ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 304 (2006) 633–665
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Ring hulls and applications

Gary F. Birkenmeier [a,*], Jae Keol Park [b], S. Tariq Rizvi [c]

[a] *Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, USA*
[b] *Department of Mathematics, Busan National University, Busan 609-735, South Korea*
[c] *Department of Mathematics, Ohio State University, Lima, OH 45804-3576, USA*

Received 16 February 2005

Available online 21 August 2006

Communicated by Efim Zelmanov

Abstract

Our goal is to develop methods that enable one to select a class $\mathfrak{K}$ of rings and then to describe all right essential overrings or all right rings of quotients of a given ring R which lie in $\mathfrak{K}$. Our major method of attack is to determine the existence and/or uniqueness of right ring hulls of R in $\mathfrak{K}$ and to use these to characterize the right essential overrings of R which are in $\mathfrak{K}$. Some applications are: (1) a characterization of the right rings of quotients of the 2-by-2 upper triangular matrix ring over a PID which are either Baer or right extending; (2) a characterization of a continuous ring hull for a commutative ring whose singular ideal has finite uniform dimension; (3) a characterization of the right extending rings which have the 2-by-2 matrix ring over a given division ring for their maximal right ring of quotients; (4) a characterization of the intermediate right extending rings between the 2-by-2 upper triangular matrix ring and the 2-by-2 matrix ring over a large class of local right finitely Σ-extending rings; (5) a characterization of the classical right ring of quotients as a ring hull from a certain class of rings.

Keywords: Ring of quotients; Essential overring; (quasi-)Baer ring; (FI-)Extending ring; (semi-)Hereditary ring; (quasi-)Continuous ring

* Corresponding author.
E-mail addresses: gfb1127@louisiana.edu (G.F. Birkenmeier), jkpark@pusan.ac.kr (J.K. Park), rizvi.1@osu.edu (S.T. Rizvi).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.06.034

0. Introduction

Throughout this paper all rings are associative with unity and R denotes such a ring. Subrings and overrings preserve the unity of the base ring. Ideals without the adjective "right" or "left" mean two-sided ideals. All modules are unital and we use M_R (respectively, ${}_RM$) to denote a right (respectively, left) R-module. If N_R is a submodule of M_R, then N_R is *essential* (respectively, *dense* also called *rational*) in M_R if for any $0 \neq x \in M$, there exists $r \in R$ such that $0 \neq xr \in N$ (respectively, for any $x, y \in M$ with $x \neq 0$, there exists $r \in R$ such that $xr \neq 0$, and $yr \in N$). Recall that a *right ring of quotients* T of R is an overring of R such that R_R is dense in T_R. The maximal right (respectively, left) ring of quotients of R is denoted by $Q(R)$ (respectively, $Q^\ell(R)$). We say that T is a *right essential overring of* R if T is an overring of R such that R_R is essential in T_R. The right injective hull of R is denoted by $E(R_R)$ and we use $\mathcal{E}_R$ to denote $\mathrm{End}(E(R_R))$. Unless noted otherwise, we work with right sided concepts. However most of the results and concepts have left sided analogues.

One of the major efforts in Ring Theory has been, for a given ring R, to find a "well behaved" overring Q in the sense that it has better properties than R such that a rich information transfer between R and Q can take place. Alternatively, given a "well behaved" ring, to find conditions which describe those subrings for which there is some fruitful transfer of information.

Our general goal is to develop methods that enable one to select a specific class $\mathfrak{K}$ of rings and then to describe all right essential overrings or all right rings of quotients of a given ring R which lie in $\mathfrak{K}$. Our major approach is to determine existence and/or uniqueness results for right essential overrings which are, in some sense, "minimal" with respect to belonging to $\mathfrak{K}$. Then, by capitalizing on the hull-like behavior of these "minimal" ones, we describe or characterize the right essential overrings belonging to $\mathfrak{K}$ in terms of the "minimal" such rings in $\mathfrak{K}$.

Motivation for our outlook can be seen from the following examples. First take $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$, where $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the ring of rational numbers, respectively. The ring R is neither right nor left Noetherian and its prime radical is nonzero. However, $Q(R)$ is simple Artinian. Next take R to be a domain which does not satisfy the right Ore condition. Then $Q(R)$ is a simple right self-injective regular ring which has an infinite set of orthogonal idempotents and an unbounded nilpotent index. The sharp disparity between R and $Q(R)$ in the aforementioned examples limits the transfer of information between R and $Q(R)$. These examples illustrate a need to find overrings of a given ring that have some weaker versions of the properties traditionally associated with right rings of quotients such as semisimple Artinian, right Artinian, right Noetherian, right self-injective regular, or right self-injective. Furthermore, this need is reinforced when one studies classes of rings for which $R = Q(R)$ (e.g., right Kasch rings). For these classes the theory of right rings of quotients is virtually useless. However our theory which considers right essential overrings is still applicable.

Using the traditional class of right self-injective rings as a model, we introduce the notion of a $\mathfrak{D}$-$\mathfrak{E}$ class. Although the concept of a $\mathfrak{D}$-$\mathfrak{E}$ class encompasses many generalizations of right self-injectivity, it retains the advantage that its members have an abundance of idempotents for their distinguished right ideals which is crucial in various structural considerations. Recall the definitions of some classes that generalize the class of right self-injective rings or regular right self-injective rings. A ring R is: right (*FI-*)*extending* if every (ideal) right ideal of R is essential in a right ideal generated by an idempotent (classes denoted by $\mathfrak{FI}$, $\mathfrak{E}$) (see [14,15,17,23] for $\mathfrak{E}$; and see [5,9,11] and [18, Corollary 33B] for $\mathfrak{FI}$); right (*quasi-*)*continuous* if R is right extending and (if A_R and B_R are direct summands of R_R with $A \cap B = 0$, then $A_R \oplus B_R$ is a direct summand of R_R) R satisfies the (C_2) condition, that is, if X and Y are right ideals of R with

$X_R \cong Y_R$ and X_R is a direct summand of R_R, then Y_R is a direct summand of R_R (classes denoted by $\mathfrak{qCon}$, $\mathfrak{Con}$) (see [20,33,36] for $\mathfrak{Con}$ and $\mathfrak{qCon}$); *(quasi-)Baer* if the right annihilator of every (ideal) nonempty subset of R is an idempotent generated right ideal (classes denoted by $\mathfrak{qB}$, $\mathfrak{B}$) (see [3,24,25,42] for $\mathfrak{B}$; and see [6–8,10,16,34] for $\mathfrak{qB}$). These classes have their roots in the study of right self-injective rings and in Operator Theory, especially in the study of von Neumann algebras. We delineate our techniques by obtaining results for the aforementioned classes. However, our methods are not limited to these classes and can be applied to many other classes of rings.

Since the right essential overrings (which are, in some sense, "minimal" with respect to a specific class of rings) are important tools in our investigations, we define several types of ring hulls to accommodate some notions of "minimality." Our search for such minimal overrings for a given ring R includes the seemingly unexplored region that lies between $Q(R)$ and $E(R_R)$ (e.g., when $R = Q(R)$). We consider two basic types (the others are their derivatives). Let S be a right essential overring of R and $\mathfrak{K}$ be a specific class of rings. We say that S is a $\mathfrak{K}$ *right ring hull* of R if S is minimal among the right essential overrings of R belonging to the class $\mathfrak{K}$ (i.e., whenever T is a subring of S where T is a right essential overring of R in the class $\mathfrak{K}$, then $T = S$). In the other basic type, we generate S with R and certain subsets of $E(R_R)$ so that S is in $\mathfrak{K}$ in some "minimal" fashion. This leads to our concepts of a $\mathfrak{C}$ pseudo and of a $\mathfrak{C}$ ρ pseudo right ring hull of R, where $\mathfrak{C}$ is a $\mathfrak{D}$-$\mathfrak{E}$ class of rings and ρ is an equivalence relation on a certain set of idempotents from $\mathcal{E}_R$. These ring hull concepts are "tool" concepts in that they appear in the proofs of various results but do not appear in the statement of the results (e.g., Theorems 3.1 and 3.7).

In this paper, we: (1) begin the development of a general theory of $\mathfrak{K}$ right ring hulls and $\mathfrak{C}$ pseudo right ring hulls, where $\mathfrak{C}$ is a $\mathfrak{D}$-$\mathfrak{E}$ class of rings; (2) apply our theory to several classes of rings; (3) characterize the right essential overrings from various classes of rings for certain subrings of matrix rings.

1. Preliminaries

This section is mainly devoted to background information and preliminary results. We present results indicating interconnections between those classes used to illustrate our theory. Various conditions are presented which transfer from a ring to its right rings of quotients or to its right essential overrings. For example, we indicate that if R is right extending (respectively, right FI-extending), then T is right extending (respectively, right FI-extending) where T is a right ring of quotients (respectively, a right essential overring) of R. We define the concept of a $\mathfrak{D}$-$\mathfrak{E}$ class and illustrate it with several concrete examples. Finally we formulate two problems which give direction and motivation to our work.

For a ring R, we use $\mathbf{I}(R)$, $\mathbf{B}(R)$, $\mathbf{U}(R)$, $Z(R_R)$, $\mathrm{Cen}(R)$, and $\mathbf{J}(R)$ to denote the idempotents, central idempotents, units, right singular ideal, center, and Jacobson radical of R, respectively. For ring extensions of R, we use $Q^r_{c\ell}(R)$, $R\mathbf{B}(Q(R))$, $\mathrm{Mat}_n(R)$, and $T_n(R)$ to denote the classical right ring of quotients, idempotents closure (i.e., the subring of $Q(R)$ generated by R and $\mathbf{B}(R)$ [2]), n-by-n matrix ring, and n-by-n upper triangular matrix ring over R, respectively. For a nonempty subset X of a ring R, the symbols $r_R(X)$, $\ell_R(X)$, and $\langle X\rangle_R$ denote the right annihilator of X in R, the left annihilator of X in R, and the subring of R generated by X, respectively. When the context is clear, we use $r(X)$ and $\ell(X)$ for $r_R(X)$ and $\ell_R(X)$, respectively. Also $\mathbb{Z}_n$ denotes the ring of integers modulo n. Recall that a ring R is called *reduced* if R has no nonzero nilpotent elements and *Abelian* if $\mathbf{I}(R) = \mathbf{B}(R)$.

The notation $N_R \leqslant M_R$, $N_R \leqslant^{\text{ess}} M_R$, and $N_R \trianglelefteq M_R$ ($I \trianglelefteq R$) symbolize that N_R is a submodule, an essential submodule, and a fully invariant submodule of M_R (I is an ideal of R), respectively. A module M_R is said to be *(FI-)extending* if every (fully invariant) submodule of M is essential in a direct summand of M_R (see [9,14,23]). Terminology not defined here can be found in a text such as [21,27,28], or [37].

Let $\mathcal{Q}_R = \text{End}({}_{\mathcal{E}_R}E(R_R))$. Observe that $Q(R) = 1 \cdot \mathcal{Q}_R$ (i.e., the canonical image of $\mathcal{Q}_R$ in $E(R_R)$) and that $\mathbf{B}(\mathcal{Q}_R) = \mathbf{B}(\mathcal{E}_R)$ [28, pp. 94–96]. So we may write elements of $\mathbf{B}(\mathcal{Q}_R)$ on the right of their arguments in using them as elements of $\mathcal{E}_R$. Also, $\mathbf{B}(Q(R)) = \{b(1) \mid b \in \mathbf{B}(\mathcal{Q}_R)\}$ [27, p. 366]. Thus $R\mathbf{B}(\mathcal{E}_R) = R\mathbf{B}(Q(R))$. In the sequel, we implicitly use the fact that if $c \in \mathbf{B}(\text{End}(M_R))$ and $e \in \mathbf{I}(\text{End}(M_R))$ with $eM_R = cM_R$, then $e = c$.

Definition 1.1. We say that a ring R is *right essentially Baer* (respectively, *right essentially quasi-Baer*) if the right annihilator of any nonempty subset (respectively, ideal) of R is essential in a right ideal generated by an idempotent. We use $\mathfrak{eB}$ (respectively, $\mathfrak{eqB}$) to denote the class of right essentially Baer (respectively, right essentially quasi-Baer) rings.

It can be seen that $\mathfrak{eB}$ (respectively, $\mathfrak{eqB}$) properly contains $\mathfrak{E}$ (respectively, $\mathfrak{FI}$) and $\mathfrak{B}$ (respectively, $\mathfrak{qB}$): If $R = A \oplus B$, where A is a domain which is not right Ore and B is a right uniform prime ring with $Z(B_B) \neq 0$ [13, Example 4.4], then R is neither right extending nor Baer. But $R \in \mathfrak{eB}$. Now take $T = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. From [11, Theorems 1.4 and 3.2], the ring T is neither right FI-extending nor quasi-Baer. However $T \in \mathfrak{eqB}$.

Proposition 1.2. *Assume that R is a right nonsingular ring.*

(i) *If $R \in \mathfrak{eB}$ (respectively, $R \in \mathfrak{eqB}$), then $R \in \mathfrak{B}$ (respectively, $R \in \mathfrak{qB}$).*
(ii) *If $R \in \mathfrak{FI}$, then $R \in \mathfrak{qB}$.*

Proof. (i) First assume that $R \in \mathfrak{eB}$. Let $\emptyset \neq X \subseteq R$. Then there is $e \in \mathbf{I}(R)$ with $r(X)_R \leqslant^{\text{ess}} eR_R$. So $\ell(r(X)) = \ell(eR) = R(1-e)$. Hence $r(X) = r[\ell(r(X))] = r(\ell(eR)) = r(R(1-e)) = eR$. Therefore $R \in \mathfrak{B}$. For the case when $R \in \mathfrak{eqB}$, take X to be an ideal of R and follow the above proof.

(ii) See [9, Proposition 4.4]. □

Proposition 1.3. ([4, Lemma 2.2] and [9, Theorem 4.7].) *Assume that R is a semiprime ring. Then $R \in \mathfrak{FI}$ if and only if $R \in \mathfrak{qB}$ if and only if $R \in \mathfrak{eqB}$.*

Lemma 1.4. *Let T be a right ring of quotients of R.*

(i) *For right ideals X and Y of T, if $X_T \leqslant^{\text{ess}} Y_T$, then $X_R \leqslant^{\text{ess}} Y_R$.*
(ii) *If $X_R \trianglelefteq T_R$, then $X_R \leqslant^{\text{ess}} TXT_R$.*

Proof. (i) Let $0 \neq y \in Y$. Then there is $t \in T$ with $0 \neq yt \in X$. Since R_R is dense in T_R, there exists $r \in R$ such that $ytr \neq 0$, and $tr \in R$. Now $ytr \in X$. Thus $X_R \leqslant^{\text{ess}} Y_R$.

(ii) Define $f_\alpha : T_R \to T_R$ by $f_\alpha(t) = \alpha t$ where $\alpha \in T$. Then $f_\alpha \in \text{End}(T_R)$. Hence $TX \subseteq X$. Let $0 \neq y \in TXT$. Then $y = x_1t_1 + \cdots + x_nt_n$ where $x_i \in X$, $t_i \in T$, and $x_it_i \neq 0$ for each i, $1 \leqslant i \leqslant n$. Since R_R is dense in T_R, there is $r_1 \in R$ with $0 \neq yr_1$, and $t_1r_1 \in R$. Again there exists $r_2 \in R$ such that $0 \neq yr_1r_2$, and $t_2r_1r_2 \in R$. Continuing this process, we get $r \in R$ such that $0 \neq yr \in X$. Thus $X_R \leqslant^{\mathrm{ess}} TXT_R$. $\square$

Proposition 1.5. *Let T be a right ring of quotients of R. Then*:

(i) *$T \in \mathfrak{FI}$ if and only if T_R is FI-extending*;
(ii) *$T \in \mathfrak{E}$ if and only if T_R is extending.*

Proof. (i) Assume that $T \in \mathfrak{FI}$. Let $X_R \trianglelefteq T_R$. By Lemma 1.4(ii), $X_R \leqslant^{\mathrm{ess}} TXT_R$. There exists $e = e^2 \in T$ such that $TXT_T \leqslant^{\mathrm{ess}} eT_T$. Thus $TXT_R \leqslant^{\mathrm{ess}} eT_R$ from Lemma 1.4(i). Therefore $X_R \leqslant^{\mathrm{ess}} eT_R$. Consequently, T_R is FI-extending.

Conversely, assume that T_R is FI-extending. Take $Y \trianglelefteq T$. Then $Y_R \trianglelefteq T_R$ since $\mathrm{End}(T_R) = \mathrm{End}(T_T) \cong T$ from [28, p. 95]. So there is $e = e^2 \in \mathrm{End}(T_R) = \mathrm{End}(T_T) \cong T$ with $Y_R \leqslant^{\mathrm{ess}} eT_R$. Hence $Y_T \leqslant^{\mathrm{ess}} eT_T$. Thus $T \in \mathfrak{FI}$.

(ii) The proof of this part is similar to that of part (i). $\square$

Definition 1.6. Let $\mathfrak{R}$ be a class of rings, $\mathfrak{K}$ a subclass of $\mathfrak{R}$, and $\mathfrak{X}$ a class containing all subsets of every ring. We say that $\mathfrak{K}$ is a class *determined by a property on right ideals* if there exist an assignment $\mathfrak{D}_{\mathfrak{K}} : \mathfrak{R} \to \mathfrak{X}$ such that $\mathfrak{D}_{\mathfrak{K}}(R) \subseteq \{$right ideals of $R\}$ and a property P such that each element of $\mathfrak{D}_{\mathfrak{K}}(R)$ has P if and only if $R \in \mathfrak{K}$.

If $\mathfrak{K}$ is such a class where P is the property that a right ideal is essential in an idempotent generated right ideal, then we say that $\mathfrak{K}$ is a $\mathfrak{D}$-$\mathfrak{E}$ *class* (i.e., distinguished extending class) and use $\mathfrak{C}$ to designate a $\mathfrak{D}$-$\mathfrak{E}$ class.

Some examples illustrating Definition 1.6 are:

(1) $\mathfrak{K}$ is the class of right Noetherian rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{$right ideals of $R\}$, and P is the property that a right ideal is finitely generated;
(2) $\mathfrak{K}$ is the class of regular rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{$principal right ideals of $R\}$, and P is the property that a right ideal is generated by an idempotent as a right ideal;
(3) $\mathfrak{K} = \mathfrak{B}$, $\mathfrak{D}_{\mathfrak{B}}(R) = \{r_R(X) \mid \emptyset \neq X \subseteq R\}$, and P is the property that a right ideal is generated by an idempotent;
(4) $\mathfrak{C} = \mathfrak{E}$ (respectively, $\mathfrak{C} = \mathfrak{FI}$, $\mathfrak{C} = \mathfrak{eB}$), $\mathfrak{D}_{\mathfrak{E}}(R) = \{I \mid I_R \leqslant R_R\}$ (respectively, $\mathfrak{D}_{\mathfrak{FI}}(R) = \{I \mid I \trianglelefteq R\}$, $\mathfrak{D}_{\mathfrak{eB}}(R) = \{r_R(X) \mid \emptyset \neq X \subseteq R\}$).

In this paper our primary focus is on classes of rings which are either $\mathfrak{D}$-$\mathfrak{E}$ classes or subclasses of $\mathfrak{D}$-$\mathfrak{E}$ classes. Note that any $\mathfrak{D}$-$\mathfrak{E}$ class always contains the class of right extending (and hence all right self-injective) rings. Moreover, many known classes of rings are subclasses of a $\mathfrak{D}$-$\mathfrak{E}$ class (e.g., $\mathfrak{B}$ is a subclass of $\mathfrak{eB}$).

Theorem 1.7 illustrates the generality achieved by working in the context of a $\mathfrak{D}$-$\mathfrak{E}$ class, while Corollary 1.8 demonstrates its application to concrete $\mathfrak{D}$-$\mathfrak{E}$ classes.

Theorem 1.7. *Assume that $\mathfrak{C}$ is a $\mathfrak{D}$-$\mathfrak{E}$ class of rings.*

(i) *Let T be a right essential overring of R. Suppose that for each $Y \in \mathfrak{D}_{\mathfrak{C}}(T)$ there exist $X_R \leqslant R_R$ and $e \in \mathbf{I}(T)$ such that $X_R \leqslant^{\mathrm{ess}} eR_R$, $X_R \leqslant^{\mathrm{ess}} Y_R$, and $eY \subseteq Y$. Then $T \in \mathfrak{C}$.*
(ii) *Let T be a right ring of quotients of R and $R \in \mathfrak{C}$. If $Y \in \mathfrak{D}_{\mathfrak{C}}(T)$ implies $Y \cap R \in \mathfrak{D}_{\mathfrak{C}}(R)$, then $T \in \mathfrak{C}$.*

Proof. (i) First note that $Y = eY \oplus (1-e)Y$. Assume that there exists $0 \neq y \in (1-e)Y$. Then $0 \neq yr \in X$ for some $r \in R$. But $yr = (1-e)yr \in (1-e)X = 0$, a contradiction. So $Y_R \leqslant^{\text{ess}} eT_R$, hence $Y_T \leqslant^{\text{ess}} eT_T$. Therefore $T \in \mathfrak{C}$.

(ii) *Claim.* $\ell_R(Y) = \ell_R(Y \cap R)$: First we note that $\ell_R(Y) \subseteq \ell_R(Y \cap R)$. Now let $a \in \ell_R(Y \cap R)$ and suppose that there exists $y \in Y$ such that $ay \neq 0$. Since R_R is dense in T_R, there exists $r \in R$ such that $ayr \neq 0$, and $yr \in Y \cap R$, a contradiction. Thus $\ell_R(Y) = \ell_R(Y \cap R)$. This proves the claim. Next, since $Y \cap R \in \mathfrak{D}_{\mathfrak{C}}(R)$, there exists $e \in \mathbf{I}(R)$ with $(Y \cap R)_R \leqslant^{\text{ess}} eR_R$. Then $1-e \in \ell_R(Y \cap R)$. Hence $(1-e)Y = 0$. Note that $Y_R \leqslant eT_R$ and $eR_R \leqslant^{\text{ess}} eT_R$. So $Y_R = eY_R \leqslant^{\text{ess}} eT_R$. Therefore $T \in \mathfrak{C}$. □

As a consequence of Theorem 1.7, the next corollary exhibits the transfer of the right (FI-)extending property from R to its (right essential overrings) right rings of quotients. Also note that whenever a property is carried from R to its (right essential overrings) right rings of quotients, then a Zorn's lemma argument can be used to show that R has a (right essential overring) right ring of quotients which is maximal with respect to having that property.

Corollary 1.8.

(i) *Any right essential overring of a right FI-extending ring is right FI-extending.*
(ii) *Any right ring of quotients of a right extending ring is right extending.*
(iii) *Any right ring of quotients of a right finitely Σ-extending ring is right finitely Σ-extending.*
(iv) *Any right ring of quotients of a right uniform extending ring is right uniform extending.*

Proof. (i) Let T be a right essential overring of a right FI-extending ring R and $Y \in \mathfrak{D}_{\mathfrak{FI}}(T)$. Since $Y \trianglelefteq T$, $Y \cap R \trianglelefteq R$. So there exists $e \in \mathbf{I}(R)$ such that $(Y \cap R)_R \leqslant^{\text{ess}} eR_R$. The result now follows from Theorem 1.7(i), where we take $X = Y \cap R$.

(ii) This part is a direct consequence of Theorem 1.7(ii) since $\mathfrak{E}$ is a $\mathfrak{D}$-$\mathfrak{C}$ class and $\mathfrak{D}_{\mathfrak{E}}(R)$ is the set of all right ideals of R.

(iii) Let T be a right ring of quotients of a right finitely Σ-extending ring R. Then by [17, Lemma 12.8], $\text{Mat}_n(R) \in \mathfrak{E}$ for any positive integer n. Since $\text{Mat}_n(T)$ is a right ring of quotients of $\text{Mat}_n(R)$, $\text{Mat}_n(T) \in \mathfrak{E}$ by part (ii). Thus T is a right finitely Σ-extending ring by [17, Lemma 12.8].

(iv) The proof follows directly from Theorem 1.7(ii). □

Motivated by Theorem 1.7, we introduce the following notations which will be used in the sequel. Let T be a right essential overring of R. For a class $\mathfrak{K}$ of rings, we use:

(i) $\mathfrak{D}_{\mathfrak{K}}(T \to R)$ to denote the condition that for each $Y \in \mathfrak{D}_{\mathfrak{K}}(T)$ there exists $X \in \mathfrak{D}_{\mathfrak{K}}(R)$ such that $X_R \leqslant^{\text{ess}} Y_R$;
(ii) $\mathfrak{D}_{\mathfrak{K}}(R \to T)$ to denote the condition that for each $X \in \mathfrak{D}_{\mathfrak{K}}(R)$ there exists $Y \in \mathfrak{D}_{\mathfrak{K}}(T)$ such that $X_R \leqslant^{\text{ess}} Y_R$;
(iii) $\mathfrak{D}_{\mathfrak{K}}(T \# R)$ to denote the condition $Y \in \mathfrak{D}_{\mathfrak{K}}(T)$ implies $Y \cap R \in \mathfrak{D}_{\mathfrak{K}}(R)$.

It is easy to see that the condition $\mathfrak{D}_{\mathfrak{K}}(T \# R)$ implies $\mathfrak{D}_{\mathfrak{K}}(T \to R)$, while the converse does not hold. Also observe that if $\mathfrak{K}$ is either $\mathfrak{E}$ or $\mathfrak{FI}$, then $\mathfrak{D}_{\mathfrak{K}}(T \# R)$ holds, while if T is a right ring of quotients of R and $\mathfrak{K} = \mathfrak{E}$ then $\mathfrak{D}_{\mathfrak{K}}(R \to T)$ holds.

Theorem 1.9.

(i) *Let T be a right and left essential overring of R. If $R \in \mathfrak{q}\mathfrak{B}$, then $T \in \mathfrak{q}\mathfrak{B}$.*
(ii) *Let T be a right essential overring of R which is also a left ring of quotients of R. If $R \in \mathfrak{B}$ (respectively, $R \in \mathfrak{e}\mathfrak{q}\mathfrak{B}$), then $T \in \mathfrak{B}$ (respectively, $T \in \mathfrak{e}\mathfrak{q}\mathfrak{B}$).*
(iii) *Let T be a right and left ring of quotients of R. If $R \in \mathfrak{e}\mathfrak{B}$, then $T \in \mathfrak{e}\mathfrak{B}$.*

Proof. (i) Assume that $R \in \mathfrak{q}\mathfrak{B}$. Let $Y \trianglelefteq T$ and $X = Y \cap R$. There exists $e \in \mathbf{I}(R)$ such that $r_R(X) = eR$. Let $a \in r_T(Y)$. Assume $(1-e)a \neq 0$. Since $R_R \leqslant^{\text{ess}} T_R$, there exists $r \in R$ with $0 \neq (1-e)ar \in R$. Note that $X(1-e)ar = 0$. Hence $(1-e)ar \in r_R(X)$, a contradiction. So $(1-e)r_T(Y) = 0$. Therefore $r_T(Y) \subseteq eT$. Now assume that there exists $y \in Y$ such that $0 \neq ye$. Since T is a left essential overring of R, there is $s \in R$ with $0 \neq sye \in R$. Hence $sye \in Y \cap R = X$. But $sye \in Xe = 0$, a contradiction. Hence $Ye = 0$. Therefore $r_R(Y) = eT$, hence $T \in \mathfrak{q}\mathfrak{B}$.

(ii) First assume that $R \in \mathfrak{B}$. Let A be a nonempty subset of T and $Y = TA$. Then $r_T(A) = r_T(Y)$. Let $X = Y \cap R$. Then there is $e \in \mathbf{I}(R)$ with $r_R(X) = eR$. First to show that $r_T(Y) \subseteq eT$, assume that there exists $a \in r_T(Y)$ such that $(1-e)a \neq 0$. Then since $R_R \leqslant^{\text{ess}} T_R$, there is $r \in R$ with $0 \neq (1-e)ar \in R$. So $X(1-e)ar = Xar = 0$, hence $0 \neq (1-e)ar \in r_R(X) = eR$, a contradiction. Thus $r_T(Y) \subseteq eT$. Next assume that $ye \neq 0$ for some $y \in Y$. Since ${}_RR$ is dense in ${}_RT$, there is $s \in R$ such that $sye \neq 0$, and $sy \in R$. So $sy \in X$. Hence $0 \neq sye \in Xe = 0$, a contradiction. Thus $e \in r_T(Y)$, so $eT \subseteq r_T(Y)$. Therefore $r_T(A) = r_T(Y) = eT$. Thus $T \in \mathfrak{B}$.

Next assume that $R \in \mathfrak{e}\mathfrak{q}\mathfrak{B}$. Let $Y \trianglelefteq T$ and $X = Y \cap R$. There exists $e \in \mathbf{I}(R)$ such that $r_R(X)_R \leqslant^{\text{ess}} eR_R$. As in the proof of (i), we obtain $r_T(Y) \subseteq eT$. Now let $0 \neq et \in eT$ with $t \in T$. Then there exists $s \in R$ with $0 \neq ets \in r_R(X)$. Assume that there is $y \in Y$ such that $0 \neq yets$. Since ${}_RR$ is dense in ${}_RT$, there exists $d \in R$ satisfying $dyets \neq 0$, and $dy \in R$. But $dy \in Y \cap R = X$, a contradiction. Hence $0 \neq ets \in r_T(Y)$. Therefore $r_T(Y)_T \leqslant^{\text{ess}} eT_T$, so $T \in \mathfrak{e}\mathfrak{q}\mathfrak{B}$.

(iii) Assume that $R \in \mathfrak{e}\mathfrak{B}$. Let A be a nonempty subset of R and $Y = TA$. Then $r_T(A) = r_T(Y)$. Take $X = Y \cap R$. There exists $e \in \mathbf{I}(R)$ with $r_R(X)_R \leqslant^{\text{ess}} eR_R$. Let $a \in r_T(Y)$. Assume $(1-e)a \neq 0$. Since R_R is dense in T_R, there is $r \in R$ such that $(1-e)ar \neq 0$, and $ar \in R$. But $ar \in r_R(X)$, a contradiction. Therefore $r_T(Y) \subseteq eT$. To show that $r_T(Y)_T \leqslant^{\text{ess}} eT_T$, use the corresponding part of the proof in part (ii). □

The following corollary generalizes the well-known result that a right ring of quotients of a Prüfer domain is a Prüfer domain [19, pp. 321–323].

Corollary 1.10. *Let T be a right and left ring of quotients of R. If R is right semihereditary and every finitely generated free right R-module satisfies the ACC on direct summands, then T is right and left semihereditary.*

Proof. The proof follows from Theorem 1.9(ii) and [30, pp. 233–235]. □

Results 1.7 through 1.10 show that under suitable conditions and for certain classes of rings if R is any ring with a right essential overring S from one of these classes, then every other right essential overring of R which contains S as a subring, is also from that class. These results provide some motivation for the study of the following problems:

Problem I. Assume that a ring R and a class of rings $\mathfrak{K}$ are given.

(i) Find conditions to ensure the existence of right rings of quotients and that of right essential overrings of R which are, in some sense, "minimal" with respect to belonging to the class $\mathfrak{K}$.

(ii) Characterize the right rings of quotients and the right essential overrings of R which are in the class $\mathfrak{K}$, possibly by using the "minimal" ones obtained in part (i).

Problem II. Given a ring S and a class $\mathfrak{K}$, determine those rings T such that $Q(T) = S$ and $T \in \mathfrak{K}$.

2. Existence and uniqueness of ring hulls

In this section we introduce several types of ring hulls and begin to develop a general theory. After illustrating the ring hull concept with various classes of rings, we develop some technical machinery which enables us to verify the existence of hulls for various $\mathfrak{D}$-$\mathfrak{E}$ classes. Equivalence relations ρ are used to refine and reduce the size of the subsets of $E(R_R)$ which are utilized to generate $\mathfrak{C}\ \rho$ pseudo right ring hulls. We exhibit examples to distinguish the difference between right ring hulls and pseudo right ring hulls.

We henceforth assume that whenever a ring R is given, all right essential overrings of R are considered to be contained as right R-modules in a fixed injective hull $E(R_R)$ of R_R and all right rings of quotients of R are considered to be subrings of a fixed maximal right ring of quotients $Q(R)$ of R.

In our next definition we exploit the notion of a right essential overring which is minimal with respect to belonging to a class $\mathfrak{K}$ of rings.

Definition 2.1. Let $\mathfrak{K}$ denote a class of rings. For R, let S be a right essential overring of R and T be an overring of R. Consider the following conditions.

(i) $S \in \mathfrak{K}$.
(ii) If $T \in \mathfrak{K}$ and T is a subring of S, then $T = S$.
(iii) If S and T are subrings of a ring V and $T \in \mathfrak{K}$, then S is a subring of T.
(iv) If $T \in \mathfrak{K}$ and T is a right essential overring of R, then S is a subring of T.

If S satisfies (i) and (ii), then we say S is a $\mathfrak{K}$ *right ring hull* of R, denoted by $\widetilde{Q}_{\mathfrak{K}}(R)$. If S satisfies (i) and (iii), then we say S is the $\mathfrak{K}$ *absolute to V right ring hull* of R, denoted by $Q^V_{\mathfrak{K}}(R)$; for the $\mathfrak{K}$ absolute to $Q(R)$ right ring hull, we use the notation $\widehat{Q}_{\mathfrak{K}}(R)$. If S satisfies (i) and (iv), then we say S is the $\mathfrak{K}$ *absolute right ring hull* of R, denoted by $Q_{\mathfrak{K}}(R)$. Observe that if $Q(R) = E(R_R)$, then $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$. We see that the concept of a $\mathfrak{K}$ absolute right ring hull was already implicit in [33] from their definition of a type III continuous (module) hull.

Next, we consider generating a right essential overring in a class $\mathfrak{K}$ from a base ring R and some subset of $\mathcal{E}_R$. By using equivalence relations, we can effectively reduce the size of the subsets of $\mathcal{E}_R$ needed to generate a right essential overring of R in $\mathfrak{K}$.

Definition 2.2. Let $\mathfrak{R}$ denote a class of rings and $\mathfrak{X}$ a class of subsets of rings such that for each $R \in \mathfrak{R}$ all subsets of $\mathcal{E}_R$ are contained in $\mathfrak{X}$. Let $\mathfrak{K}$ be a subclass of $\mathfrak{R}$ such that there exists an assignment $\delta_{\mathfrak{K}} : \mathfrak{R} \to \mathfrak{X}$ such that $\delta_{\mathfrak{K}}(R) \subseteq \mathcal{E}_R$ and $\delta_{\mathfrak{K}}(R)(1) \subseteq R$ implies $R \in \mathfrak{K}$, where $\delta_{\mathfrak{K}}(R)(1) = \{h(1) \in E(R_R) \mid h \in \delta_{\mathfrak{K}}(R)\}$. Let S be a right essential overring of R and ρ an equivalence relation on $\delta_{\mathfrak{K}}(R)$.

(i) If $\delta_{\mathfrak{K}}(R)(1) \subseteq S$ and $\langle R \cup \delta_{\mathfrak{K}}(R)(1)\rangle_S \in \mathfrak{K}$, then we call $\langle R \cup \delta_{\mathfrak{K}}(R)(1)\rangle_S$ the $\delta_{\mathfrak{K}}$ *pseudo right ring hull of R with respect to S* and denote it by $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$. If $S = R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$, then we say that S is a $\delta_{\mathfrak{K}}$ *pseudo right ring hull of R*.

(ii) If $\delta^{\rho}_{\mathfrak{K}}(R)(1) \subseteq S$ and $\langle R \cup \delta^{\rho}_{\mathfrak{K}}(R)(1)\rangle_S \in \mathfrak{K}$, then we call $\langle R \cup \delta^{\rho}_{\mathfrak{K}}(R)(1)\rangle_S$ a $\delta_{\mathfrak{K}}$ ρ *pseudo right ring hull of R with respect to S* and denote it by $R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$, where $\delta^{\rho}_{\mathfrak{K}}(R)$ is a set of representatives of all equivalence classes of ρ and $\delta^{\rho}_{\mathfrak{K}}(R)(1) = \{h(1) \in E(R_R) \mid h \in \delta^{\rho}_{\mathfrak{K}}(R)\}$. If $S = R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$, then we say that S is a $\delta_{\mathfrak{K}}$ ρ *pseudo right ring hull of R*.

If the $\delta_{\mathfrak{K}}$ has been fixed for a class $\mathfrak{K}$, then in the above nomenclature we replace $\delta_{\mathfrak{K}}$ (respectively, $\delta_{\mathfrak{K}}$ ρ) with $\mathfrak{K}$ (respectively, $\mathfrak{K}$ ρ) (e.g., $\delta_{\mathfrak{K}}$ pseudo right ring hull becomes $\mathfrak{K}$ pseudo right ring hull) and delete $\delta_{\mathfrak{K}}$ from the notation (e.g., $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$ becomes $R(\mathfrak{K}, S)$).

Throughout the remainder of this paper take $\mathfrak{R}$ to be the class of all rings unless indicated otherwise. Some examples illustrating Definition 2.2 are:

(1) $\mathfrak{K} = \mathfrak{SI} = \{\text{right self-injective rings}\}$, $\delta_{\mathfrak{SI}}(R) = \mathcal{E}_R$.
(2) $\mathfrak{K} = \mathfrak{qCon}$, $\delta_{\mathfrak{qCon}}(R) = \mathbf{I}(\mathcal{E}_R)$.
(3) $\mathfrak{K} = \{\text{right } P\text{-injective rings}\}$, $\delta_{\mathfrak{K}}(R) = \{h \in \mathcal{E}_R \mid$ there exist $a \in R$ and an R-homomorphism $f : aR \to R$ such that $h|_{aR} = f\}$.
(4) Let $\mathfrak{R} = \{\text{right nonsingular rings}\}$, $\mathfrak{K} = \mathfrak{B}$, $\delta_{\mathfrak{B}}(R) = \{e \in \mathbf{I}(\mathcal{E}_R) \mid$ there exists $\emptyset \neq X \subseteq R$ such that $r_{Q(R)}(X) = eQ(R)\}$.

Also note that Definition 2.2 allows us the flexibility to consider any right essential overring S of a ring R, such that $S \in \mathfrak{K}$ and $S = \langle R \cup \delta(1)\rangle_S$, to be a $R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$ where $\emptyset \neq \delta \subseteq \delta_{\mathfrak{K}}(R)$ and $\delta(1) = \{e(1) \mid e \in \delta\}$. To see this, choose $f \in \delta$. Let $X = \delta_{\mathfrak{K}}(R) \setminus \{e \mid e \in \delta \text{ and } e \neq f\}$. Then $\{X\} \cup \{\{e\} \mid e \in \delta \text{ and } e \neq f\}$ is a partition of $\delta_{\mathfrak{K}}(R)$. Let ρ be the equivalence relation induced on $\delta_{\mathfrak{K}}(R)$ by this partition and take $\delta^{\rho}_{\mathfrak{K}}(R)(1) = \delta(1)$. Then $S = R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$. (See Definition 2.14 for other useful equivalence relations.)

In our first result of this section we use several well-known theorems to illustrate Definitions 2.1 and 2.2, and Problem I from Section 1.

Recall that a ring is called *right duo* if every right ideal is an ideal.

Proposition 2.3.

(i) *Let $\mathfrak{A}$ be the class of semisimple Artinian rings and R a right nonsingular ring with finite right uniform dimension. Then $Q_{\mathfrak{A}}(R) = Q(R)$.*
(ii) *If $Q(R) = E(R_R)$, then $Q_{\mathfrak{SI}}(R) = Q(R) = R(\mathfrak{SI}, \delta_{\mathfrak{SI}}, Q(R))$.*
(iii) *If $Q(R) = E(R_R)$, then $Q_{\mathfrak{qCon}}(R) = \langle R \cup \mathbf{I}(Q(R))\rangle_{Q(R)} = R(\mathfrak{qCon}, \delta_{\mathfrak{qCon}}, Q(R))$.*
(iv) *If R is a commutative semiprime ring, then $Q_{\mathfrak{B}}(R) = \langle R \cup \mathbf{I}(Q(R))\rangle_{Q(R)} = Q_{\mathfrak{qCon}}(R)$.*
(v) *Assume that R has finite right uniform dimension and S is a right ring of quotients of R. Then $\mathrm{Mat}_n(S) = \widetilde{Q}_{\mathfrak{B}}(\mathrm{Mat}_n(R))$ for all positive integers n if and only if S is a right and left semihereditary right ring hull of R.*
(vi) *If R is a right Ore domain, then R has a right duo absolute right ring hull.*

Proof. (i) By Gabriel's theorem [27, p. 378], $Q(R) \in \mathfrak{A}$. Let T be a right essential overring of R such that $T \in \mathfrak{A}$. Since $Z(R_R) = 0$, T is a subring of $Q(R) = E(R_R)$. Hence $T = Q(T) = Q(R)$. Therefore $Q_{\mathfrak{A}}(R) = Q(R)$.

(ii) From Johnson's theorem [27, p. 376], $Q(R) = E(R_R) \in \mathfrak{S}\mathfrak{I}$. Let T be a right essential overring of R such that $T \in \mathfrak{S}\mathfrak{I}$. Then T is a right ring of quotients of R. By an argument similar to that in [28, p. 95, Proposition 2], $E(R_R) = E(T_T) = T$. Hence $Q_{\mathfrak{S}\mathfrak{I}}(R) = Q(R)$.

(iii) This part follows from [20, Theorem 1.1] and [28, pp. 94–95].

(iv) Part (iii) and [32, Proposition 2.5] yield this part.

(v) This follows from [30, pp. 233–235] and [38].

(vi) Note that $Q(R)$ is right duo since it is a division ring. Let S be the intersection of all right duo right rings of quotients of R. Assume that T and V are right duo right rings of quotients of R. Suppose $s, x \in S$ with $x \neq 0$. Then there exist $t \in T$ and $u \in V$ such that $sx = xt = xu$. Hence $x(t-u) = 0$, so $t = u$. Thus t (or u) $\in T \cap V$. Since T and V are arbitrary right duo right rings of quotients of R, $t \in S$. So $sx = xt \in xS$. Therefore S is the right duo absolute right ring hull of R. □

For Proposition 2.3(vi), the next example is that of a right Ore domain R which is *not* right duo, but it has a right duo absolute right ring hull properly between R and $Q(R)$.

Example 2.4. Take $A = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$, the integer quaternions. Let $P = 5\mathbb{Z}$ and $\widehat{\mathbb{Z}}_P$ the P-adic completion of $\mathbb{Z}$. Also let $R = \widehat{\mathbb{Z}}_P + \widehat{\mathbb{Z}}_P i + \widehat{\mathbb{Z}}_P j + \widehat{\mathbb{Z}}_P k$. Then R is a right Ore domain. Note that R is not right duo because $(3+i)R$ is not a left ideal. Take $\lambda = (1/2)(1+i+j+k) \in Q(A) = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$. Let $S = A + \lambda A$. Then by [35, p. 131, Exercise 2] S is a maximal $\mathbb{Z}$-order in $Q(A)$. Thus the P-adic completion $\widehat{S}_P = \widehat{\mathbb{Z}}_P \otimes_{\mathbb{Z}} S$ of S is a maximal $\widehat{\mathbb{Z}}_P$-order in $Q(R) = Q(\widehat{\mathbb{Z}}_P) \otimes_{\mathbb{Q}} Q(A)$ by [35, p. 134, Corollary 11.6]. Since $\widehat{\mathbb{Z}}_P$ is a complete discrete valuation ring and $Q(R)$ is a division ring, $\widehat{S}_P$ is the unique maximal $\widehat{\mathbb{Z}}_P$-order in $Q(R)$, thus $\widehat{S}_P$ is right duo by [35, p. 139, Theorem 13.2]. So $\widehat{S}_P$ is a proper intermediate right duo ring between R and $Q(R)$. Thus, by Proposition 2.3(vi), there exists a right duo absolute right ring hull properly between R and $Q(R)$.

Lemma 2.5. *Let T be a right essential overring of R. Then $Q(R) \cap T$ is a subring of $Q(R)$ and of T (i.e., the ring multiplications of $Q(R)$ and of T coincide on $Q(R) \cap T$).*

Proof. Since $Q(R)$ and T have the same addition, $Q(R) \cap T$ is an additive subgroup of $Q(R)$ and T. Let $\cdot$ and $*$ denote the ring multiplications of $Q(R)$ and T, respectively. Take $t \in T$. Define $f_t : T \to E(R_R)$ by $f_t(x) = t * x$ for $x \in T$. Let juxtaposition denote scalar multiplication (by R or $Q(R)$). Hence f_t is an R-homomorphism and extends to an element $\bar{f}_t$ of $\mathcal{E}_R$. By [28, p. 95, Proposition 2], $\bar{f}_t$ is a $Q(R)$-homomorphism. Now for $q_1, q_2 \in Q(R) \cap T$, $q_1 * q_2 = f_{q_1}(q_2) = \bar{f}_{q_1}(1)q_2 = (q_1 * 1)q_2 = (q_1)q_2 = q_1 \cdot q_2$. □

From Lemma 2.5, we have that if $\delta_{\mathfrak{K}}(R)(1) \subseteq Q(R)$ and S is a right essential overring of R such that $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$ exists, then $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S) = R(\mathfrak{K}, \delta_{\mathfrak{K}}, Q(R))$.

Let $\mathfrak{U}$ denote the class $\{R \mid R \cap \mathbf{U}(Q(R)) = \mathbf{U}(R)\}$ of rings.

Lemma 2.6.

(i) *$T \in \mathfrak{U}$ if and only if $T = \langle T \cup \{q \in \mathbf{U}(Q(T)) \mid q^{-1} \in T\}\rangle_{Q(T)}$.*
(ii) *$\langle R \cup \{q \in \mathbf{U}(Q(R)) \mid q^{-1} \in T\}\rangle_{Q(R)} \subseteq T$ for $T \in \mathfrak{U}$ and $R \subseteq T \subseteq Q(R)$.*

Proof. The proof is straightforward. □

Recall from [21] that R is called *directly finite* if one-sided inverses of R are two-sided. Note that if R has finite right uniform dimension, or if R satisfies the condition that $r_R(x) = 0$ implies $\ell_R(x) = 0$, or if R is Abelian, then R is directly finite.

For our next result, let $i < j$ be ordinal numbers. We define $R_1 = \langle R \cup \{q \in \mathbf{U}(Q(R)) \mid q^{-1} \in R\}\rangle_{Q(R)}$, $R_j = R_i \cup \{q \in \mathbf{U}(Q(R)) \mid q^{-1} \in R_i\rangle_{Q(R)}$ for $j = i + 1$, and $R_j = \bigcup_{i<j} R_i$ for j a limit ordinal. The following theorem characterizes $Q^r_{c\ell}(R)$ as a $\mathfrak{U}$ absolute to $Q(R)$ right ring hull.

Theorem 2.7.

(i) *$\widehat{Q}_{\mathfrak{U}}(R)$ exists and $\widehat{Q}_{\mathfrak{U}}(R) = R_j$ for any j with $|j| > |Q(R)|$.*
(ii) *Assume that T is a directly finite right essential overring of R and T_T satisfies* (C_2). *Then $\widehat{Q}_{\mathfrak{U}}(R)$ is a subring of T.*
(iii) *If R is a right Ore ring, then $\widehat{Q}_{\mathfrak{U}}(R) = Q^r_{c\ell}(R)$.*

Proof. (i) Using Lemma 2.6, it can be seen that the intersection of all $T \in \mathfrak{U}$ which are intermediate rings between R and $Q(R)$ is again in $\mathfrak{U}$. Thus $\widehat{Q}_{\mathfrak{U}}(R)$ exists. By transfinite induction $\widehat{Q}_{\mathfrak{U}}(R) = R_j$ for any j with $|j| > |Q(R)|$.

(ii) Take $q \in Q(R)$ such that $q^{-1} = x \in R$. Define $f : T \to xT$ by $f(t) = xt$ for $t \in T$. Clearly, f is a T-epimorphism. Note that $r_R(x) = 0$. Thus $r_T(x) = 0$ since $R_R \leqslant^{\text{ess}} T_R$. Therefore f is a T-monomorphism, hence $T_T \cong xT_T$. So xT_T is a direct summand of T_T by the (C_2) property of T_T. Thus there exists $e \in \mathbf{I}(T)$ with $xT = eT$. Since T is directly finite, $xT = eT = T$. So x is right invertible in T. Hence x is invertible in T.

Let $*$ denote the multiplication of T and $\cdot$ the multiplication of $Q(R)$. Let $v \in T$ be the inverse of x in T. We *claim* that $v = q$. To see this, assume to the contrary that $v - q \neq 0$. Then there exists $r \in R$ such that $0 \neq (v - q)r \in R$. Say $r_1 = (v - q)r = v * r - q \cdot r$. So $v * r = r_1 + q \cdot r \in Q(R) \cap T$. Similarly, $q \cdot r \in Q(R) \cap T$. Since $r_R(x) = 0$, $xr_1 \neq 0$. Now by Lemma 2.5, $0 \neq xr_1 = x(v - q)r = x(v * r - q \cdot r) = x * (v * r) - x \cdot (q \cdot r) = (x * v) * r - (x \cdot q) \cdot r = r - r = 0$, a contradiction. Hence $q = v \in T$. Thus the ring $\langle R \cup \{q \in \mathbf{U}(Q(R)) \mid q^{-1} \in R\}\rangle_{Q(R)}$ is a subring of T by Lemma 2.5. The result now follows from part (i).

(iii) Routine arguments using Lemma 2.6 show that $\mathbf{U}(Q(R)) \subseteq Q^r_{c\ell}(R)$, $Q^r_{c\ell}(R) \subseteq R_1$, and $Q^r_{c\ell}(R) \in \mathfrak{U}$. Thus $\widehat{Q}_{\mathfrak{U}}(R) = Q^r_{c\ell}(R)$. □

Lemma 2.8.

(i) *Let R be a right Ore ring such that $r_R(x) = 0$ implies $\ell_R(x) = 0$. If $e \in \mathbf{B}(Q^r_{c\ell}(R))$ and $h : eQ^r_{c\ell}(R) \to Y \subseteq Q^r_{c\ell}(R)$ is a $Q^r_{c\ell}(R)$-isomorphism, then $eQ^r_{c\ell}(R) = Y$.*
(ii) *If R is Abelian and right extending, then $Q(R)$ is Abelian and right extending.*

Proof. (i) There exists $y \in Y$ such that $h(e) = y = ye = ey$. Hence $Y = yQ^r_{c\ell}(R) \subseteq eQ^r_{c\ell}(R)$. Thus there are $s, t, c, d \in R$ with $y = st^{-1}$ and $1 - e = cd^{-1}$. Then $s = yt \in Y$ and $(1 - e)d = c \in R$. Let $a \in r_R(s + (1 - e)d)$. So $(s + (1 - e)d)a = sa + (1 - e)da = 0$. Hence $sa \in eQ^r_{c\ell}(R) \cap (1 - e)Q^r_{c\ell}(R) = 0$. Thus $0 = sa = yta = h(e)ta = h(eta)$. Hence $eta = 0$. Then $ta = (1 - e)ta$, so $a = t^{-1}(1 - e)ta = (1 - e)a$. Therefore $0 = (s + (1 - e)d)a = (1 - e)da = d(1 - e)a = da$. Then $a = d^{-1}0 = 0$. Hence there is $u \in Q^r_{c\ell}(R)$ such that $(s + (1 - e)d)u = 1$. Thus $e = sue + (1 - e)de = sue \in Y$. Therefore $eQ^r_{c\ell}(R) = Y$.

(ii) By Corollary 1.8(ii), $Q(R)$ is right extending. Let $e \in \mathbf{I}(Q(R))$ and $X = R \cap eQ(R)$. Then there exists $c \in \mathbf{I}(R)$ such that $X_R \leqslant^{\text{ess}} cR_R$. So $c \in \mathbf{B}(R) \subseteq \mathbf{B}(Q(R))$. Hence $X_R \leqslant^{\text{ess}} eQ(R) \cap cQ(R) = ecQ(R)$, so $eQ(R) = ecQ(R) = ceQ(R) = cQ(R)$. Thus $e = c \in \mathbf{B}(Q(R))$. □

The next few results are inspired by the work on continuous module hulls in [33] or [36].

Proposition 2.9. *Assume that R is a right Ore right ring such that $r_R(x) = 0$ implies $\ell_R(x) = 0$ for $x \in R$. If $Q^r_{c\ell}(R)$ is Abelian and right extending, then $\widehat{Q}_{\mathfrak{Con}}(R) = Q^r_{c\ell}(R)$.*

Proof. By Lemma 2.8(i), $Q^r_{c\ell}(R)$ has the (C_2) property. So $Q^r_{c\ell}(R)$ is right continuous. Since $Q(R) = Q(Q^r_{c\ell}(R))$, $Q(R)$ is Abelian by Lemma 2.8(ii). Hence every intermediate ring between R and $Q(R)$ is directly finite. So, by Theorem 2.7, $\widehat{Q}_{\mathfrak{Con}}(R) = Q^r_{c\ell}(R)$. □

Corollary 2.10. *Let R be a right Ore ring. If at least one of the following conditions is satisfied, then $\widehat{Q}_{\mathfrak{Con}}(R) = Q^r_{c\ell}(R)$.*

(i) *R is Abelian, right extending, and $r_R(x) = 0$ implies $\ell_R(x) = 0$.*
(ii) *R is right uniform and $r_R(x) = 0$ implies $\ell_R(x) = 0$.*
(iii) *R is Abelian, right extending, and $Z(R_R) = 0$.*

Proof. Note that if R is right uniform, then R is Abelian and right extending. Also note that a reduced ring satisfies the condition $r_R(x) = 0$ implies $\ell_R(x) = 0$. Thus the result is a consequence of Lemma 2.8 and Proposition 2.9. □

The following theorem is an adaptation of [36, Theorem 4.25].

Theorem 2.11. *Let R be a right nonsingular ring and S the intersection of all right continuous right rings of quotients of R. Then $Q_{\mathfrak{Con}}(R) = S$.*

Proof. From [40, Theorems 2 and 4], $Q(R) = A \oplus B$ (ring direct sum), where $A = eQ(R)$ is a strongly regular right self-injective ring, $e \in \mathbf{B}(Q(R))$, and B is a regular right self-injective ring generated by idempotents. By Proposition 2.3(iii), $e \in Q_{\mathfrak{qCon}}(R)$ and $B \subseteq Q_{\mathfrak{qCon}}(R)$. Let T be a right continuous right ring of quotients of R. Then $Q_{\mathfrak{qCon}}(R)$ is a subring of T and $T = eT \oplus B$. From [41, Lemma 4.1], T is regular since T is right nonsingular. Hence eT is strongly regular. Then $S = eS \oplus B$ and $eS = \bigcap eT$, where T is a right continuous right ring of quotients of R. Since $Q_{\mathfrak{qCon}}(R)$ is a subring of S, $S \in \mathfrak{E}$ by Corollary 1.8(ii). From [22, Proposition 3.6 and Corollary 13.4], $eS \in \mathfrak{Con}$. Since B is right self-injective, $S = eS \oplus B \in \mathfrak{Con}$. Therefore $Q_{\mathfrak{Con}}(R) = S$. □

Theorem 2.12. *Let R be a ring such that $Q(R)$ is Abelian.*

(i) *$Q(R) \in \mathfrak{E}$ if and only if $\widehat{Q}_{\mathfrak{E}}(R) = \widehat{Q}_{\mathfrak{qCon}}(R) = R\mathbf{B}(Q(R))$.*
(ii) *Assume that R is a right Ore ring such that $r_R(x) = 0$ implies $\ell_R(x) = 0$ for $x \in R$ and $Z(R_R)$ has finite right uniform dimension. Then $Q(R) \in \mathfrak{E}$ if and only if $\widehat{Q}_{\mathfrak{Con}}(R)$ exists and $\widehat{Q}_{\mathfrak{Con}}(R) = H_1 \oplus H_2$ (ring direct sum), where H_1 is a right continuous strongly regular ring and H_2 is a direct sum of right continuous local rings.*

Proof. (i) Assume that $Q(R) \in \mathfrak{E}$. Let $X_R \leqslant R\mathbf{B}(Q(R))_R$. Since $Q(R) \in \mathfrak{E}$, there exists $e \in \mathbf{I}(Q(R))$ such that $XQ(R)_{Q(R)} \leqslant^{\text{ess}} eQ(R)_{Q(R)}$. Thus $XQ(R)_R \leqslant^{\text{ess}} eQ(R)_R$ by Lemma 1.4(i). So $X_R \leqslant^{\text{ess}} XQ(R)_R \leqslant^{\text{ess}} eQ(R)_R$. Since $e \in R\mathbf{B}(Q(R))$, $R\mathbf{B}(Q(R))_R$ is extending. From Proposition 1.5(ii), $R\mathbf{B}(Q(R)) \in \mathfrak{E}$. Assume that S is a right extending right ring of quotients of R and $b \in \mathbf{B}(Q(R))$. Let $Y = S \cap bQ(R)$. Then $Y_R \leqslant^{\text{ess}} bQ(R)_R$ and there is $c \in \mathbf{I}(S)$ with $Y_S \leqslant^{\text{ess}} cS_S$. Thus $Y_R \leqslant^{\text{ess}} cS_R$ by Lemma 1.4(i). Therefore $Y_R \leqslant^{\text{ess}} bQ(R)_R \cap cQ(R)_R$. Since $b, c \in \mathbf{B}(Q(R))$, then $b = c$. Hence $R\mathbf{B}(Q(R))$ is a subring of S. Thus $R\mathbf{B}(Q(R)) = \widehat{Q}_{\mathfrak{E}}(R)$. Since $Q(R)$ is Abelian, every right extending right ring of quotients is right quasi-continuous. Therefore $\widehat{Q}_{\mathfrak{E}}(R) = \widehat{Q}_{\mathfrak{qCon}}(R)$. The converse follows from Corollary 1.8(ii).

(ii) Assume that $Q(R) \in \mathfrak{E}$. Since $Q(R)$ is Abelian and right extending, $Q(R) = Q_1 \oplus Q_2$ (ring direct sum), where Q_1 is right nonsingular and $Z(Q(R)_{Q(R)})_{Q(R)} \leqslant^{\text{ess}} Q_{2\,Q(R)}$. Now there are $e, f \in \mathbf{B}(Q(R))$ with $Q_1 = eQ(R)$ and $Q_2 = fQ(R)$. A routine argument yields that $Q_1 = Q(eRe)$. Let H_1 be the intersection of all right extending regular subrings of Q_1 which contain eRe. By Theorem 2.11, $H_1 = Q_{\mathfrak{Con}}(eRe)$. Since H_1 is Abelian, it is strongly regular.

Note that $Z(R_R)_R \leqslant^{\text{ess}} Z(Q(R)_{Q(R)})_R$. Hence Q_2 has finite right uniform dimension. Thus there exists a complete set of primitive idempotents $\{f_1, \dots, f_n\}$ for Q_2. So $f = f_1 + \cdots + f_n$. In this case, $Q_2 = Q(fRf)$ and $f_iQ_2 = Q(f_iRf_i)$ for each i. Since $Q_2 \in \mathfrak{E}$, each $f_iQ_2 \in \mathfrak{E}$ for each i. Therefore $f_iQ_{2\,f_iR}$ is extending for each i by Proposition 1.5. We show that $f_iR_{f_iR}$ is uniform. For this, take $0 \neq I_{f_iR} \leqslant f_iR_{f_iR}$. Then $I_{f_iR} \leqslant^{\text{ess}} J_{f_iR}$ for a direct summand J_{f_iR} of $f_iQ_{2\,f_iR}$ because $f_iQ_{2\,f_iR}$ is extending. Thus there exists $g_i = g_i^2 \in \text{End}(f_iQ_{2\,f_iR})$ such that $J_{f_iR} = g_i(f_iQ_2)_{f_iR}$. Note that $\text{End}(f_iQ_{2\,f_iR}) = \text{End}(f_iQ_{2\,f_iQ_2}) = f_iQ_2$ from [28, p. 95]. Since f_iQ_2 is right uniform, $g_i = f_i$, so $J_{f_iR} = f_iQ_{2\,f_iR}$. Thus $I_{f_iR} \leqslant^{\text{ess}} f_iQ_{2\,f_iR}$, so $I_{f_iR} \leqslant^{\text{ess}} f_iR_{f_iR}$. Hence f_iR is a right uniform ring. Also each f_iR is a right Ore ring. By Corollary 2.10(ii), $Q^r_{c\ell}(f_iR) = Q_{\mathfrak{Con}}(f_iR)$ for each i. Let $H_2 = Q^r_{c\ell}(f_1R) \oplus \cdots \oplus Q^r_{c\ell}(f_nR)$.

We *claim* that $H_2 = \widehat{Q}_{\mathfrak{Con}}(fRf)$. To see this, let T be a right continuous ring of quotients of fRf. Since Q_2 is Abelian and right extending, $(fRf)\mathbf{B}(Q_2) \subseteq T$ by part (i). So $f_1, f_2, \dots, f_n$ are in T. Now f_iT is a right continuous right ring of quotients of f_iR. Thus $Q^r_{c\ell}(f_iR)$ is a subring of f_iT. Hence H_2 is a subring of $f_1T \oplus \cdots \oplus f_nT = T$. Therefore $H_2 = \widehat{Q}_{\mathfrak{Con}}(fRf)$. Since a right uniform right continuous ring is local, H_2 is a direct sum of local rings.

Let V be any right continuous right ring of quotients of R. By part (i), $\mathbf{B}(Q(R)) \subseteq V$. Hence $V = eV \oplus fV$ (ring direct sum). Thus H_1 is a subring of eU and H_2 is a subring of fV. Therefore $H_1 \oplus H_2 = \widehat{Q}_{\mathfrak{Con}}(R)$. From Corollary 1.8(ii), we obtain the converse. □

For commutative rings, the preceding results yield the following corollary which is related to [33, Corollaries 3 and 7].

Corollary 2.13. *Let R be a commutative ring.*

(i) *If R or $Q^r_{c\ell}(R)$ is extending, then $\widehat{Q}_{\mathfrak{Con}}(R) = Q^r_{c\ell}(R)$.*
(ii) *If R is uniform, then $Q_{\mathfrak{Con}}(R) = Q^r_{c\ell}(R)$ and is also a local ring.*
(iii) *If $Z(R_R) = 0$, then $Q_{\mathfrak{Con}}(R) = \bigcap\{T \mid \mathbf{B}(Q(R)) \subseteq T$ and T is a regular right ring of quotients of $R\}$.*
(iv) *Assume that $Z(R_R)$ has finite uniform dimension. Then $Q(R)$ is right extending if and only if $\widehat{Q}_{\mathfrak{Con}}(R)$ exists and $\widehat{Q}_{\mathfrak{Con}}(R) = H_1 \oplus H_2$ (ring direct sum), where H_1 is a continuous regular ring and H_2 is a direct sum of continuous local rings.*

We observe that in Corollary 2.13(i), the hypothesis "R or $Q^r_{c\ell}(R)$ is extending" is not superfluous. Let T be a countably infinite direct product of copies of a field F. Take $R = \langle \bigoplus_{i=1}^{\infty} F_i \cup \{1\}\rangle_T$. Then $Q^r_{c\ell}(R)$ is the subring of T whose elements are eventually constant. It can be seen that neither R nor $Q^r_{c\ell}(R)$ is extending. Hence $Q^r_{c\ell}(R)$ is not continuous. Also, in general, R may not satisfy the (C_2) property (e.g., take $F = \mathbb{Q}$); but $Q^r_{c\ell}(R)$ does satisfy the (C_2) property since it is regular.

To develop the theory of pseudo hulls for $\mathfrak{D}$-$\mathfrak{E}$ classes $\mathfrak{C}$, we fix $\mathfrak{D}_{\mathfrak{C}}(R)$ for each ring R and define

$$\delta_{\mathfrak{C}}(R) = \bigl\{e \in \mathbf{I}(\mathcal{E}_R) \bigm| X_R \leqslant^{\text{ess}} eE(R_R) \text{ for some } X \in \mathfrak{D}_{\mathfrak{C}}(R)\bigr\}.$$

To find a right essential overring S of R such that $S \in \mathfrak{C}$, one might naturally look for a right essential overring T of R with $\delta_{\mathfrak{C}}(R)(1) \subseteq T$. Then take $S = \langle R \cup \delta_{\mathfrak{C}}(R)(1)\rangle_T$. Indeed, under some mild conditions, this choice of S is in $\mathfrak{C}$. However, in order to obtain a right essential overring with some hull-like behavior, we need to determine subsets Λ of $\delta_{\mathfrak{C}}(R)(1)$ for which $\langle R \cup \Lambda\rangle_T \in \mathfrak{C}$ in some minimal sense. Moreover, to facilitate the transfer of information between R and $\langle R \cup \Lambda\rangle_T$, one would want to include in Λ enough of $\delta_{\mathfrak{C}}(R)(1)$ so that for all (or almost all) $X \in \mathfrak{D}_{\mathfrak{C}}(R)$ there is $e \in \delta_{\mathfrak{C}}(R)$ with $X_R \leqslant^{\text{ess}} e(1) \cdot \langle R \cup \Lambda\rangle_T$ and $e(1) \in \Lambda$. To accomplish this, we use equivalence relations on $\delta_{\mathfrak{C}}(R)$.

Since we have fixed the $\delta_{\mathfrak{C}}$ assignment for all $\mathfrak{D}$-$\mathfrak{E}$ classes $\mathfrak{C}$, we will use the terminology $\mathfrak{C}$ (respectively, $\mathfrak{C}\rho$) pseudo right ring hull for $\delta_{\mathfrak{C}}$ pseudo right ring hull and use $R(\mathfrak{C}, S)$ (respectively, $R(\mathfrak{C}, \rho, S)$) for $R(\mathfrak{C}, \delta_{\mathfrak{C}}, S)$ (respectively, $R(\mathfrak{C}, \delta_{\mathfrak{C}}, \rho, S)$).

The next two equivalence relations are particularly important to our study.

Definition 2.14.

(i) Let A be a ring and let $\delta \subseteq \mathbf{I}(A)$. We define an equivalence relation α on δ by $e\ \alpha\ c$ if and only if $ce = e$ and $ec = c$.
(ii) We define an equivalence relation β on $\delta_{\mathfrak{C}}(R)$ by $e\ \beta\ c$ if and only if there exists $X_R \leqslant R_R$ such that $X_R \leqslant^{\text{ess}} eE(R_R)$ and $X_R \leqslant^{\text{ess}} cE(R_R)$.

Note that for $e, c \in \delta_{\mathfrak{C}}(R)$, $e\ \alpha\ c$ implies $e\ \beta\ c$. Also note that $\alpha = \beta$ if and only if every element of $\mathfrak{D}_{\mathfrak{C}}(R)$ has a unique essential closure in $E(R_R)$. So if $Z(R_R) = 0$, then $\alpha = \beta$.

The following example indicates the independence of Definitions 2.1 and 2.2 for $\mathfrak{D}$-$\mathfrak{E}$ classes. Hence these definitions provide distinct tools for investigating a ring and its right essential overrings. Recall from [27, Corollary 8.28], a ring R is right *Kasch* if the left annihilator of every maximal right ideal of R is nonzero.

Example 2.15. [39, 1.1] For a field F, let $T = F[x]/x^4F[x]$ and $\bar{x}$ be the canonical image of x in T. Then $T = F + F\bar{x} + F\bar{x}^2 + F\bar{x}^3$. Let $R = F + F\bar{x}^2 + F\bar{x}^3$ which is a subring of T. Now R and T have the following properties.

(i) R is right Kasch, so $R = Q(R)$ by [27, Corollary 13.24].
(ii) T is a QF right essential overring of R. There is no proper intermediate ring between R and T. Hence $T = \widetilde{Q}_{\mathfrak{C}}(R) = \widetilde{Q}_{\mathfrak{S}\mathfrak{I}}(R)$.
(iii) T is not a $\mathfrak{C}\ \rho$ pseudo right ring hull of R for any choice of $\mathfrak{C}$ and any equivalence relation ρ on $\delta_{\mathfrak{C}}(R)$. Indeed, there is no $c \in \delta_{\mathfrak{C}}(R)$ such that $c(1) \in T \setminus R$ and $I_R \leqslant^{\text{ess}} cE(R_R)$ for any nonzero ideal I of R.

(iv) T_R is not FI-extending (hence not extending). In fact, $\bar{x}^3 R_R \trianglelefteq T_R$. But there does not exist $e \in \mathbf{I}(\mathrm{End}(T_R))$ such that $\bar{x}^3 R_R \leqslant^{\mathrm{ess}} eT_R$. Thus the hypothesis in Proposition 1.5 that T is a right ring of quotients of R is necessary.
(v) Since T_T is injective, T is maximal among right extending right essential overrings of R.
(vi) By [29, Theorem 4], $E(R_R)$ has no ring multiplication which extends its R-module scalar multiplication.

By Example 2.15, there is a ring R with a right extending right ring hull which is not a right extending ρ pseudo right ring hull for any choice of ρ. However, there are $\mathfrak{D}$-$\mathfrak{E}$ classes of rings for which $R(\mathfrak{C}, Q(R))$ always exists (see [12, Theorem 2.8]).

Lemma 2.16. *Let T be a right essential overring of R.*

(i) *For $e \in \mathbf{I}(T)$, there exists $c \in \mathbf{I}(\mathcal{E}_R)$ such that $c|_T \in \mathrm{End}(T_T)$ and $c(1) = e$.*
(ii) *For $b \in \mathbf{I}(\mathcal{E}_R)$, if $b|_T \in \mathrm{End}(T_T)$, then $b(1) \in \mathbf{I}(T)$.*
(iii) *For $b \in \mathbf{I}(\mathcal{E}_R)$, if $b(1) \in Q(R)$, then $b(1) \in \mathbf{I}(Q(R))$.*

Proof. (i) First we see that $E(T_R) = E(eT_R) \oplus E((1-e)T_R)$. Let c be the projection of $E(T_R)$ onto $E(eT_R)$. Then $c(t) = c(et) + c((1-e)t) = c(et) = et$ for all $t \in T$. Hence $c(1) = e$. If $s \in T$, then $c(ts) = ets = c(t)s$. Therefore $c|_T \in \mathrm{End}(T_T)$.

(ii) Note that $b(1) = b(b(1)) = b(1b(1)) = b(1)b(1)$. Thus $b(1) \in \mathbf{I}(T)$.

(iii) From [28, p. 95], each element of $\mathcal{E}_R$ is a $Q(R)$-homomorphism. Thus if $b(1) \in Q(R)$, then we can see that $b(1) \in \mathbf{I}(Q(R))$ as in the proof of part (ii). □

The following result may seem somewhat technical, however its usefulness is demonstrated by its application in many of the remaining results of this paper.

Theorem 2.17. *Assume that $\mathfrak{C}$ is a $\mathfrak{D}$-$\mathfrak{E}$ class of rings. Let T be a right essential overring of R, $\delta \subseteq \delta_{\mathfrak{C}}(R)$ with $\delta(1) = \{c(1) \mid c \in \delta\} \subseteq T$, and set $S = \langle R \cup \delta(1)\rangle_T$. Suppose that $\mathfrak{D}_{\mathfrak{C}}(S \to R)$ holds.*

(i) *Let $\delta = \delta_{\mathfrak{C}}(R)$ or some $\delta^{\alpha}_{\mathfrak{C}}(R)$, respectively. Assume that $c|_S \in \mathrm{End}(S_S)$ for each $c \in \delta$. Then $S = R(\mathfrak{C}, T)$ or $R(\mathfrak{C}, \alpha, T)$, respectively.*
(ii) *Let $\delta = \delta_{\mathfrak{C}}(R)$, some $\delta^{\alpha}_{\mathfrak{C}}(R)$, or some $\delta^{\beta}_{\mathfrak{C}}(R)$, respectively. Assume that $\delta(1) \subseteq \mathbf{I}(T)$ and that either $\alpha = \beta$ or each $Y \in \mathfrak{D}_{\mathfrak{C}}(S)$ satisfies the condition that $(Y \cap fE(R_R))_R \leqslant^{\mathrm{ess}} Y_R$ for some $f \in \delta$ implies $f(1)\cdot Y \subseteq Y$. Then $S = R(\mathfrak{C}, T)$, $R(\mathfrak{C}, \alpha, T)$, or $R(\mathfrak{C}, \beta, T)$, respectively.*

Proof. (i) Since S is a right essential overring of R and $c|_S \in \mathrm{End}(S_S)$ for all $c \in \delta$, $\delta(1) \subseteq \mathbf{I}(S)$ by Lemma 2.16(ii). To show that $S = R(\mathfrak{C}, T)$ or $R(\mathfrak{C}, \alpha, T)$, respectively, let $Y \in \mathfrak{D}_{\mathfrak{C}}(S)$. Since $\mathfrak{D}_{\mathfrak{C}}(S \to R)$ holds, there is $X \in \mathfrak{D}_{\mathfrak{C}}(R)$ with $X_R \leqslant^{\mathrm{ess}} Y_R$. Thus $X_R \leqslant^{\mathrm{ess}} Y_R \leqslant^{\mathrm{ess}} E(Y_R) = eE(R_R)$ for some $e \in \mathbf{I}(\mathcal{E}_R)$. Hence $e \in \delta_{\mathfrak{C}}(R)$, so there exists $c \in \delta^{\alpha}_{\mathfrak{C}}(R)$ satisfying $eE(R_R) = cE(R_R)$. Hence $Y_R \leqslant^{\mathrm{ess}} c(S)_R = c(1)S_R$ with $c(1) \in \mathbf{I}(S)$ because $c|_S \in \mathrm{End}(S_S)$. Therefore $S = R(\mathfrak{C}, T)$ or $R(\mathfrak{C}, \alpha, T)$, respectively.

(ii) Assume that $\delta(1) \subseteq \mathbf{I}(T)$. We prove the case for $\delta = \delta^{\beta}_{\mathfrak{C}}(R)$; the other cases are similar.

First suppose that $\alpha = \beta$. Let $Y \in \mathfrak{D}_{\mathfrak{C}}(S)$. Then as in the proof of part (i), $Y_R \leqslant^{\mathrm{ess}} eE(R_R)$ and $e \in \delta_{\mathfrak{C}}(R)$. There is $b \in \delta^{\beta}_{\mathfrak{C}}(R)$ with $e\ \beta\ b$. Hence $e\ \alpha\ b$, so $eE(R_R) = bE(R_R)$ and $b(1) \in \mathbf{I}(S) \subseteq \mathbf{I}(T)$. Lemma 2.16(i) yields $c \in \mathbf{I}(\mathcal{E}_R)$ such that $c|_T \in \mathrm{End}(T_T)$ and $c(1) = b(1)$. We show that $cE(R_R) \cap R = bE(R_R) \cap R$. To see this, let $c(x) \in cE(R_R) \cap R$ with $x \in E(R_R)$. Then $c(x) = c(c(x)) = c(1 \cdot c(x)) = c(1)c(x) = b(1)c(x) = b(1 \cdot c(x)) = b(c(x)) \in bE(R_R) \cap R$. Thus $cE(R_R) \cap R \subseteq bE(R_R) \cap R$. Similarly, $bE(R_R) \cap R \subseteq cE(R_R) \cap R$. So $cE(R_R) \cap R = bE(R_R) \cap R$. Thus $c \,\beta\, b$. Since $\alpha = \beta$, we have that $c \,\alpha\, b$, so $cE(R_R) = bE(R_R)$. Consequently,

$$Y_R \leqslant^{\text{ess}} c(S)_R = c(1) \cdot S_R.$$

Therefore $S = R(\mathfrak{C}, T)$, $R(\mathfrak{C}, \alpha, T)$, or $R(\mathfrak{C}, \beta, T)$, respectively.

Next we suppose the other condition, that is, each $Y \in \mathfrak{D}_{\mathfrak{C}}(S)$ satisfies the condition that $(Y \cap fE(R_R))_R \leqslant^{\text{ess}} Y_R$ for some $f \in \delta$ implies $f(1) \cdot Y \subseteq Y$. Now take $Y \in \mathfrak{D}_{\mathfrak{C}}(S)$. Then there exist $e \in \mathbf{I}(\mathcal{E}_R)$ and $X \in \mathfrak{D}_{\mathfrak{C}}(R)$ such that $X_R \leqslant^{\text{ess}} Y_R \leqslant^{\text{ess}} eE(R_R)$. Hence $e \in \delta_{\mathfrak{C}}(R)$, so there is $f \in \delta_{\mathfrak{C}}^{\beta}(R)$ with $f \beta e$. Thus there is $X'_R \leqslant R_R$ satisfying $X'_R \leqslant^{\text{ess}} eE(R_R)$ and $X'_R \leqslant^{\text{ess}} fE(R_R)$. So $(X \cap X')_R \leqslant^{\text{ess}} eE(R_R)$. Since $X \cap X' \subseteq Y \cap fE(R_R) \subseteq eE(R_R)$, we have that $(Y \cap fE(R_R))_R \leqslant^{\text{ess}} eE(R_R)$. Hence $(Y \cap fE(R_R))_R \leqslant^{\text{ess}} Y_R$. Therefore $f(1) \cdot Y \subseteq Y$ by the assumption.

Let $K = X \cap fE(R_R)$. Since $(Y \cap fE(R_R))_R \leqslant^{\text{ess}} Y_R$ and $X_R \leqslant^{\text{ess}} Y_R$, we have that $K_R \leqslant^{\text{ess}} Y_R$. Now $K_R \leqslant^{\text{ess}} fE(R_R)$. To see this, let $0 \neq f(t) \in fE(R_R)$ with $t \in E(R_R)$. Since $X'_R \leqslant^{\text{ess}} fE(R_R)$, there exists $r \in R$ such that $0 \neq f(t)r \in X'$. Also since $(X \cap X')_R \leqslant^{\text{ess}} eE(R_R)$, there is $a \in R$ with $0 \neq f(t)ra \in X \cap X' \subseteq X \cap fE(R_R) = K$. So $K_R \leqslant^{\text{ess}} fE(R_R)$.

Note that $K_R \leqslant R_R$ and $K = X \cap fE(R_R)$. Therefore $K = f(K) = f(1) \cdot K_R \leqslant f(1) \cdot Y_R$. Since $f(1) \cdot Y \subseteq Y$ and $f(1) \in \mathbf{I}(S)$, we have that $f(1) \cdot Y_R$ is a direct summand of Y_R. Also note that $K_R \leqslant^{\text{ess}} Y_R$ and $K_R \leqslant f(1) \cdot Y_R \leqslant Y_R$. Thus $f(1) \cdot Y = Y$. Now

$$Y_R = f(1) \cdot Y_R \leqslant^{\text{ess}} f(1) \cdot S_R.$$

To prove this, note that $K_R \leqslant^{\text{ess}} f(R_R) \leqslant fE(R_R)$ and $f(R_R) = f(1) \cdot R_R \leqslant^{\text{ess}} f(1) \cdot S_R$. So $K_R \leqslant^{\text{ess}} f(1) \cdot S_R$. Therefore $Y_R = f(1) \cdot Y_R \leqslant^{\text{ess}} f(1) \cdot S_R$ because $K_R \leqslant Y_R$.

From $f \in \delta_{\mathfrak{C}}^{\beta}(R)$ and $f(1) \in \mathbf{I}(S)$, it follows that $S = R(\mathfrak{C}, \beta, T)$. $\square$

If, in Theorem 2.17(i), T is a right ring of quotients of R, then $R(\mathfrak{C}, T) = R(\mathfrak{q}\mathfrak{Con}, T)$. In general, if $R \in \mathfrak{C}$ it is not necessarily true that R is the $\mathfrak{C}$ pseudo right ring hull of R itself. For example, let $R = T_2(F)$, where F is a field. Then $R \in \mathfrak{C}$, so $R = Q_{\mathfrak{C}}(R)$ but $R(\mathfrak{C}, Q(R)) = \mathrm{Mat}_2(F) = Q_{\mathfrak{q}\mathfrak{Con}}(R)$.

Corollary 2.18. *Assume that $\mathfrak{C}$ is a $\mathfrak{D}$-$\mathfrak{E}$ class of rings. Let T be a right essential overring of R, $\delta \subseteq \delta_{\mathfrak{C}}(R)$ such that $\delta(1) \subseteq T$, and take $S = \langle R \cup \delta(1)\rangle_T$. Suppose that $\mathfrak{D}_{\mathfrak{C}}(S \to R)$ holds.*

- (i) *Let δ be some $\delta_{\mathfrak{C}}^{\alpha}(R)$. If $\delta(1) \subseteq Q(R)$, then $S = R(\mathfrak{C}, \alpha, T)$. If T is an intermediate ring between S and $Q(R)$ such that $\mathfrak{D}_{\mathfrak{C}}(T \# R)$ holds, then $T \in \mathfrak{C}$.*
- (ii) *Let δ be some $\delta_{\mathfrak{C}}^{\beta}(R)$ and $\delta(1) \subseteq \mathbf{I}(T)$ (e.g., if $\delta(1) \subseteq Q(R)$). If either $\alpha = \beta$, $\delta(1) \subseteq \mathrm{Cen}(T)$, or $\mathfrak{D}_{\mathfrak{C}}(S) \subseteq \{Y \mid Y \trianglelefteqslant S\}$, then $S = R(\mathfrak{C}, \beta, T)$.*

Proof. (i) If $\delta(1) \subseteq Q(R)$, then by Lemma 2.16(iii), $\delta(1) \subseteq \mathbf{I}(S)$ because S is a subring of both T and $Q(R)$ by Lemma 2.5. Thus by Theorem 2.17(i), $S = R(\mathfrak{C}, \alpha, T)$. The remainder of part (i) follows from Theorem 1.7(ii).

(ii) The condition either $\delta(1) \subseteq \mathrm{Cen}(T)$ or the condition $\mathfrak{D}_{\mathfrak{C}}(S) \subseteq \{Y \mid Y \trianglelefteqslant S\}$ ensures that the condition "$f(1) \cdot Y \subseteq Y$" in Theorem 2.17(ii) is satisfied. Therefore, by Theorem 2.17(ii), $S = R(\mathfrak{C}, \beta, T)$. $\square$

Lemma 2.19. *Let $c \in \mathbf{I}(\mathcal{E}_R)$. Then we have the following.*

(i) *If $c \in \mathbf{B}(\mathcal{E}_R)$, then $[c]_\alpha = [c]_\beta = \{c\}$ and $c(1) \in \mathbf{B}(Q(R))$.*
(ii) *If T is a right ring of quotients of R with $c(1) \in \mathbf{B}(T)$, then $[c]_\alpha = [c]_\beta = \{c\}$.*

Proof. (i) Let $b \in [c]_\beta$. Then there exists $X_R \leqslant R_R$ such that $X_R \leqslant^{\text{ess}} bE(R_R)$ and $X_R \leqslant^{\text{ess}} cE(R_R)$. But $bc \in \mathbf{I}(\mathcal{E}_R)$ and $X_R \leqslant bcE(R_R) = bE(R_R) \cap cE(R_R)$. Hence $bE(R_R) = cE(R_R)$. Since $c \in \mathbf{B}(\mathcal{E}_R)$, $b = c$. Therefore $\{c\} = [c]_\alpha = [c]_\beta$. From [28, pp. 94–95], $c \in \mathbf{B}(\mathcal{E}_R)$ if and only if $c(1) \in \mathbf{B}(Q(R))$.

(ii) Note that $\text{Cen}(T) \subseteq \text{Cen}(Q(R))$. The remainder follows from part (i). □

The following is the left sided version of the claim in the proof of Theorem 1.7(ii).

Lemma 2.20. *Let T be a left ring of quotients of R and X a left ideal of T. Then $r_R(X) = r_R(X \cap R)$.*

Our next result shows that when $Q(R) = E(R_R)$ the α pseudo right ring hulls and β pseudo right ring hulls also exist, respectively for the right FI-extending and the right essentially quasi-Baer properties.

Corollary 2.21. *Assume that $Q(R) = E(R_R)$.*

(i) *For each $\delta^\alpha_{\mathfrak{E}}(R)$ (respectively, $\delta^\beta_{\mathfrak{F}\mathfrak{I}}(R)$), $R(\mathfrak{E}, \alpha, Q(R))$ (respectively, $R(\mathfrak{F}\mathfrak{I}, \beta, Q(R))$) exists. Moreover, every right ring of quotients of R containing $R(\mathfrak{E}, \alpha, Q(R))$ (respectively, $R(\mathfrak{F}\mathfrak{I}, \beta, Q(R))$) is right extending (respectively, right FI-extending).*
(ii) *Let $S = \langle R \cup \delta(1)\rangle_{Q(R)}$. If $\delta(1) = \delta^\alpha_{\mathfrak{e}\mathfrak{B}}(R)(1)$ (respectively, $\delta(1) = \delta^\beta_{\mathfrak{e}\mathfrak{q}\mathfrak{B}}(R)(1)$) and S is a left ring of quotients of R, then $R(\mathfrak{e}\mathfrak{B}, \alpha, Q(R))$ (respectively, $R(\mathfrak{e}\mathfrak{q}\mathfrak{B}, \beta, Q(R))$) exists. Moreover, any right and left ring of quotients of R which also lies between $R(\mathfrak{e}\mathfrak{B}, \alpha, Q(R))$ (respectively, $R(\mathfrak{e}\mathfrak{q}\mathfrak{B}, \beta, Q(R))$) and $Q(R)$ is right essentially Baer (respectively, right essentially quasi-Baer). If $Z(R_R) = 0$, then these intermediate rings are Baer (respectively, quasi-Baer).*

Proof. (i) This is a consequence of Lemma 2.16(iii), Corollaries 2.18, and 1.8.

(ii) Let $\delta(1) = \delta^\alpha_{\mathfrak{e}\mathfrak{B}}(R)(1)$. Take $r_S(K) \in \mathfrak{D}_{\mathfrak{e}\mathfrak{B}}(S)$ with $\emptyset \neq K \subseteq S$ and let $Y = SK$. Then $r_S(Y) = r_S(K)$ and $r_R(Y) = r_S(Y) \cap R$. Now $r_R(Y) = r_R(Y \cap R) \in \mathfrak{D}_{\mathfrak{e}\mathfrak{B}}(R)$ by Lemma 2.20. Hence $r_R(Y \cap R)_R \leqslant^{\text{ess}} r_S(Y)_R$, so $\mathfrak{D}_{\mathfrak{e}\mathfrak{B}}(S \to R)$ holds. By Corollary 2.18(i), $S = R(\mathfrak{e}\mathfrak{B}, \alpha, Q(R))$. Also any right and left ring of quotients of R which is also intermediate between $R(\mathfrak{e}\mathfrak{B}, \alpha, Q(R))$ and $Q(R)$ is right essentially Baer by Theorem 1.9.

If $\delta(1) = \delta^\beta_{\mathfrak{e}\mathfrak{q}\mathfrak{B}}(R)(1)$, then $\mathfrak{D}_{\mathfrak{e}\mathfrak{q}\mathfrak{B}}(S \to R)$ holds by an argument similar to that used in the above argument. Thus, by Corollary 2.18(ii), $S = R(\mathfrak{e}\mathfrak{q}\mathfrak{B}, \beta, Q(R))$. Let T be a right and left ring of quotients of R which is intermediate between $R(\mathfrak{e}\mathfrak{q}\mathfrak{B}, \beta, Q(R))$ and $Q(R)$. Then by Theorem 1.9, $T \in \mathfrak{e}\mathfrak{q}\mathfrak{B}$. The rest of the proof follows from Proposition 1.2. □

Lemma 2.22. *Let $\mathfrak{C}$ be a $\mathfrak{D}$-$\mathfrak{E}$ class of rings and T a right essential overring of R.*

(i) *If for each $X \in \mathfrak{D}_{\mathfrak{C}}(R)$ there exists $e \in \mathbf{I}(T)$ satisfying $X_R \leqslant^{\text{ess}} eT_R$, then there is a $\delta^\beta_{\mathfrak{C}}(R)$ such that $c|_T \in \text{End}(T_T)$ and $c(1) \in \mathbf{I}(T)$ for each $c \in \delta^\beta_{\mathfrak{C}}(R)$.*

(ii) *Let* $\delta \subseteq \mathbf{I}(T)$ *such that for each* $e \in \delta$ *there exists* $X \in \mathfrak{D}_{\mathfrak{C}}(R)$ *with* $X_R \leqslant^{\text{ess}} eT_R$. *If* $S = \langle R \cup \delta \rangle_T$ *and* $S \in \mathfrak{C}$, *then* $S = R(\mathfrak{C}, \rho, T)$ *for some* ρ.

(iii) $R \in \mathfrak{C}$ *if and only if* $R = R(\mathfrak{C}, \beta, T)$.

Proof. (i) Let $b \in \delta_{\mathfrak{C}}(R)$. Then there is $X \in \mathfrak{D}_{\mathfrak{C}}(R)$ with $X_R \leqslant^{\text{ess}} bE(R_R)$. Thus, by assumption, there exists $e \in \mathbf{I}(T)$ satisfying $X_R \leqslant^{\text{ess}} eT_R$. By Lemma 2.16(i) there is $c \in \mathbf{I}(\mathcal{E}_R)$ such that $c|_T \in \text{End}(T_T)$ and $c(1) = e$. Hence $X_R \leqslant^{\text{ess}} cE(R_R)$. So $b\ \beta\ c$.

(ii) This follows from Lemma 2.16(i).

(iii) This part follows from parts (i) and (ii). □

We remark that the $\mathfrak{K}$ absolute (absolute to $Q(R)$) right ring hull of R is the intersection of all right essential overrings (of all right rings of quotients) of R which are in $\mathfrak{K}$. Our next result shows that under suitable conditions, these intersections coincide with the intersections of the α pseudo or the β pseudo right ring hulls for various $\mathfrak{D}$-$\mathfrak{C}$ classes (e.g., $\mathfrak{E}$, $\mathfrak{FI}$, $\mathfrak{eB}$, and $\mathfrak{eqB}$). Also under these conditions a $\mathfrak{C}$ right ring hull will be a $\mathfrak{C}$ α or a $\mathfrak{C}$ β pseudo right ring hull. We note that the condition $X \trianglelefteq R$ implies $XT \trianglelefteq T$ holds for example when T is a centralizing extension of R or when R is a right Noetherian ring and T is a right ring of quotients of R contained in $Q^r_{c\ell}(R)$ [27, pp. 314–315]. This condition is useful in the following result.

Corollary 2.23. *Assume that* T *is a right ring of quotients of* R.

(i) *Suppose that either* $\alpha = \beta$ *or some* $\delta^{\beta}_{\mathfrak{E}}(R)(1) \subseteq \text{Cen}(T)$. *Then* $T \in \mathfrak{E}$ *if and only if there exists a* $R(\mathfrak{E}, \alpha, Q(R))$ *which is a subring of* T.

(ii) *If* $X \trianglelefteq R$ *implies* $XT \trianglelefteq T$, *then* $T \in \mathfrak{FI}$ *if and only if there exists a* $R(\mathfrak{FI}, \beta, Q(R))$ *which is a subring of* T.

(iii) *Suppose that either* $\alpha = \beta$ *or some* $\delta^{\beta}_{\mathfrak{eB}}(R)(1) \subseteq \text{Cen}(T)$. *If* T *is also a left ring of quotients of* R, *then* $T \in \mathfrak{eB}$ *if and only if there exists a* $R(\mathfrak{eB}, \alpha, Q(R))$ *which is a subring of* T.

(iv) *If* T *is also a left ring of quotients of* R *and* $X \trianglelefteq R$ *implies* $TX \trianglelefteq T$, *then* $T \in \mathfrak{eqB}$ *if and only if there exists a* $R(\mathfrak{eqB}, \beta, Q(R))$ *which is a subring of* T.

Proof. (i) Let $T \in \mathfrak{E}$. Suppose that $\alpha = \beta$. Say $X \in \mathfrak{D}_{\mathfrak{E}}(R)$. As in the proof of Lemma 1.4(ii), $X_R \leqslant^{\text{ess}} XT_R$. Take $Y = XT$. Since T is right extending, there is $e \in \mathbf{I}(T)$ with $Y_T \leqslant^{\text{ess}} eT_T$. Thus $Y_R \leqslant^{\text{ess}} eT_R$ by Lemma 1.4(i), so $X_R \leqslant^{\text{ess}} Y_R \leqslant^{\text{ess}} eT_R$. By Lemma 2.22(i), there exists $\delta^{\beta}_{\mathfrak{E}}(R)$ such that $c|_T \in \text{End}(T_T)$ and $c(1) \in \mathbf{I}(T)$ for each $c \in \delta^{\beta}_{\mathfrak{E}}(R)$. Take $S = \langle R \cup \delta^{\beta}_{\mathfrak{E}}(R)(1)\rangle_T$ $(= \langle R \cup \delta^{\beta}_{\mathfrak{E}}(R)(1)\rangle_{Q(R)})$. Since $\alpha = \beta$ and $\mathfrak{D}_{\mathfrak{E}}(S \to R)$ holds, $S = \langle R \cup \delta^{\alpha}_{\mathfrak{E}}(R)(1)\rangle_{Q(R)} = R(\mathfrak{E}, \alpha, Q(R))$ by Corollary 2.18. Clearly S is a subring of T.

Next, suppose that $\delta^{\beta}_{\mathfrak{E}}(R)(1) \subseteq \text{Cen}(T)$ for some β. Since $\text{Cen}(T) \subseteq \text{Cen}(Q(R))$, it follows that $\delta^{\beta}_{\mathfrak{E}}(R)(1) \subseteq \text{Cen}(Q(R))$. Thus by Lemma 2.19, $\beta = \alpha$. From Corollary 2.18, $S = \langle R \cup \delta^{\beta}_{\mathfrak{E}}(R)(1)\rangle_T = \langle R \cup \delta^{\beta}_{\mathfrak{E}}(R)(1)\rangle_{Q(R)} = R(\mathfrak{E}, \alpha, Q(R))$ and it is a subring of T. The converse follows from Corollary 1.8(ii).

(ii) Assume that $T \in \mathfrak{FI}$. Take $X \in \mathfrak{D}_{\mathfrak{FI}}(R)$ (i.e., $X \trianglelefteq R$). Since T is right FI-extending and $XT \trianglelefteq T$, there is $e \in \mathbf{I}(T)$ with $XT_T \leqslant^{\text{ess}} eT_T$. Thus $XT_R \leqslant^{\text{ess}} eT_R$ from Lemma 1.4(i). Also by a modification of the proof of Lemma 1.4(ii), we can see that $X_R \leqslant^{\text{ess}} XT_R$. Therefore $X_R \leqslant^{\text{ess}} eT_R$. Let $Y = XT$. Then $X_R \leqslant^{\text{ess}} Y_R \leqslant^{\text{ess}} eT_R$ with $Y \in \mathfrak{D}_{\mathfrak{FI}}(T)$ and $e \in \mathbf{I}(T)$. Now from Lemma 2.22(i), there exists $\delta^{\beta}_{\mathfrak{FI}}(R)$ such that $c|_T \in \text{End}(T_T)$ and $c(1) \in \mathbf{I}(T)$

for each $c \in \delta^{\beta}_{\mathfrak{FI}}(R)$. Then Corollary 2.18(ii) yields $S = \langle R \cup \delta^{\beta}_{\mathfrak{FI}}(R)(1)\rangle_T = R(\mathfrak{FI}, \beta, T)$ $(= R(\mathfrak{FI}, \beta, Q(R)))$ which is a subring of T. The converse follows from Corollary 1.8(i).

(iii) Assume that $T \in \mathfrak{eB}$. First, note that if $X \in \mathfrak{D}_{\mathfrak{eB}}(R)$ then there exists a nonempty set $K \subseteq R$ such that $r_R(K) = X$. Therefore $r_R(K) = r_T(K) \cap R$. Hence $(r_R(K))_R \leqslant^{\text{ess}} (r_T(K))_R$. For, if $0 \neq t \in r_T(K)$, then there is $r \in R$ with $0 \neq tr \in R$ because $R_R \leqslant^{\text{ess}} T_R$. Thus $Ktr = 0$, so $0 \neq tr \in r_R(K)$.

Since $T \in \mathfrak{eB}$, there is $e \in \mathbf{I}(T)$ with $r_T(K)_T \leqslant^{\text{ess}} eT_T$. So $r_T(K)_R \leqslant^{\text{ess}} eT_R$ from Lemma 1.4(i). Thus $X_R = r_R(K)_R \leqslant^{\text{ess}} r_T(K)_R \leqslant^{\text{ess}} eT_R$ with $r_T(K) \in \mathfrak{D}_{\mathfrak{eB}}(T)$ and $e \in \mathbf{I}(T)$. Hence it yields the hypothesis of Lemma 2.22(i). Thus there exists $\delta^{\beta}_{\mathfrak{eB}}(R)$ such that $c|_T \in \text{End}(T_T)$ and $c(1) \in \mathbf{I}(T)$ for each $c \in \delta_{\mathfrak{eB}}(R)$. Let $S = \langle R \cup \delta^{\beta}_{\mathfrak{eB}}(R)(1)\rangle_T$ $(= \langle R \cup \delta^{\beta}_{\mathfrak{eB}}(R)(1)\rangle_{Q(R)})$. Then $\mathfrak{D}_{\mathfrak{eB}}(S \to R)$ holds. To see this, take $Y \in \mathfrak{D}_{\mathfrak{eB}}(S)$. Then $Y = r_S(L)$ for some $\emptyset \neq L \subseteq S$. Now note that $Y = r_S(SL)$ and $r_R(SL) = r_S(SL) \cap R_R \leqslant^{\text{ess}} r_S(SL)_R$. Also note that S is a left ring of quotients of R, hence $r_R(SL) = r_R(SL \cap R)$ by Lemma 2.20. Therefore $\mathfrak{D}_{\mathfrak{eB}}(S \to R)$ holds since $r_R(SL \cap R) \in \mathfrak{D}_{\mathfrak{eB}}(R)$.

By an argument similar to that used in part (i), $S = R(\mathfrak{eB}, \alpha, Q(R))$ is the desired subring of T, by Corollary 2.18(ii). The converse follows from Theorem 1.9(iii).

(iv) Assume that $T \in \mathfrak{eqB}$. Let $K \in \mathfrak{D}_{\mathfrak{eqB}}(R)$. Then there is $X \trianglelefteq R$ with $K = r_R(X)$. Noting that $TX = TXT$ by assumption, $K = r_R(X)_R \leqslant^{\text{ess}} r_T(X)_R = r_T(TX)_R = r_T(TXT)_R$. Now $r_T(TXT) \in \mathfrak{D}_{\mathfrak{eqB}}(T)$. Thus there is $e \in \mathbf{I}(T)$ such that $r_T(TXT)_T \leqslant^{\text{ess}} eT_T$, so $r_T(TXT)_R \leqslant^{\text{ess}} eT_R$ since R_R is dense in T_R. Hence, by Lemma 2.22(i), there is $\delta^{\beta}_{\mathfrak{eqB}}(R)$ satisfying $c|_T \in \text{End}(T_T)$ and $c(1) \in \mathbf{I}(T)$ for each $c \in \delta^{\beta}_{\mathfrak{eqB}}(R)$. Let $S = \langle R \cup \delta^{\beta}_{\mathfrak{eqB}}(R)(1)\rangle_T$ $(= \langle R \cup \delta^{\beta}_{\mathfrak{eqB}}(R)(1)\rangle_{Q(R)})$. Since T is a left ring of quotients of R, so is S. By an argument similar to that used in the proof of part (iii), $\mathfrak{D}_{\mathfrak{eqB}}(S \to R)$ holds. By Corollary 2.18(ii), $S = R(\mathfrak{eqB}, \beta, Q(R))$. The converse follows from Theorem 1.9(ii). □

Corollary 2.24. *Assume that $E(R_R) = Q(R)$, $Q(R)$ is a left ring of quotients of R, and T is a right ring of quotients of R. Then*:

(i) $\delta_{\mathfrak{E}}(R) = \delta_{\mathfrak{eB}}(R)$.
(ii) *Assume that $\alpha = \beta$ or some $\delta^{\beta}_{\mathfrak{E}}(R)(1) \subseteq \text{Cen}(T)$. Then $T \in \mathfrak{E}$ if and only if $T \in \mathfrak{eB}$. Also every right extending α pseudo right ring hull of R is a right essentially Baer α pseudo right ring hull of R and conversely.*
(iii) *Assume that $Z(R_R) = 0$. Then $T \in \mathfrak{E}$ if and only if $T \in \mathfrak{B}$. Moreover, every right extending α pseudo right ring hull of R is an essentially Baer α pseudo right ring hull of R which is Baer and conversely.*

Proof. Obviously, $\delta_{\mathfrak{eB}}(R) \subseteq \delta_{\mathfrak{E}}(R)$. Let $c \in \delta_{\mathfrak{E}}(R)$ and $e = c(1)$. By Lemma 2.20, $r_R(Q(R) \times (1 - e) \cap R) = r_R(Q(R)(1 - e)) = r_{Q(R)}(Q(R)(1 - e)) \cap R = (eQ(R) \cap R)_R \leqslant^{\text{ess}} eQ(R)_R = c(1)Q(R)_R = cQ(R)_R$. Hence $c \in \delta_{\mathfrak{eB}}(R)$. Thus $\delta_{\mathfrak{eB}}(R) = \delta_{\mathfrak{E}}(R)$. The remainder of the proof follows from Corollary 2.23 and Proposition 1.2. □

3. Applications to matrix and generalized triangular matrix rings

In this section, we apply our theory to provide answers to Problems I and II of Section 1, when the class $\mathfrak{K}$ is $\mathfrak{B}$, $\mathfrak{E}$, $\mathfrak{FI}$, or the class of right (semi)hereditary rings and the ring R (in Problem I) or the ring T (in Problem II) is a subring of a 2-by-2 matrix ring.

Our first result of the section characterizes any right extending ring whose maximal right ring of quotients is the 2-by-2 matrix ring over a division ring.

Theorem 3.1. *Let D be a division ring and assume that T is a ring such that $Q(T) = \mathrm{Mat}_2(D)$ (respectively, $Q(T) = Q^{\ell}(T) = \mathrm{Mat}_2(D)$). Then $T \in \mathfrak{E}$ (respectively, $T \in \mathfrak{B}$) if and only if the following conditions are satisfied*:

(i) *there exist $v, w \in D$ such that $\left(\begin{smallmatrix} 1 & v \\ 0 & 0 \end{smallmatrix}\right) \in T$ and $\left(\begin{smallmatrix} 0 & 0 \\ w & 1 \end{smallmatrix}\right) \in T$; and*
(ii) *for each $0 \neq d \in D$ at least one of the following conditions is true*:
  (1) $\left(\begin{smallmatrix} 0 & d \\ 0 & 1 \end{smallmatrix}\right) \in T$,
  (2) $\left(\begin{smallmatrix} 1 & 0 \\ d^{-1} & 0 \end{smallmatrix}\right) \in T$, *or*
  (3) *there exists $a \in D$ such that $a - a^2 \neq 0$ and $\left(\begin{smallmatrix} a & (1-a)d \\ d^{-1}a & d^{-1}(1-a)d \end{smallmatrix}\right) \in T$.*

Proof. Routine calculations show that any nontrivial idempotent of $Q(T)$ has one of the following forms where $a, b, f \in D$ with $a - a^2 \neq 0$ and $b \neq 0$:

$$\begin{pmatrix} 1 & f \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ f & 1 \end{pmatrix}, \begin{pmatrix} 0 & f \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ f & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ b^{-1}(1-a)a & b^{-1}(1-a)b \end{pmatrix}.$$

By using Definition 2.14(i), we obtain:

$$\text{for } f \in D,\ \begin{pmatrix} 1 & f \\ 0 & 0 \end{pmatrix} \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ f & 1 \end{pmatrix} \alpha \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\text{for } 0 \neq f,\ 0 \neq g \in D,\ \begin{pmatrix} 0 & f \\ 0 & 1 \end{pmatrix} \alpha \begin{pmatrix} 1 & 0 \\ g & 0 \end{pmatrix} \text{ if and only if } g = f^{-1}; \text{ and}$$

$$\text{for } 0 \neq f \in D,\ \begin{pmatrix} 0 & f \\ 0 & 1 \end{pmatrix} \alpha \begin{pmatrix} a & b \\ b^{-1}(1-a)a & b^{-1}(1-a)b \end{pmatrix} \text{ if and only if } b = (1-a)f.$$

Thus for some $v, w \in D$, let

$$\begin{aligned} Y = {}& \left\{0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ w & 1 \end{pmatrix}\right\} \\ & \cup \left\{ K \in \mathrm{Mat}_2(D) \;\middle|\; \text{for each } 0 \neq d \in D,\ K \text{ has exactly one of the following forms:} \right. \\ & \left. \begin{pmatrix} 0 & d \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ d^{-1} & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} a & (1-a)d \\ d^{-1}a & d^{-1}(1-a)d \end{pmatrix} \text{ for some } a \in D \text{ with } a - a^2 \neq 0 \right\}. \end{aligned}$$

Then $Y = \delta_{\mathfrak{E}}^{\alpha}(T)(1)$. Since $Z(T_T) = 0$, $\alpha = \beta$. Hence the result is now a direct consequence of Corollaries 2.23 and 2.24, where R in the corollaries coincides with T in the present result. □

We observe that one can generalize Theorem 3.1 by replacing $\mathrm{Mat}_2(D)$ with a semisimple Artinian ring $S = \bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(D_i)$, where n_i is a positive integer and D_i is a division ring. To see this, one can develop a generalized proof of Theorem 3.1 as follows:

(i) calculate $\mathbf{I}(\mathrm{Mat}_{n_i}(D_i))$ for each i;

(ii) note that if $X_T \leqslant T_T$, then $(\bigoplus_{i=1}^{k}(X \cap S_i))_T \leqslant^{\text{ess}} X_T$, where $S_i = \text{Mat}_{n_i}(D_i)$;
(iii) find a $\delta_{\mathfrak{C}}^{\alpha}(T \cap \text{Mat}_{n_i}(D_i))(1)$ for each i;
(iv) take $\delta_{\mathfrak{C}}^{\alpha}(T)(1)$ to be the Cartesian product of the $\delta_{\mathfrak{C}}^{\alpha}(T \cap \text{Mat}_{n_i}(D_i))(1)$ by using (ii) and (iii).

Now the generalized version is a direct consequence of Corollaries 2.23 and 2.24 as in the proof of Theorem 3.1.

Theorem 3.1 provides the following elementwise characterization of a Prüfer domain.

Corollary 3.2. *Let A be a commutative domain with F as its field of fractions. Then the following conditions are equivalent.*

(i) *A is a Prüfer domain.*
(ii) *For each $d \in F$ with $d \notin A$ and $d^{-1} \notin A$, there exists $a \in A$ such that $d^{-1}a \in A$ and $(1-a)d \in A$.*

Proof. The proof follows from [25, p. 17, Exercise 3] and Theorem 3.1, where we take $T = \text{Mat}_2(A)$ in Theorem 3.1. □

As in the comment before Corollary 1.10, we remark that any ring of quotients of a Prüfer domain is a Prüfer domain follows immediately from Corollary 3.2.

Corollary 3.3.

(i) *Let T be a ring such that $Q(T) = \text{Mat}_2(D)$, where D is a division ring and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in T$. If either $\left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right) \subseteq T$ or $\left(\begin{smallmatrix} 0 & 0 \\ D & 0 \end{smallmatrix}\right) \subseteq T$, then T is right extending and Baer.*
(ii) *Let A be a right Ore domain with $D = Q_{c\ell}^{r}(A)$. Then $\left(\begin{smallmatrix} A & D \\ 0 & A \end{smallmatrix}\right)$ is a right extending right ring hull of $T_2(A)$ and it is Baer.*

Proof. Part (i) is a direct consequence of Theorem 3.1 and [15, Theorem 1.1]. Part (ii) follows from part (i). □

As a consequence of Corollary 3.3, our next example provides a right extending generalized 2-by-2 triangular matrix ring T such that $Q(T) = \text{Mat}_2(D)$, where $D = Q_{c\ell}^{r}(A)$ and A is a right Ore domain, but T is not necessarily an overring of $T_2(A)$.

Example 3.4. Let A be a right Ore domain with $D = Q_{c\ell}^{r}(A)$ and B any subring of D. Then $T = \left(\begin{smallmatrix} B & D \\ 0 & A \end{smallmatrix}\right) \in \mathfrak{C}$ and $Q(T) = \text{Mat}_2(D)$. For an explicit example, take $A = \mathbb{Z}[x]$ or $\mathbb{Q}[x]$, and $B = \mathbb{Z}$.

From [25, p. 16, Exercise 2] it is well known that if A is a commutative domain with F as its field of fractions and $A \neq F$, then $T_n(A)$ $(n > 1)$ is not Baer, but by Theorem 1.9 any right ring of quotients of $T_n(A)$ which contains $T_n(F)$ is Baer. This result motivates the question: *If A is a commutative domain, can we find $\mathfrak{C}$ right ring hulls or $\mathfrak{C}$ ρ pseudo right ring hulls for $T_n(A)$ and use these to describe all $\mathfrak{C}$ right rings of quotients of $T_n(A)$ when $\mathfrak{C}$ is a class related to the Baer class*? (See Problems I and II, Section 1.) In the next five results, we answer this question when A is either a PID or a Bezout domain (i.e., every finitely generated ideal is principal [19]).

Lemma 3.5. *Let A be a commutative Bezout domain and T a right ring of quotients of $T_2(A)$ such that $\left(\begin{smallmatrix} A & A \\ aA & A \end{smallmatrix}\right) \subseteq T$ for some $0 \neq a \in A$.*

(i) *If $\left(\begin{smallmatrix} 0 & a^{-1} \\ 0 & 0 \end{smallmatrix}\right) \in T$, then T is right extending.*
(ii) *If $a = p_1^{k_1} \cdots p_m^{k_m}$ where each p_i is a distinct prime, each k_i is a positive integer, and*

$$\begin{pmatrix} 0 & (p_1^{k_1-1} \cdots p_m^{k_m-1})^{-1} \\ 0 & 0 \end{pmatrix} \in T,$$

then T is right extending.

Proof. Let $c, d \in A$ such that $c \neq 0$ and $d \neq 0$. Assume that $\left(\begin{smallmatrix} 0 & cd^{-1} \\ 0 & 0 \end{smallmatrix}\right) \notin T$ and $\left(\begin{smallmatrix} 0 & 0 \\ dc^{-1} & 0 \end{smallmatrix}\right) \notin T$. Let $\gcd(c, d) = z$. Then $c = c_1 z$, $d = d_1 z$, and $\gcd(c_1, d_1) = 1$ for some $c_1, d_1, z \in A$. By noting that $cd^{-1} = c_1 d_1^{-1}$ because A is a GCD domain [26], we may assume that $\gcd(c, d) = 1$. Let $g = \gcd(d, a)$. Then $a = sg$, $d = tg$, and $\gcd(s, t) = 1$ for some $s, t \in A$.

(i) Since $\gcd(s, t) = 1 = \gcd(c, t)$, then $\gcd(t, cs) = 1$. Hence there are x, $y \in A$ with $1 = csx + ty$. Take $b = csx$. If $b = 0$, then $1 = ty$, thus $t^{-1} = y \in A$. So $cd^{-1} = c(tg)^{-1} = cyg^{-1} \in g^{-1}A = a^{-1}sA \subseteq a^{-1}A$, a contradiction. Hence $b \neq 0$. If $b = 1$, then $c^{-1} = sx$. So $dc^{-1} = dsx = tgsx = tax \in aA$, a contradiction. Thus $b - b^2 \neq 0$. Note that $dc^{-1}b = dsx = tgsx = tax \in aA$ and $(1 - b)cd^{-1} = (ty)c(tg)^{-1} = g^{-1}cy \in g^{-1}A = a^{-1}sA \subseteq a^{-1}A$. Therefore $\left(\begin{smallmatrix} b & (1-b)cd^{-1} \\ dc^{-1}b & 1-b \end{smallmatrix}\right) \in T$. From Theorem 3.1, $T \in \mathfrak{E}$.

(ii) We can consider the divisibility of d with respect to each p_i and obtain

$$d = p_1^{h_1} \cdots p_m^{h_m} q,$$

where each h_i is a nonnegative integer such that $h_i \leqslant k_i$, and $h_i < k_i$ implies $\gcd(p_i, q) = 1$.

Case 1. Assume that whenever $h_i \neq 0$, then $h_i = k_i$. We *claim* that $\gcd(d, s) = 1$. First, if $h_1 = \cdots = h_m = 0$, then $d = q$. Thus $g = \gcd(d, a) = 1$. So $a = sg = s$ and $d = tg = t$. Since $\gcd(s, t) = 1$, $\gcd(s, d) = 1$. Now suppose that not all h_i are zero. Assume to the contrary that $\gcd(d, s) \neq 1$. Then there exists $j \in \{1, \ldots, m\}$ such that $p_j \mid s$. Thus $s = s_1 p_j$ for some $s_1 \in A$. Since $at = sd$, we have that

$$p_1^{k_1} \cdots p_{j-1}^{k_{j-1}} p_{j+1}^{k_{j+1}} \cdots p_m^{k_m} t = p_1^{h_1} \cdots p_j^{h_j - k_j} \cdots p_m^{h_m} q p_j s_1$$

and $h_j - k_j = 0$. Hence $p_j \mid t$, a contradiction to $\gcd(s, t) = 1$. Therefore $\gcd(d, s) = 1$.

Since $\gcd(c, d) = 1$, we have that $\gcd(cs, d) = 1$. Thus there exist $x, y \in A$ such that $1 = csx + dy$. Let $b = csx$. If $b = 0$, then $dy = 1$, hence $d^{-1} = y \in A$. So $cd^{-1} = cy \in A$, a contradiction. Thus $b \neq 0$. If $b = 1$, then $1 = csx$, so $c^{-1} = sx$. Thus $dc^{-1} = dsx = tgsx = tax \in aA$, a contradiction. So $b \neq 1$, hence $b - b^2 \neq 0$. Now $dc^{-1}b = dsx = tgsx = tax \in aA$ and $(1 - b)cd^{-1} = dycd^{-1} = yc \in A$. Therefore

$$\begin{pmatrix} b & (1-b)cd^{-1} \\ dc^{-1}b & 1-b \end{pmatrix} \in \begin{pmatrix} A & A \\ aA & A \end{pmatrix} \subseteq T.$$

Case 2. Assume that there exists $\ell \in \{1, \ldots, m\}$ such that $1 \leqslant h_\ell < k_\ell$. Let $I = \{i \in \{1, \ldots, m\} \mid h_i < k_i\}$ and $v = |I|$. Also let $J = \{1, \ldots, m\} \setminus I$ and $w = |J|$. Denote $I = \{i_1, \ldots, i_v\}$ and

$J=\{j_1,\dots,j_w\}$. In this case, note $J=\{j\in\{1,2,\dots,m\}\mid h_j=0 \text{ or } h_j=k_j\}$. Then there exist $\pi,\sigma\in A$ such that

$$\pi c p_{i_1}^{\theta_{i_1}}\cdots p_{i_v}^{\theta_{i_v}}+\sigma p_{j_1}^{k_{j_1}}\cdots p_{j_w}^{k_{j_w}}q=1,$$

where $\theta_{i_1}=k_{i_1}-h_{i_1},\ \dots,\ \theta_{i_v}=k_{i_v}-h_{i_v}$. Take $b=\pi c p_{i_1}^{\theta_{i_1}}\cdots p_{i_v}^{\theta_{i_v}}$. Then $bc^{-1}d\in aA$ and $(1-b)cd^{-1}=\sigma p_{j_1}^{k_{j_1}}\cdots p_{j_w}^{k_{j_w}}qc(p_1^{h_1}\cdots p_m^{h_m}q)^{-1}=\sigma c(p_{i_1}^{h_{i_1}}\cdots p_{i_v}^{h_{i_v}})^{-1}(p_{j_1}^{k_{j_1}-1}\cdots p_{j_w}^{k_{j_w}-1})^{-1}$ is in $(p_1^{k_1-1}\cdots p_m^{k_m-1})^{-1}A$. Clearly $b-b^2\neq 0$. Thus $\left(\begin{smallmatrix} b & (1-b)cd^{-1}\\ dc^{-1}b & 1-b\end{smallmatrix}\right)\in T$. Therefore, by Theorem 3.1, $T\in\mathfrak{E}$. □

Lemma 3.6. *Let A be a commutative Bezout domain and T a right ring of quotients of $T_2(A)$ such that $T\cap \mathrm{Mat}_2(A)=\left(\begin{smallmatrix} A & A\\ aA & A\end{smallmatrix}\right)$, where $0\neq a=p_1^{k_1}\cdots p_m^{k_m}\in A$, each p_i is a distinct prime, and each k_i is a positive integer. If T is right extending, then*

$$\begin{pmatrix} A & (p_1^{k_1-1}\cdots p_m^{k_m-1})^{-1}A\\ aA & A\end{pmatrix}$$

is a subring of T.

Proof. Assume that $T\in\mathfrak{E}$ and that F denotes the field of fractions of A. Let $d=p_1^{k_1-1}\cdots p_m^{k_m-1}\in A$. Note that $\left(\begin{smallmatrix} 0 & 0\\ d & 0\end{smallmatrix}\right)\notin T$. Thus, by Theorem 3.1, either $\left(\begin{smallmatrix} 0 & d^{-1}\\ 0 & 0\end{smallmatrix}\right)\in T$ or there exists $b\in F$ such that $b-b^2\neq 0$ and $\left(\begin{smallmatrix} b & (1-b)d^{-1}\\ db & 1-b\end{smallmatrix}\right)\in T$. Suppose there is $b\in F$ with $b-b^2\neq 0$ and $\left(\begin{smallmatrix} b & (1-b)d^{-1}\\ db & 1-b\end{smallmatrix}\right)\in T$. Now there are $x,y\in A$ with $b=xy^{-1}$ and $\gcd(x,y)=1$. Hence $\left(\begin{smallmatrix} x & (y-x)d^{-1}\\ dx & y-x\end{smallmatrix}\right)\in T$. Since A is a commutative Bezout domain and $\gcd(x,y)=1$, there are $v,w\in A$ such that $xv+yw=1$. Note that $\left(\begin{smallmatrix} wx & (wy-wx)d^{-1}\\ wdx & wy-wx\end{smallmatrix}\right)\in T$ and $\left(\begin{smallmatrix} vx & 0\\ vdx & 0\end{smallmatrix}\right)\in T$. Since $wy=1-vx$, it follows that $\left(\begin{smallmatrix} (v+w)x & (1-(v+w)x)d^{-1}\\ (v+w)xd & 1-(v+w)x\end{smallmatrix}\right)\in T$. Also since $\left(\begin{smallmatrix} 0 & 0\\ xd & 0\end{smallmatrix}\right)\in T\cap\mathrm{Mat}_2(A)$, we have that $xd\in aA$. Thus x is not a unit of A. Hence $1-(v+w)x\neq 0$. If $(v+w)x=0$, then $\left(\begin{smallmatrix} 0 & d^{-1}\\ 0 & 0\end{smallmatrix}\right)\in T$. Assume $(v+w)x\neq 0$. Take $g=(v+w)x$. Then $\left(\begin{smallmatrix} g & (1-g)d^{-1}\\ gd & 1-g\end{smallmatrix}\right)\in T$, where $g\in A$, $g-g^2\neq 0$, and $gd\in aA$. Thus $p_i\mid g$ for $i=1,\dots,m$. So $\gcd(1-g,d)=1$. Hence there exist $\pi,\sigma\in A$ with $(1-g)\sigma+d\pi=1$. Thus $(1-g)\sigma d^{-1}+\pi=d^{-1}$. Since $\left(\begin{smallmatrix} 0 & (1-g)d^{-1}\\ 0 & 0\end{smallmatrix}\right)\in T$, then $\left(\begin{smallmatrix} 0 & (1-g)\sigma d^{-1}+\pi\\ 0 & 0\end{smallmatrix}\right)\in T$. Thus in all cases, we have that $\left(\begin{smallmatrix} 0 & d^{-1}\\ 0 & 0\end{smallmatrix}\right)\in T$. Consequently, $\left(\begin{smallmatrix} A & (p_1^{k_1-1}\cdots p_m^{k_m-1})^{-1}A\\ aA & A\end{smallmatrix}\right)$ is a subring of T. □

Theorem 3.7. *Let A be a commutative Bezout domain with F as its field of fractions, $A\neq F$, and T be a right ring of quotients of $T_2(A)$. If at least one of the following conditions holds, then T is right extending and Baer.*

(i) $\left(\begin{smallmatrix} A & F\\ 0 & A\end{smallmatrix}\right)$ *is a subring of T.*
(ii) $\left(\begin{smallmatrix} A & a^{-1}A\\ aA & A\end{smallmatrix}\right)$ *is a subring of T for some $0\neq a\in A$.*
(iii) $\left(\begin{smallmatrix} A & (p_1^{k_1-1}\cdots p_m^{k_m-1})^{-1}A\\ aA & A\end{smallmatrix}\right)$ *is a subring of T for some $0\neq a\in A$, where $a=p_1^{k_1}\cdots p_m^{k_m}$, each p_i is a distinct prime, and each k_i is a positive integer.*

Proof. This result follows from [15, Theorem 1.1], Theorem 3.1, and Lemma 3.5. □

Lemma 3.8. *Let A be a commutative PID with F as its field of fractions, $A \neq F$, and*

$$V = \begin{pmatrix} A & (p_1^{k_1-1} \cdots p_m^{k_m-1})^{-1} A \\ p_1^{k_1} \cdots p_m^{k_m} A & A \end{pmatrix},$$

where each p_i is a distinct prime of A. Then V is a right hereditary ring.

Proof. Define $\sigma : V \to \begin{pmatrix} A & A \\ p_1 \cdots p_m A & A \end{pmatrix} = W$ by

$$\sigma\left[\begin{pmatrix} a & (p_1^{k_1-1} \cdots p_m^{k_m-1})^{-1} b \\ p_1^{k_1} \cdots p_m^{k_m} c & d \end{pmatrix}\right] = \begin{pmatrix} a & b \\ p_1 \cdots p_m c & d \end{pmatrix},$$

where $a, b, c, d \in A$. Then we can see that σ is a ring isomorphism. Thus to show that V is right hereditary, we need to prove that the ring W is right hereditary.

For this, let $P = pA$ be a nonzero prime ideal of A. Then P is a maximal ideal of A. If $p \notin \{p_1, p_2, \ldots, p_m\}$, then

$$W_P = \begin{pmatrix} A_P & A_P \\ p_1 p_2 \cdots p_m A_P & A_P \end{pmatrix} = \begin{pmatrix} A_P & A_P \\ A_P & A_P \end{pmatrix}$$

is right hereditary, where A_P and W_P are localizations of A and the A-algebra W at P, respectively. Next suppose that $p \in \{p_1, p_2, \ldots, p_m\}$. Say $p = p_1$, so $P = p_1 A$. Thus,

$$W_P = \begin{pmatrix} A_P & A_P \\ pA_P & A_P \end{pmatrix}$$

is right hereditary by [31, p. 155, Example 5.11(i)]. So W_P is right hereditary for any maximal ideal P of A. Thus W is right hereditary by [35, p. 41, Theorem 3.28]. Therefore V is right hereditary. □

The following corollary illustrates how both Definitions 2.1 and 2.2 can be used to characterize all right rings of quotients from a class $\mathfrak{C}$ (see Problem I in Section 1).

Corollary 3.9. *Let A be a commutative PID with F as its field of fractions, $A \neq F$, and let $R = T_2(A)$.*

(i) *Let T be a right ring of quotients of R. Then T is right extending if and only if either the ring $U = \begin{pmatrix} A & F \\ 0 & A \end{pmatrix}$ is a subring of T, or the ring $V = \begin{pmatrix} A & (p_1^{k_1-1} \cdots p_m^{k_m-1})^{-1} A \\ aA & A \end{pmatrix}$ is a subring of T for some nonzero $a = p_1^{k_1} \cdots p_m^{k_m}$, where each p_i is a distinct prime of A.*

(ii) $\begin{pmatrix} A & F \\ 0 & A \end{pmatrix}$ *is the unique right extending right ring hull of R.*

(iii) *R has no right extending absolute right ring hull.*

(iv) *In* (i)–(iii) *we can replace "right extending" with "Baer," "right PP," or "right semihereditary."*

Proof. (i) Assume that $T \in \mathfrak{E}$. If U is not a subring of T, then $T \cap \mathrm{Mat}_2(A) = \left(\begin{smallmatrix} A & A \\ aA & A \end{smallmatrix}\right)$ for some $0 \neq a \in A$. For, if $T \cap \mathrm{Mat}_2(A) = T_2(A)$, then $\left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right) \subseteq T$ by Theorem 3.1. So U is a subring of T, a contradiction. Now V is a subring of T by Lemma 3.6. The converse follows from Theorem 3.7.

(ii) If there is another distinct right extending right ring hull H of R, then from part (i) there exists $0 \neq a = p_1^{k_1} \cdots p_m^{k_m} \in A$ such that each p_i is a distinct prime of A and $H = V$. Let p be a prime in A with $\gcd(p, p_1 \cdots p_m) = 1$. Take

$$H_1 = \begin{pmatrix} A & (p_1^{k_1-1} \cdots p_m^{k_m-1})^{-1} A \\ paA & A \end{pmatrix}.$$

Then, by part (i), H_1 is a right extending right ring of quotients of R such that H_1 is a proper subring of H, a contradiction.

(iii) If R has a right extending absolute right ring hull S, then $S \subseteq U$ from part (i). Let p be a prime element of A. Then $\left(\begin{smallmatrix} A & A \\ pA & A \end{smallmatrix}\right) \in \mathfrak{E}$ by part (i). Thus $S \subseteq \left(\begin{smallmatrix} A & A \\ pA & A \end{smallmatrix}\right)$, hence $S \subseteq \left(\begin{smallmatrix} A & F \\ 0 & A \end{smallmatrix}\right) \cap \left(\begin{smallmatrix} A & A \\ pA & A \end{smallmatrix}\right) = T_2(A) = R$. So $S = R$. Therefore $R \in \mathfrak{E}$, which is a contradiction.

(iv) Corollary 2.24 and the fact that a right PP ring with no infinite set of orthogonal idempotents is a Baer ring [38] yield that "right extending" can be replaced by "Baer" or "right PP." To see that "right extending" can be replaced by "right semihereditary," we first note that a right semihereditary ring is right PP. Hence if T is right semihereditary, then it must have either U or V as a subring.

Next we *claim* that the ring U is right semihereditary. First note that U is Baer by Theorem 3.7(i). Thus U is right PP. Now suppose that I is a finitely generated right ideal of U generated by $\left(\begin{smallmatrix} a_i & q_i \\ 0 & b_i \end{smallmatrix}\right)$, where $i = 1, 2, \ldots, k$.

Case 1. If there exists i with $a_i \neq 0$. Then there are $a, b \in A$ such that $a \neq 0$, $aA = a_1 A + \cdots + a_k A$, and $bA = b_1 A + \cdots + b_k A$. Moreover, it follows that

$$I = \begin{pmatrix} a_1 & q_1 \\ 0 & b_1 \end{pmatrix} A + \cdots + \begin{pmatrix} a_k & q_k \\ 0 & b_k \end{pmatrix} A = \begin{pmatrix} aA & F \\ 0 & bA \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} U.$$

Now since U is right PP, I is projective as a right U-module.

Case 2. $a_i = 0$ for each i. Then

$$I = \left\{ \begin{pmatrix} 0 & q_1 r_1 \\ 0 & n_1 r_1 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & q_k r_k \\ 0 & n_k r_k \end{pmatrix} \,\middle|\, r_1, \ldots, r_k \in A \right\}.$$

Since A is a commutative PID, there exist $\left(\begin{smallmatrix} 0 & s_j \\ 0 & t_j \end{smallmatrix}\right) \in I$ with $j = 1, \ldots, \ell$ such that $I = \left(\begin{smallmatrix} 0 & s_1 \\ 0 & t_1 \end{smallmatrix}\right) A \oplus \cdots \oplus \left(\begin{smallmatrix} 0 & s_\ell \\ 0 & t_\ell \end{smallmatrix}\right) A$, where the scalar multiplication is $\left(\begin{smallmatrix} 0 & s_j \\ 0 & t_j \end{smallmatrix}\right) \cdot r = \left(\begin{smallmatrix} 0 & s_j r \\ 0 & t_j r \end{smallmatrix}\right)$ for $r \in A$. Thus we see that $I = \left(\begin{smallmatrix} 0 & s_1 \\ 0 & t_1 \end{smallmatrix}\right) U \oplus \cdots \oplus \left(\begin{smallmatrix} 0 & s_\ell \\ 0 & t_\ell \end{smallmatrix}\right) U$. Since U is right PP, each $\left(\begin{smallmatrix} 0 & s_j \\ 0 & t_j \end{smallmatrix}\right) U$ is projective, so I is a projective right ideal. Hence U is right semihereditary.

By Lemma 3.8, V is right hereditary. Since both U and V have finite right uniform dimension, Corollary 1.10 yields that if T has U or V as a subring, then T must be right semihereditary. Now arguments for parts (ii) and (iii) hold when "right extending" is replaced by "right semihereditary." □

We remark that U and V, in Corollary 3.9, are right extending α pseudo right ring hulls of R, whereas $Q(R) = R(\mathfrak{E}, Q(R))$. Moreover, if $\{p_1, p_2, \ldots\}$ is an infinite set of distinct primes of A and

$$V_i = \begin{pmatrix} A & A \\ p_1 \cdots p_i A & A \end{pmatrix},$$

then $V_1 \supseteq V_2 \supseteq \cdots$ forms an infinite descending chain of right extending α pseudo right ring hulls none of which contains U. Thus no V_i is a right extending right ring hull.

Corollary 3.10. *Let A be a commutative PID with F as its field of fractions, $A \neq F$, and let T be a right ring of quotients of $R = T_2(A)$. Take*

$$S = \begin{pmatrix} A & F \\ 0 & F \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} A & (p_1^{k_1-1} \cdots p_m^{k_m-1})^{-1} A \\ p_1^{k_1} \cdots p_m^{k_m} A & A \end{pmatrix},$$

where each p_i is a distinct prime of A.

(i) *If T is right hereditary, then either S or V is a subring of T. The converse holds when T is right Noetherian.*
(ii) *S is the unique right hereditary right ring hull of R; but R has no right hereditary absolute right ring hull.*

Proof. (i) This is a direct consequence of Corollary 3.9(i) and (iv).

(ii) A routine argument shows that S is right Noetherian [21, p. 114, Exercise 15]. By part (i), S is right hereditary. Assume that Λ is a right hereditary right ring of quotients of R such that Λ is a proper subring of S. Then $\Lambda = \left(\begin{smallmatrix} A & F \\ 0 & B \end{smallmatrix}\right)$, where B is a proper subring of F which is an overring of A. By Corollary 3.9, Λ is right extending. From [17, Corollary 10.6(i)], Λ is right Noetherian. Let $0 \neq b \in B$ such that $b^{-1} \notin B$. We see that the B-submodules of F of the type $b^{-n}B$ form a strictly ascending chain (as n increases where n is a positive integer). Thus $\left(\begin{smallmatrix} 0 & b^{-n}B \\ 0 & 0 \end{smallmatrix}\right)$ form a strictly ascending chain of right ideals of Λ, a contradiction. Therefore S is a right hereditary right ring hull of R.

For uniqueness, let H be another distinct right hereditary right ring hull of R. By part (i), $H = V$ for some nonzero $a = p_1^{k_1} \cdots p_m^{k_m}$ where each p_i is a distinct prime of A. Take H_1 as in the proof of Corollary 3.9(ii). Then by the same method as was used in the proof of Lemma 3.8, H_1 is isomorphic to W. Hence H_1 is right hereditary. But H_1 is a proper subring of H, a contradiction. By an argument similar to that used in Corollary 3.9(iii), R has no right hereditary absolute right ring hull. □

The following result provides an answer to Problem I of Section 1 for the case when $\mathfrak{K} = \mathfrak{E}$ and $R = T_2(W)$ by characterizing the right extending right rings of quotients which are intermediate between $T_2(W)$ and $\mathrm{Mat}_2(W)$, where W is from a large class of local right finitely Σ-extending rings (see [17] for finitely Σ-extending modules).

Theorem 3.11. *Let W be a local ring, V a subring of W with $\mathbf{J}(W) \subseteq V$, $R = \left(\begin{smallmatrix} V & W \\ 0 & W \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} V & W \\ \mathbf{J}(W) & W \end{smallmatrix}\right)$, and $T = \mathrm{Mat}_2(W)$. Then we have the following.*

(i) *For each $e \in \mathbf{I}(T)$, there exists $f \in \mathbf{I}(S)$ such that $e \,\alpha\, f$.*
(ii) *$S \in \mathfrak{E}$ if and only if $T \in \mathfrak{E}$ if and only if $S = R(\mathfrak{E}, \rho, T)$ for some ρ.*
(iii) *If W is right self-injective, then $S = R(\mathfrak{E}, \alpha, T)$, and $Q_{\mathfrak{q}\mathfrak{Con}}(R) = R(\mathfrak{E}, T) = T$.*

(iv) *If* $T \in \mathfrak{E}$ (*respectively,* W *is right self-injective*) *and at least one of the following conditions is satisfied, then* $S = Q^T_{\mathfrak{E}}(R)$ (*respectively,* $S = Q_{\mathfrak{E}}(R)$):
  (a) $\mathbf{J}(W) \subseteq \mathrm{Cen}(W)$;
  (b) $\mathbf{U}(W) \subseteq \mathrm{Cen}(W)$;
  (c) $\mathbf{J}(W)$ *is nil*;
  (d) W *is right nonsingular.*

(v) *Assume that* $S = Q^T_{\mathfrak{E}}(R)$ *and* M *is an intermediate ring between* R *and* T. *Then* $M \in \mathfrak{E}$ *if and only if* $M = \left(\begin{smallmatrix} A & W \\ \mathbf{J}(W) & W \end{smallmatrix}\right)$ *or* $M = T$, *where* A *is an intermediate ring between* V *and* W.

(vi) $R \in \mathfrak{F}\mathfrak{I}$ *if and only if* $W \in \mathfrak{F}\mathfrak{I}$.

Proof. Since $B = \left(\begin{smallmatrix} 0 & W \\ 0 & W \end{smallmatrix}\right)$ is a left ideal of T such that $\ell_T(B) = 0$ and $B \subseteq R$, [27, pp. 380–381, Exercise 9] yields that $Q(R) = Q(S) = Q(T)$ and $E(R_R) = E(S_S) = E(T_T)$.

(i) Let $e = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{I}(T)$. Then

(1) $a^2 + bc = a$;
(2) $d^2 + cb = d$;
(3) $ab + bd = b$; and
(4) $ca + dc = c$.

We need only consider the following cases.

Case 1. Assume that c is invertible. Then $e = \left(\begin{smallmatrix} a & b \\ c & c(1-a)c^{-1} \end{smallmatrix}\right)$ from (4). Take $f = \left(\begin{smallmatrix} 0 & ac^{-1} \\ 0 & 1 \end{smallmatrix}\right)$. Then $f \in \mathbf{I}(R)$ and $e \cdot f = \left(\begin{smallmatrix} 0 & a^2c^{-1}+b \\ 0 & 1 \end{smallmatrix}\right) = f$ because $a^2c^{-1} + b = (a^2 + bc)c^{-1} = ac^{-1}$. Also $f \cdot e = \left(\begin{smallmatrix} a & a(1-a)c^{-1} \\ c & c(1-a)c^{-1} \end{smallmatrix}\right) = e$ since $a(1-a)c^{-1} = (a - a^2)c^{-1} = bcc^{-1} = b$. Therefore $e \,\alpha\, f$.

Case 2. Assume that $c \in \mathbf{J}(W)$ and $a \notin V$. Then $bc = a(1-a) \in \mathbf{J}(W)$. Since a is invertible, $1 - a \in \mathbf{J}(W) \subseteq V$. But $1 \in V$, so $a \in V$, a contradiction.

(ii) If $S \in \mathfrak{E}$, then $T \in \mathfrak{E}$ by Corollary 1.8(ii). Now assume that $T \in \mathfrak{E}$ and $X_S \leqslant S_S$. Then there exists $e \in \mathbf{I}(T)$ such that $XT_T \leqslant^{\text{ess}} eT_T$. From Lemma 1.4(i), $XT_S \leqslant^{\text{ess}} eT_S$. Since S_S is dense in T_S, $X_S \leqslant^{\text{ess}} XT_S$ by an argument similar to that used in the proof of Lemma 1.4(ii). By part (i), there is $f \in \mathbf{I}(S)$ with $e \,\alpha\, f$ in $\mathbf{I}(T)$. Hence $eT = fT$. So $X_S \leqslant^{\text{ess}} fS_S$. Thus $S \in \mathfrak{E}$. The remainder of the proof of this part follows from Lemma 2.22(ii) and that

$$S = \left\langle R \cup \left\{ \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix} \,\middle|\, c \in \mathbf{J}(W) \right\} \right\rangle_T .$$

(iii) Observe that $\left(\begin{smallmatrix} 1 & 0 \\ x & 0 \end{smallmatrix}\right) \alpha \left(\begin{smallmatrix} 1 & 0 \\ y & 0 \end{smallmatrix}\right)$ if and only if $x = y$. Thus, by part (i), there is a $\delta^{\alpha}_{\mathfrak{E}}(R)(1)$ with

$$\left\{ \begin{pmatrix} 1 & 0 \\ d & 0 \end{pmatrix} \,\middle|\, d \in \mathbf{J}(W) \right\} \subseteq \delta^{\alpha}_{\mathfrak{E}}(R)(1) \subseteq \mathbf{I}(S).$$

Hence $S = \langle R \cup \delta^{\alpha}_{\mathfrak{E}}(R)(1)\rangle_S$. So, from Corollary 2.18(i), $S = R(\mathfrak{E}, \alpha, T)$. By Proposition 2.3(iii),

$$\left\{ \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix} \,\middle|\, c \in W \right\} \subseteq Q_{\mathfrak{q}\mathfrak{Con}}(R).$$

Thus $Q_{\mathfrak{q}\mathfrak{Con}}(R) = T = R(\mathfrak{E}, T)$.

(iv) Let M be an intermediate ring between R and T. Then there is an additive subgroup I of W with $IA+WI \subseteq I$ and $A+WI \subseteq A$. Thus I is a left ideal of W. Also A is a subring of W with $V \subseteq A$ and $M = \left(\begin{smallmatrix} A & W \\ I & W \end{smallmatrix}\right)$. Assume that $M \in \mathfrak{E}$ and that $S \nsubseteq M$. Then $I \subsetneq \mathbf{J}(W)$ since W is local. Let $t \in \mathbf{J}(W)$ such that $t \notin I$. Then there is $e = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{I}(M)$ with $\left(\begin{smallmatrix} 0 & 1 \\ 0 & t \end{smallmatrix}\right)M_M \leqslant^{\text{ess}} eM_M$. Thus $e\left(\begin{smallmatrix} 0 & 1 \\ 0 & t \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & t \end{smallmatrix}\right)$. Hence a, b, c, d satisfy Eqs. (1)–(4) in the proof of part (i) and the equations:

(5) $a + bt = 1$ and
(6) $c + dt = t$.

From (5), $bt = 1 - a \in \mathbf{J}(W)$, so a is invertible. By (6), $c = (1-d)t \in \mathbf{J}(W)$. If $1-d$ is invertible, then $t \in Wc \subseteq WI \subseteq I$, a contradiction. Hence $1 - d \in \mathbf{J}(W)$, so d is invertible. If b is invertible, then $d = b^{-1}(1-a)b \in \mathbf{J}(W)$ by (3), a contradiction. Thus $b \in \mathbf{J}(W)$.

Claim 1. *If $bc = 0$, then $S \subseteq M$.*

Proof. From (1), $a = 1$. Then (3) implies $bd = 0$. Since d is invertible, $b = 0$. Hence (2) implies that $d = 1$. Then (4) yields that $c = 0$. Thus $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Note that $\left(\begin{smallmatrix} 0 & 1 \\ 0 & t \end{smallmatrix}\right)M \cap \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)M = 0$. So $\left(\begin{smallmatrix} 0 & 1 \\ 0 & t \end{smallmatrix}\right)M_M$ is not essential in $eM_M = M_M$, a contradiction. Therefore if $M \in \mathfrak{E}$, then $S \subseteq M$. □

Claim 2. *If $\mathbf{J}(W) \subseteq \mathrm{Cen}(W)$ or $\mathbf{U}(W) \subseteq \mathrm{Cen}(W)$, then $bc = 0$.*

Proof. Multiply both right sides of (3) by c and use the fact that $\mathbf{J}(W) \subseteq \mathrm{Cen}(W)$ or $\mathbf{U}(W) \subseteq \mathrm{Cen}(W)$ to obtain $bc(a+d) = bc$. Hence $bc(1-(a+d)) = 0$. Suppose that $1-(a+d) = j \in \mathbf{J}(W)$. Then $d = (1-a) - j \in \mathbf{J}(W)$, a contradiction. Hence $1-(a+d)$ is invertible. Therefore $bc = 0$. □

Claim 3. *If $\mathbf{J}(W)$ is nil, then $bc = 0$.*

Proof. From (3) and (5), $bd = (1-a)b = btb$. Hence $b = btbd^{-1} = (btbd^{-1})tbd^{-1} = b(tbd^{-1})^n$. Since $\mathbf{J}(W)$ is nil, $b = 0$. Thus $bc = 0$. □

Claim 4. *If W is right nonsingular, then $bc = 0$.*

Proof. Since $T \in \mathfrak{E}$, $W \in \mathfrak{E}$ [17, Lemma 12.8]. So W is a domain. As in the proof of Claim 3, $bd = btb$. If $b \neq 0$, then $d = tb \in \mathbf{J}(W)$, a contradiction. Therefore $0 = b = bc$. □

Thus by Claims 1–4, $S = Q^T_{\mathfrak{E}}(R)$ (respectively, $S = Q_{\mathfrak{E}}(R)$) when either $\mathbf{J}(W) \subseteq \mathrm{Cen}(W)$, $\mathbf{J}(W)$ is nil, $\mathbf{U}(W) \subseteq \mathrm{Cen}(W)$, or W is right nonsingular.

(v) This part follows from Corollary 1.8(ii).

(vi) This is a consequence of [11, Corollary 1.6]. □

Recall that if W is a local ring such that W_W is finitely Σ-extending [17, Lemma 12.8], then $\mathrm{Mat}_n(W) \in \mathfrak{E}$ for all positive integers n. The class of local commutative Prüfer domains is a class of local rings which are finitely Σ-extending [17, Corollary 12.10].

Corollary 3.12. *Let $D[\![x, \psi]\!]$ be the skew formal power series ring over a division ring D with ψ an automorphism of D and V a subring of D. If*

$$R=\begin{pmatrix} V+xD[\![x,\psi]\!] & D[\![x,\psi]\!] \\ 0 & D[\![x,\psi]\!] \end{pmatrix}, \quad \textit{then}$$

$$Q_{\mathfrak{E}}^{T}(R)=\begin{pmatrix} V+xD[\![x,\psi]\!] & D[\![x,\psi]\!] \\ xD[\![x,\psi]\!] & D[\![x,\psi]\!] \end{pmatrix}, \quad \textit{where } T=\mathrm{Mat}_2\big(D[\![x,\psi]\!]\big).$$

Proof. Since $D[\![x,\psi]\!]$ is a right hereditary Noetherian local domain, [17, Corollary 12.18] yields that $D[\![x,\psi]\!]$ is right finitely Σ-extending. The result is now a consequence of [17, Lemma 12.8] and Theorem 3.11. □

We note that $Q_{\mathfrak{E}}^{T}(R)$ in Corollary 3.12 is a Baer ring by [15, Theorem 1.1].

Corollary 3.13. *Assume that W is a local ring and V is a subring of W. Let $R=\left(\begin{smallmatrix} V & W \\ 0 & W\end{smallmatrix}\right)$. Then the following are equivalent.*

(i) *R is right extending.*
(ii) *$T_2(W)$ is right extending.*
(iii) *W is a division ring.*

Proof. As in the proof of Theorem 3.11, $Q(\mathrm{Mat}_2(W))=Q(R)$.

(i) ⇒ (ii). This follows from Corollary 1.8(ii).

(ii) ⇒ (iii). Suppose that $0\neq d\in \mathbf{J}(W)$. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix} T_2(W)\cap \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T_2(W)=0,$$

$\left(\begin{smallmatrix} 0 & 1 \\ 0 & d\end{smallmatrix}\right)T_2(W)_{T_2(W)}$ is not essential in $T_2(W)_{T_2(W)}$. So there is $e=\left(\begin{smallmatrix} 0 & y \\ 0 & 1\end{smallmatrix}\right)\in \mathbf{I}(T_2(W))$ with

$$\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix} T_2(W)_{T_2(W)} \leqslant^{\mathrm{ess}} \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} T_2(W)_{T_2(W)}.$$

Then

$$\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix} = e\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & yd \\ 0 & d \end{pmatrix}.$$

Hence $yd=1$, a contradiction. Thus $\mathbf{J}(W)=0$, so W is a division ring.

(iii) ⇒ (i). Since W is a division ring, $\mathrm{Mat}_2(W)\in\mathfrak{E}$. From Theorem 3.11(ii), $R\in\mathfrak{E}$. □

The following example illustrates Theorem 3.11 and right ring hulls for several classes of rings.

Example 3.14. In Theorem 3.11, take $V=W$ to be a local right self-injective ring with $\mathbf{J}(W)$ nilpotent and nonzero (e.g., W can be a local QF-ring with $\mathbf{J}(W)\neq 0$). Let $S_0=\left(\begin{smallmatrix} W & \mathbf{J}(W) \\ 0 & W\end{smallmatrix}\right)$, $S_1=R$, $S_2=S$ and $S_3=T$. Then $S_0\subseteq S_1\subseteq S_2\subseteq S_3$ is an ascending chain of subrings of S_3 where each ring is a right essential overring of its predecessor. Note that ${S_0}_{S_0}\leqslant^{\mathrm{ess}} {S_1}_{S_0}$ and ${S_1}_{S_1}$ is dense in ${S_3}_{S_1}$, but ${S_0}_{S_0}$ is not essential in ${S_2}_{S_0}$. By [11, Theorem 1.4], S_0 is not right FI-extending; but it is right Kasch, so $S_0=Q(S_0)$ by [27, Corollary 13.24]. From [11, Theorem 1.4] and Theorem 3.11, we have that $S_1 = \widetilde{Q}_{\mathfrak{FI}}(S_0)$, $S_2 = Q_{\mathfrak{E}}(S_1)$, and $S_3 = Q_{\mathfrak{SI}}(S_1) = Q_{\mathfrak{SI}}(S_2)$. Observe that this chain of right essential overrings must terminate at S_3 since S_3 is right self-injective.

Note that in Theorem 3.11 and in the following result we can construct some $\mathfrak{C}$ right ring hulls for rings which may or may not be right nonsingular.

Proposition 3.15. *Let $A \in \mathfrak{FI}$, $M = W = \bigoplus_{i=1}^{n} A_i$, $A_i = A$ for each i, and S a subring of W containing $D = \{(a_1, \ldots, a_n) \in W \mid$ for some $a \in A$, $a_i = a$ for all $i = 1, \ldots, n\}$. Then the ring $H = \left(\begin{smallmatrix} W & M \\ 0 & A \end{smallmatrix}\right)$ is a right FI-extending right ring hull of $R = \left(\begin{smallmatrix} S & M \\ 0 & A \end{smallmatrix}\right)$.*

Proof. Assume that ${}_W N_A \leqslant {}_W M_A$. Then $N = \bigoplus_{i=1}^{n} I_i$, where each $I_i \trianglelefteq A_i$. Since $A \in \mathfrak{FI}$, there is $e_i \in \mathbf{I}(A)$ with $I_{i\,A} \leqslant^{\text{ess}} e_i A_A$. Let $e = (e_1, \ldots, e_n) \in W$. Then $N_A \leqslant^{\text{ess}} eM_A$. By [11, Corollary 1.6], $H \in \mathfrak{FI}$. Note that H is a right essential overring of R. Next assume that U is a right FI-extending intermediate ring between R and H. Then $U = \left(\begin{smallmatrix} V & M \\ 0 & A \end{smallmatrix}\right)$, where V is a subring of W. So A_i is a (V, A)-bisubmodule of ${}_V M_A$. By [11, Corollary 1.6], there is $f = f^2 \in V$ with $A_{i\,A} \leqslant^{\text{ess}} fM_A$. Since $A_{i\,A}$ is closed in M_A, $A_{i\,A} = fM_A$. Note that fD is the ith component of W. But $fD \subseteq V$. Hence $V = W$. Thus $U = H$, so $H = \widetilde{Q}_{\mathfrak{FI}}(R)$. $\square$

We conclude with an example illustrating Proposition 3.15.

Example 3.16. Let A be a field and

$$R = \begin{pmatrix} A & A \oplus A \\ 0 & A \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & c \end{pmatrix} \middle|\, a, c, x, y \in A \right\}.$$

(i) Then $R \in \mathfrak{q}\mathfrak{B}$ and $Z(R_R) = 0$. But, by using [11, Corollary 1.6 and Theorem 3.2], R is not right FI-extending (hence the converse of Proposition 1.2(ii) does not hold).

(ii) Let

$$H_1 = \begin{pmatrix} A \oplus A & A \oplus A \\ 0 & A \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix} \middle|\, a, b, c, x, y \in A \right\}$$

and let

$$H_2 = \left\{ \begin{pmatrix} a+b & a & x \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix} \middle|\, a, b, c, x, y \in A \right\}.$$

Note that R, H_1 and H_2 are subrings of $\text{Mat}_3(A)$. Define $\phi : H_1 \to H_2$ by

$$\phi\left[\begin{pmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix}\right] = \begin{pmatrix} a & a-b & x-y \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix}.$$

Then ϕ is a ring isomorphism. By Proposition 3.15, H_1 and H_2 are right FI-extending right ring hulls of R such that $H_1 \cap H_2 = R$. Thus, in general, the intersection of right FI-extending right ring hulls is *not* a right FI-extending absolute right ring hull.

(iii) Note that

$$R(\mathfrak{FI}, Q(R)) = \begin{pmatrix} A & A & A \\ A & A & A \\ 0 & 0 & A \end{pmatrix} \neq \mathrm{Mat}_3(A) = Q_{\mathfrak{qCon}}(R) = E(R_R).$$

(iv) We see that H_1 and H_2 are properly contained in $T_3(A) = \bigcap_\alpha R(\mathfrak{FI}, \alpha, Q(R))$. Thus, we have right FI-extending right ring hulls properly contained in the intersection of all right FI-extending α pseudo right ring hulls.

(v) Let

$$H_3 = \left\{ \begin{pmatrix} a+b & b & x \\ b & a & y \\ 0 & 0 & c \end{pmatrix} \;\middle|\; a, b, c, x, y \in A \right\}.$$

Note that H_3 can be represented by the generalized matrix ring $\left(\begin{smallmatrix} B & M \\ 0 & A \end{smallmatrix}\right)$, where $B = \{\left(\begin{smallmatrix} a+b & b \\ b & a \end{smallmatrix}\right) \mid a, b \in A\}$ is a ring; and $M = \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mid x, y \in A\}$ is a (B, A)-bimodule. From [11, Corollary 1.6], H_3 is a right FI-extending right ring hull of R if and only if either:

(a) ${}_B M_A$ has 0 as its only proper (B, A)-bisubmodule; or
(b) there is $0 \neq {}_B N_A \leqslant {}_B M_A$ and $f \in \mathbf{I}(B)$ with $N = fM$ and $\dim(N_A) = 1$.

Thus if $A = \mathbb{Z}_2$, then H_3 is also a right FI-extending right ring hull of R. But H_3 is *not* a right FI-extending ρ pseudo right ring hull of R for any equivalence relation ρ on $\delta_{\mathfrak{FI}}(R)$. Also we see that $H_3 \ncong H_1$.

Open problems.

(i) Characterize the classes $\mathfrak{K}$ of rings such that each ring in $\mathfrak{K}$ has a right self-injective or right continuous (absolute) right ring hull, respectively.
(ii) Assume that T is a right essential overring of a ring R. Find some interesting property P such that if R has P, then T has P.
(iii) Fix a class $\mathfrak{K}$ of rings. Determine those rings R such that $Q(R)$ is semisimple Artinian and $R \in \mathfrak{K}$. (In particular, consider $\mathfrak{K} = \mathfrak{FI}$ or $\mathfrak{K} = \mathfrak{qB}$.)

Motivated by Example 3.14, we give the next definition for problem (iv): An overring S of R is a *generalized right essential overring of* R if there exists a finite chain $R = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S$ of subrings such that S_{i+1} is a right essential overring of S_i. Note that any such chain of right essential overrings will terminate when S_i is right self-injective.

(iv) Determine necessary and sufficient conditions for R to have a maximal generalized right essential overring.

Note added in proof

In the comment after the proof of Theorem 3.1, if the division rings D_i are fields, a method for calculating $\mathbf{I}(\mathrm{Mat}_n(D_i))$ appears in [1].

Acknowledgments

The authors thank the referee and Professor L.W. Small for their helpful suggestions and comments for improving the results of this paper. The authors appreciate the gracious hospitality they received at each others' institutions. The partial support received from the Mathematics Research Institute, Columbus, OSU-Lima grants, and Busan National University is appreciated.

References

[1] C. Barnett, V. Camillo, Idempotents in matrix rings, Proc. Amer. Math. Soc. 122 (1994) 965–969.
[2] K. Beidar, R. Wisbauer, Strongly and properly semiprime modules and rings, in: S.K. Jain, S.T. Rizvi (Eds.), Ring Theory, Proc. Ohio State-Denison Conf., World Scientific, Singapore, 1993, pp. 58–94.
[3] S.K. Berberian, Baer $*$-Rings, Springer-Verlag, Berlin, 1972.
[4] G.F. Birkenmeier, A generalization of FPF rings, Comm. Algebra 17 (1989) 855–884.
[5] G.F. Birkenmeier, When does a supernilpotent radical split off?, J. Algebra 172 (1995) 49–60.
[6] G.F. Birkenmeier, H.E. Heatherly, J.Y. Kim, J.K. Park, Triangular matrix representations, J. Algebra 230 (2000) 558–595.
[7] G.F. Birkenmeier, J.Y. Kim, J.K. Park, On quasi-Baer rings, in: D.V. Huynh, S.K. Jain, S.R. López-Permonth (Eds.), Algebras and Its Applications, in: Contemp. Math., vol. 259, Amer. Math. Soc., Providence, RI, 2000, pp. 67–92.
[8] G.F. Birkenmeier, J.Y. Kim, J.K. Park, Polynomial extensions of Baer and quasi-Baer rings, J. Pure Appl. Algebra 159 (2001) 25–42.
[9] G.F. Birkenmeier, B.J. Müller, S.T. Rizvi, Modules in which every fully invariant submodule is essential in a direct summand, Comm. Algebra 30 (2002) 1395–1415.
[10] G.F. Birkenmeier, J.K. Park, Triangular matrix representations of ring extensions, J. Algebra 265 (2003) 457–477.
[11] G.F. Birkenmeier, J.K. Park, S.T. Rizvi, Generalized triangular matrix rings and the fully invariant extending property, Rocky Mountain J. Math. 32 (2002) 1299–1319.
[12] G.F. Birkenmeier, J.K. Park, S.T. Rizvi, Ring hulls determined by central idempotents, preprint.
[13] K.A. Brown, The singular ideals of group rings, Quart. J. Math. Oxford 28 (1977) 41–60.
[14] A.W. Chatters, C.R. Hajarnavis, Rings in which every complement right ideal is a direct summand, Quart. J. Math. Oxford 28 (1977) 61–80.
[15] A.W. Chatters, S.M. Khuri, Endomorphism rings of modules over non-singular CS rings, J. London Math. Soc. 21 (1980) 434–444.
[16] W.E. Clark, Twisted matrix units semigroup algebras, Duke Math. J. 34 (1967) 417–424.
[17] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, Extending Modules, Longman, 1994.
[18] C. Faith, S. Page, FPF Ring Theory: Faithful Modules and Generators of Mod-R, London Math. Soc. Lecture Note Ser., vol. 88, Cambridge Univ. Press, Cambridge, 1984.
[19] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, New York, 1972.
[20] V.K. Goel, S.K. Jain, π-Injective modules and rings whose cyclics are π-injective, Comm. Algebra 6 (1978) 59–73.
[21] K.R. Goodearl, Ring Theory: Nonsingular Rings and Modules, Marcel Dekker, New York, 1976.
[22] K.R. Goodearl, Von Neumann Regular Rings, Krieger, Malabar, 1991.
[23] M. Harada, On modules with extending property, Osaka J. Math. 19 (1982) 203–215.
[24] S.S. Holland Jr., Remarks on type I Baer and Baer $*$-rings, J. Algebra 27 (1973) 516–522.
[25] I. Kaplansky, Rings of Operators, Benjamin, New York, 1968.
[26] I. Kaplansky, Commutative Rings, Univ. of Chicago Press, Chicago, 1974.
[27] T.Y. Lam, Lectures on Modules and Rings, Springer-Verlag, Berlin, 1999.
[28] J. Lambek, Lectures on Rings and Modules, Chelsea, New York, 1986.
[29] N.C. Lang, On ring properties of injective hulls, Canad. Math. Bull. 18 (1975) 233–239.
[30] H. Lenzing, Halberbliche Endomorphismenringe, Math. Z. 118 (1970) 219–240.
[31] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, Wiley, Chichester, 1987.
[32] A.C. Mewborn, Regular rings and Baer rings, Math. Z. 121 (1971) 211–219.
[33] B.J. Müller, S.T. Rizvi, On the existence of continuous hulls, Comm. Algebra 10 (1982) 1819–1838.
[34] A. Pollingher, A. Zaks, On Baer and quasi-Baer rings, Duke Math. J. 37 (1970) 127–138.
[35] I. Reiner, Maximal Orders, Academic Press, London, 1975.
[36] S.T. Rizvi, Contributions to the theory of continuous modules, Doctoral dissertation, McMaster University, 1981.
[37] L.H. Rowen, Ring Theory I, II, Academic Press, San Diego, 1988.

[38] L.W. Small, Semihereditary rings, Bull. Amer. Math. Soc. 73 (1967) 656–658.
[39] Y. Utumi, On quotient rings, Osaka J. Math. 8 (1956) 1–18.
[40] Y. Utumi, On continuous regular rings and semisimple self injective rings, Canad. J. Math. 12 (1960) 597–605.
[41] Y. Utumi, On continuous rings and selfinjective rings, Trans. Amer. Math. Soc. 118 (1965) 158–173.
[42] K.G. Wolfson, Baer rings of endomorphisms, Math. Ann. 143 (1961) 19–28.