream $r+1$, so, we need at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy the remaining constraints. Similarly, it requires at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy the remaining constraints in each stream i for $i \in \{r+2, r+3, \ldots, 2r+1\}$. Hence, we need at least $2r+2+\lceil \frac{p-3}{2} \rceil + 2r\lceil \frac{p-2}{2} \rceil$ vertices in all.

Now we consider the case (1B). For stream i, we have already taken i and $i+(p-1)(2r+1)$ into D, satisfying the first and the last constraints, then we need at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy the remaining constraints. For each of the other streams, we need at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy the remaining constraints. Thus, we need at least $2r+2+\lceil \frac{p-3}{2} \rceil + 2r\lceil \frac{p-2}{2} \rceil$ vertices.

We now turn to the case (1C). For stream j, we have already taken j and $j+(p-1)(2r+1)$ into D, satisfying the first and the last constraints, then we need at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy the remaining constraints. For each of the other streams, we need at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy the remaining constraints. Thus, we need at least $2r+2+\lceil \frac{p-3}{2} \rceil + 2r\lceil \frac{p-2}{2} \rceil$ vertices.

Finally we consider the case (1D). For stream i, we have already taken i and $i+(p-1)(2r+1)$ into D, satisfying the first and the last constraints, then we need at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy the remaining constraints. For stream j, similarly, we need at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy the remaining constraints. For stream $r+1$, we need at least $\lceil \frac{p-1}{2} \rceil$ vertices to satisfy its constraints. For each of the other streams, we need at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy the remaining constraints. Hence, we need at least $2r+2+2\lceil \frac{p-3}{2} \rceil + (2r-2)\lceil \frac{p-2}{2} \rceil + \lceil \frac{p-1}{2} \rceil$ vertices.

Finally, comparing the required minimum number of D in all four cases, we see that when p is even, the minimum is $\frac{(2r+1)p}{2}+1$, which is achieved in both cases (1A), (1B) and (1C) and when p is odd, the minimum is $\frac{(2r+1)(p-1)}{2}+2r$, which is achieved in case (1D).

Next, we construct an r-IC, achieving the bound as follows:

When p is even,

- from stream i: use vertices $i+z(2r+1)$, where z is even, for $i \in \{1, r+2, r+3, \ldots, 2r+1\}$;
- from stream j: use vertices $j+z(2r+1)$, where z is odd, for $j \in \{2, 3, \ldots, r+1\}$;
- add the vertex $1+(p-1)(2r+1)$.

When p is odd,

- from stream i: use vertices $i+z(2r+1)$, where z is even, for $i \in \{1, r+2, r+3, \ldots, 2r+1\}$;
- from stream j: use vertices $j+z(2r+1)$, where z is odd, for $j \in \{2, 3, \ldots, r+1\}$;
- add vertices $i+(p-1)(2r+1)$ for $i \in \{2, 3, \ldots, r\}$.

(2) If $1 \le q \le r+1$, then $r+2, r+3, \ldots, 2r+1, q+1+(p-1)(2r+1), q+2+(p-1)(2r+1), \ldots, q+r+(p-1)(2r+1) \in D$, which follows from condition (3) of Lemma 5. For $i \in \{1, 2, \ldots, q\}$, the constraint stream i is given as follows: $i \vee i+(2r+1) \vee \cdots \vee i+p(2r+1)$. For $i \in \{q+1, \ldots, 2r+1\}$, the constraint stream i is given as follows: $i \vee i+(2r+1) \vee \cdots \vee i+(p-1)(2r+1)$. To satisfy condition (4) of Lemma 5, there are four possible cases:

(2A) $i \in D$ for some $i \in \{1, \ldots, q\}$ and $j+p(2r+1) \in D$ for some $j \in \{1, \ldots, q\}$;
(2B) $i \in D$ for some $i \in \{1, \ldots, q\}$ and $j+(p-1)(2r+1) \in D$ for some $j \in \{q+r+1, \ldots, 2r+1\}$;
(2C) $i \in D$ for some $i \in \{q+1, \ldots, r+1\}$ and $j+p(2r+1) \in D$ for some $j \in \{1, \ldots, q\}$;
(2D) $i \in D$ for some $i \in \{q+1, \ldots, r+1\}$ and $j+(p-1)(2r+1) \in D$ for some $j \in \{q+r+1, \ldots, 2r+1\}$.

First consider the case (2A). We first discuss the situation $i \neq j$. For stream i, we have already taken i into D, satisfying the first constraint in stream i, and hence we need to take at least $\lceil \frac{p-1}{2} \rceil$ vertices from stream i to satisfy the remaining constraints. For stream j, we have already taken $j+p(2r+1)$ into D, satisfying the last constraint in stream j, and hence we need to take at least $\lceil \frac{p-1}{2} \rceil$ vertices from stream j to satisfy the remaining constraints. For each stream t with $t \in \{1, \ldots, q\} \setminus \{i, j\}$, we need to take at least $\lceil \frac{p}{2} \rceil$ vertices into D to satisfy its constraints. For each stream t with $t \in \{q+1, \ldots, r+1\}$, we have already taken the vertex $t+(p-1)(2r+1)$ into D, satisfying the last constraint in stream t, and there are $p-2$ remaining constraints. Hence, we need to take at least $\lceil \frac{p-2}{2} \rceil$ vertices from stream t to satisfy the remaining constraints. For each stream t with $t \in \{r+2, \ldots, r+q\}$, we have already taken t and $t+(p-1)(2r+1)$ into D, satisfying the first and the last constraints in stream t, and there are $p-3$ remaining constraints. Hence, we need to take at least $\lceil \frac{p-3}{2} \rceil$ vertices from stream t to satisfy the remaining constraints. For each stream t with $t \in \{r+q+1, \ldots, 2r+1\}$, we have already taken t into D, satisfying the first constraint in stream t. Hence, we need to take at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy remaining constraints. Therefore, we need at least $2r+2+(q-2)\lceil \frac{p}{2} \rceil + 2\lceil \frac{p-1}{2} \rceil + (2r-2q+2)\lceil \frac{p-2}{2} \rceil + (q-1)\lceil \frac{p-3}{2} \rceil$ vertices in all.

We now discuss the situation $i = j$. For stream i, we have already taken i and $i+p(2r+1)$ into D, satisfying the first and the last constraints in stream i, and hence we need to take at least $\lceil \frac{p-2}{2} \rceil$ vertices from stream i to satisfy the remaining constraints. For each stream t with $t \in \{1, \ldots, q\} \setminus \{i\}$, we need to take at least $\lceil \frac{p}{2} \rceil$ vertices into D to satisfy its constraints. For each stream t with $t \in \{q+1, \ldots, 2r+1\}$, the discussion is the same as above. Therefore, we need at least $2r+2+(q-1)\lceil \frac{p}{2} \rceil + (2r-2q+3)\lceil \frac{p-2}{2} \rceil + (q-1)\lceil \frac{p-3}{2} \rceil$ vertices in all.

We now consider the case (2B). Similarly, we need to take at least $\lceil \frac{p-1}{2} \rceil$ vertices to satisfy the remaining constraints in stream i. For each stream t with $t \in \{1, \ldots, q\} \setminus \{i\}$, we need to take at least $\lceil \frac{p}{2} \rceil$ vertices to satisfy the remaining constraints. For each stream t with $t \in \{q+1, \ldots, r+1\}$, we need to take at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy the remaining constraints. For each stream t with $t \in \{r+2, \ldots, r+q\}$, we need to take at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy the remaining constraints. For stream j, we need to take at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy the remaining constraints. For each stream t with $t \in \{q+r+1, \ldots, 2r+1\} \setminus \{j\}$, we need to take at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy the remaining constraints. Therefore, we need at least $2r+2+(q-1)\lceil \frac{p}{2} \rceil + \lceil \frac{p-1}{2} \rceil + (2r-2q+1)\lceil \frac{p-2}{2} \rceil + q\lceil \frac{p-3}{2} \rceil$ vertices in all.

We now turn to the case (2C). We need at least $\lceil \frac{p-1}{2} \rceil$ vertices to satisfy the remaining constraints in stream j. For each stream t with $t \in \{1, \ldots, q\} \setminus \{j\}$, we need at least $\lceil \frac{p}{2} \rceil$ vertices to satisfy its constraints. For stream i, we need at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy the remaining constraints. For each stream t with $t \in \{q+1, \ldots, r+1\} \setminus \{i\}$, we need at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy the remaining constraints. For each stream t with $t \in \{r+2, \ldots, r+q\}$, we need at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy the remaining constraints. For each stream t with $t \in \{r+q+1, \ldots, 2r+1\}$, we need at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy the remaining constraints. Therefore, we need at least $2r+2+(q-1)\lceil \frac{p}{2} \rceil + \lceil \frac{p-1}{2} \rceil + (2r-2q+1)\lceil \frac{p-2}{2} \rceil + q\lceil \frac{p-3}{2} \rceil$ vertices in all.

At last we consider the case (2D). For each stream t with $t \in \{1, \ldots, q\}$, we need at least $\lceil \frac{p}{2} \rceil$ vertices to satisfy its constraints. For stream i, we need at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy its remaining constraints. For each stream t with $t \in \{q+1, \ldots, r+1\} \setminus \{i\}$, we need at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy its remaining constraints. For each stream t with $t \in \{r+2, \ldots, r+q\}$, we need at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy the remaining constraints. For stream j, we need at least $\lceil \frac{p-3}{2} \rceil$ vertices to satisfy its remaining constraints. For each stream t with $t \in \{r+q+1, \ldots, 2r+1\} \setminus \{j\}$, we need at least $\lceil \frac{p-2}{2} \rceil$ vertices to satisfy the remaining constraints. Therefore, we need at least $2r+2+q\lceil \frac{p}{2} \rceil + (2r-2q)\lceil \frac{p-2}{2} \rceil + (q+1)\lceil \frac{p-3}{2} \rceil$ vertices in all.

Finally, comparing the required minimum size of D in all four cases, we see that when p is even, the minimum size of D is $\frac{(2r+1)p}{2} + q$, which is achieved in case (2A) under the situation $i = j$, and when p is odd, the minimum size of D is $\frac{(2r+1)(p-1)}{2} + 2r + 1$, which is achieved in cases (2C), (2D) and case (2A) under the situation $i \neq j$.

Next, we construct an r-IC, achieving the bound as follows:

When p is even,

- from stream i: use vertices $i + z(2r+1)$, where z is even, for $i \in \{1, r+2, r+3, \ldots, 2r+1\}$;
- from stream j: use vertices $j + z(2r+1)$, where z is odd, for $j \in \{2, 3, \ldots, r+1\}$;
- add vertices $i + (p-1)(2r+1)$ for $i \in \{r+2, r+3, \ldots, r+q\}$.

When p is odd,

- from stream i: use vertices $i + z(2r+1)$, where z is odd, for $i \in \{1, 2, \ldots, q\}$;
- from stream j: use vertices $j + z(2r+1)$, where z is even, for $j \in \{q+1, q+2, \ldots, 2r+1\}$.

(3) The proof of (3) is analogous to that of (2). We simply include the instruction for how to achieve an optimal set D in this case.

When p is even,

- from stream i: use vertices $i + z(2r+1)$, where z is even, for $i \in \{1, r+2, r+3, \ldots, 2r+1\}$;
- from stream j: use vertices $j + z(2r+1)$, where z is odd, for $j \in \{2, 3, \ldots, r+1\}$;
- add vertices $i + (p-1)(2r+1)$ for $i \in \{q+1, q+2, \ldots, 2r+1\}$ and $j + p(2r+1)$ for $j \in \{2, 3, \ldots, q-r-1\}$.

When p is odd,

- from stream i: use vertices $i + z(2r+1)$, where z is odd, for $i \in \{1, 2, \ldots, r\}$;
- from stream j: use vertices $j + z(2r+1)$, where z is even, for $j \in \{r+1, r+3, \ldots, 2r+1\}$. □

4. 2-locating–dominating sets for cycle C_n

Let A and B be two sets. Define $A \triangle B$ as $(A \setminus B) \cup (B \setminus A)$. For three vertices x, u, v, if $x \in D_r(u) \triangle D_r(v)$, then we say that $\{u, v\}$ are *r-separated by x*, or *xr-separates* $\{u, v\}$. Let D be an r-LD for C_n. Recall that only vertices not in D need to be separated and also that, as a consequence, there is no constraint on n. Two different vertices x and y not in D are *D-consecutive* if either $\{x+1, \ldots, y-1\} \subseteq D$ or $\{y+1, \ldots, x-1\} \subseteq D$ holds. Note that a pair of consecutive vertices $\{x, x+1\}$ not in D are also D-consecutive.

Lemma 7 ([1]). *Let $r \geq 1$ be an integer. Suppose D is an r-LD for C_n. For every vertex x in D, x can r-separate at most two pairs of D-consecutive vertices.*

Proof. Let ℓ and ℓ' be integers such that $0 < \ell \leq r$ and $\ell' > r$. x can at most r-separate the following two types of D-consecutive vertices: $(x \pm \ell, x + \ell')$ and $(x \pm \ell, x - \ell')$. □

Lemma 8 ([1]). *For $r \geq 2$, $n \geq 1$, $M_r^{LD}(C_n) \geq \lceil \frac{n}{3} \rceil$.*

Proof. Let D be an r-LD of C_n. By Lemma 7, and since there are $n - |D|$ pairs of D-consecutive vertices, we have $2|D| \geq n - |D|$. □

Here, we focus on $r = 2$. Our main result is the following theorem.

Theorem 9. *Let C_n be a cycle with vertex set $\{x_1, \ldots, x_n\}$.*

(1) $M_2^{LD}(C_n) = n$ *if* $n = 1$;
(2) $M_2^{LD}(C_n) = n - 1$ *if* $2 \leq n \leq 5$;
(3) $M_2^{LD}(C_n) = \lceil \frac{n}{3} \rceil + 1$ *if* $n = 6$ *or* $n = 6k + 3$ $(k \geq 1)$;
(4) $M_2^{LD}(C_n) = \lceil \frac{n}{3} \rceil$ *otherwise.*

Proof. For $n = 1$, it is obvious that $M_2^{LD}(C_n) = n$. When $2 \leq n \leq 5$, the distance between any two vertices in C_n is no more than 2. Hence, $M_2^{LD}(C_n) = n - 1$. As a set with size two has only three nonempty subsets, we know that $M_2^{LD}(C_6) \geq 3$. It is easy to see that $D = \{x_1, x_3, x_5\}$ is a 2-LD of C_6. Therefore, $M_2^{LD}(C_6) = 3$. In the following, we assume that $n \geq 7$.

$M_2^{LD}(C_n) \geq \lceil \frac{n}{3} \rceil$ holds by Lemma 8, next we construct a 2-LD achieving the lower bound in the following cases:

- $n = 6k, D = \{x_i | i = 6p + 4, p \geq 0\} \cup \{x_i | i = 6q, q \geq 1\}$;
- $n = 6k + 1$ or $6k + 2, D = \{x_i | i = 6p + 4, p \geq 0\} \cup \{x_i | i = 6q, q \geq 1\} \cup \{x_n\}$;
- $n = 6k + 4, D = \{x_i | i = 6p + 4, p \geq 0\} \cup \{x_i | i = 6q, q \geq 1\} \cup \{x_{n-2}\}$;
- $n = 6k + 5$ and $n > 11, D = \{x_i | i = 6p + 2, 0 \leq p \leq k - 2\} \cup \{x_i | i = 6q, 1 \leq q \leq k - 1\} \cup \{x_{n-8}, x_{n-7}, x_{n-2}, x_{n-1}\}$;
- $n = 11, D = \{x_1, x_2, x_5, x_9\}$.

Now we turn to the case $n = 6k + 3$ $(k \geq 1)$. By Lemma 8, we have known that $M_2^{LD}(C_n) \geq 2k + 1$. We first show that $M_2^{LD}(C_n) \geq 2k + 2$. Suppose to the contrary that D is a 2-LD for C_n with $2k + 1$ vertices. Then there are $4k + 2$ pairs of D-consecutive vertices, and hence every vertex in D 2-separates exactly two pairs of D-consecutive vertices, and these pairs are disjoint. We have the following claims. □

Claim 1. D contains at most two consecutive vertices in C_n.

Proof of Claim 1. Since each vertex in D 2-separates two pairs of D-consecutive vertices, it follows that D contains at most four consecutive vertices in C_n. Suppose that D contains four consecutive vertices in C_n, without loss of generality, we assume that $\{x_1, x_2, x_3, x_4\} \subseteq D$. Then both x_1 and x_4 2-separate a pair of D-consecutive vertices $\{x_n, x_5\}$, a contradiction. If D contains three consecutive vertices in C_n, without loss of generality, we assume that $\{x_1, x_2, x_3\} \subseteq D$, then both x_1 and x_3 2-separate a pair of D-consecutive vertices $\{x_n, x_4\}$, a contradiction. □

Assume that $D = \{x_{i_1}, x_{i_2}, \ldots, x_{i_{2k+1}}\}$ with $1 \leq i_1 < i_2 < \cdots < i_{2k+1} \leq n$. □

Claim 2. $|i_j - i_{j+1}| = 2$ or 4 for all $j \in \{1, \ldots, 2k + 1\}$.

Proof of Claim 2. Since $D_2(x) \neq \emptyset$ for any $x \notin D$, it is easy to know that $|i_j - i_{j+1}| \leq 5$. If $|i_j - i_{j+1}| = 5$ for some $j \in \{1, \ldots, 2k + 1\}$, then both x_{i_j} and $x_{i_{j+1}}$ 2-separate the pair of consecutive vertices $\{x_{i_j+2}, x_{i_j+3}\}$, a contradiction.

Suppose that $|i_j - i_{j+1}| = 1$ for some $j \in \{1, \ldots, 2k + 1\}$, without loss of generality, we assume that $x_1 \in D$ and $x_2 \in D$. By Claim 1, we know that $x_3 \notin D$ and $x_n \notin D$. If $x_4 \in D$, then either $\{x_3, x_5\}$ or $\{x_3, x_6\}$ is a pair of D-consecutive vertices. So, x_1 and x_2 2-separate the same pair of D-consecutive vertices, a contradiction. Thus $x_4 \notin D$. Similarly, $x_{n-1} \notin D$. If x_{n-2} and x_5 are both in D, then they both 2-separate the pair of D-consecutive vertices $\{x_n, x_3\}$, a contradiction. Without loss of generality, we take $x_5 \notin D$. $x_6 \in D$ implies that the pair of D-consecutive vertices $\{x_3, x_4\}$ are 2-separated by both x_1 and x_6. It is a contradiction. $x_7 \in D$ implies that the pair of D-consecutive vertices $\{x_4, x_5\}$ are 2-separated by both x_2 and x_7. It is a contradiction. Hence, $x_6 \notin D$ and $x_7 \notin D$. Thus, $D_2(x_5) = \emptyset$, a contradiction. Therefore, $|i_j - i_{j+1}| \neq 1$.

Suppose that $|i_j - i_{j+1}| = 3$ for some $j \in \{1, \ldots, 2k + 1\}$, without loss of generality, we assume that $x_1 \in D$ and $x_4 \in D$. Then $x_n \in D$ or $x_5 \in D$, which follows from the pair of D-consecutive vertices $\{x_2, x_3\}$ requiring to be 2-separated, however, it contradicts with $|i_j - i_{j+1}| \geq 2$. □

Since C_n contains $6k + 3$ vertices and there are $2k + 1$ vertices in D, thus by Claim 2, there must exist some $j \in \{1, \ldots, 2k + 1\}$ such that $|i_j - i_{j+1}| = |i_j - i_{j-1}|$. However, if $|i_j - i_{j+1}| = |i_j - i_{j-1}| = 2$, then both $x_{i_{j-1}}$ and $x_{i_{j+1}}$ 2-separate $\{x_{i_j-1}, x_{i_j+1}\}$; if $|i_j - i_{j+1}| = |i_j - i_{j-1}| = 4$, then there is no vertex in D 2-separating $\{x_{i_j-1}, x_{i_j+1}\}$. Therefore, $M_2^{LD}(C_n) \geq 2k + 2$.

Now, we construct a 2-LD for C_n with $2k + 2$ vertices as follows: $D = \{x_i | i = 6p + 1 \text{ or } 6p + 3, 0 \leq p \leq k - 1\} \cup \{x_{n-1}, x_{n-2}\}$. □

5. Conclusion

The main purpose of this paper is to give the exact value of $M_r^I(G)$ for paths and cycles for arbitrary positive integer r, and of $M_2^{LD}(C_n)$. It would be of interest to extend the latter to r-LDs for $r > 2$. Some new results on r-LDs for cycles can be found in [13].

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