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THE BINOMIAL THEOREM: A WIDESPREAD CONCEPT IN MEDIEVAL ISLAMIC MATHEMATICS

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SUMMARIES

An example illustrating the expansion of the binomial theorem up to the seventh power of the sum of two quantities can be found in a work by al-Zanjānī (d. 1262). While the binomial theorem is presumed to have been discovered by al-Karajī (ca. 1029) and utilized by several subsequent mathematicians, the elaboration of the theorem by al-Zanjānī points to the fact that it was a widespread concept in medieval Islamic mathematics. Al-Zanjānī's exposition of this theorem is herewith translated from the original Arabic.

نمونه‌ای از بسط قضیّه دوجمله‌ایها که روش محاسبه و بتوان رسانیدن مجموع دو مقدار را تا قوه هفتم نشان میدهد در نوشته‌های الزنجانی (۱۲۶۲ میلادی) میتوان بدست آورد. با اینکه قضیه فوق توسط الکرجی (۱۰۲۹ میلادی) کشف و مورد استفاده چندین ریاضی‌دان پس از او قرار گرفته است، تشریح آن توسط الزنجانی نمایانگر آنست که قضیّه دوجمله‌ایها اهمیت بسزائی در ریاضیات قرون وسطی داشته است. شرحی که الزنجانی در مورد این قضیه بزبان عربی نوشته است در اینجا ترجمه شده است.

L'oeuvre d'Al-Zanjānī (m. 1262) contient un exemple de l'extension à la septième puissance du théorème du binôme. Le théorême du binôme est présumé avoir été découvert par al-Karajī (m. circa 1029) et avoir été employé ultérieurement par plusieurs mathématiciens. L'extension du théorème par Al-Zanjānī nous amène à le supposer d'un emploi courant dans les mathématiques islamiques médiévales. Nous traduisons, du texte arabe original, le travail d'Al-Zanjānī.

The table of binomial coefficients, its formation law $C_n^i = C_n^{i-1} + C_{n-1}^i$, and the expansion $(a + b)^n = \sum_{i=0}^{n} C_n^i a^{n-i} b^n$ for integer n can be found in a text by al-Karajī (ca. 1029) reported by al-Samaw'al (d. 1175) in his *al-Bāhir*. Roshdi Rashed believes this to be the first known book in which the binomial theorem is elaborated [Rashed 1972, 3].

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Several Muslim mathematicians discussed the binomial theorem after its presumed discovery by al-Karajī. Among them were ^{c}Umar Khayyam (ca. 1131), al-Zanjānī (d. 1262), al-Ṭūsī (d. 1274), Ibn al-Bannā' (d. 1321), and Ibn Zurayq, who flourished at the end of the 14th century.

^{c}Umar Khayyām claimed to have discovered the law for the expansion of $(a + b)^n$ which he used to find the fourth, fifth, sixth, and higher roots of numbers. He asserted:

> *I have composed a book demonstrating the soundness of these methods leading to the discovery of required values and I have added methods for the solution of various other types--I refer to extraction of the sides of the square of the square, the square of the cube, and the cube of the cube, etc.--all of which is new. These proofs are arithmetical* ... [Kasir 1931, 54].

^{c}Umar Khayyām was referring to his book, *Mushkilāt al-Ḥisāb* [Problems of Arithmetic], which is yet unrecovered. However, his rule for the binomial expansion of $(a+b)^n$ up to $n = 12$ was given by al-Ṭūsī (who was greatly influenced by ^{c}Umar Khayyām) in his book, *Jawāmi^{c} al-Ḥisāb* [Youschkevitch 1970, 326].

In an Arabic manuscript on mathematics written by al-Zanjānī, I came across a page which explains the expansion of the binomial theorem up to the seventh power of the sum of two quantities. Al-Zanjānī is well known to historians as a jurisprudent, but his work in mathematics has been neglected. This particular work merits attention as an example illustrating the widespread use of the binomial theorem in Islamic mathematical circles. It is well-known that al-Zanjānī was influenced by al-Karajī. An examination of al-Zanjānī's work will shed some light on the spread of mathematical knowledge within the span of two centuries (from al-Karajī to al-Zanjānī).

Al-Zanjānī's text is entitled *Qusṭās al-Mu^{c}ādala fī ^{c}Ilm al-Jabr wa al-Muqābala* [The Balance of Equation(s) in the Science of Algebra] and is dated 696 A.H. (1297 A.D.). The author describes himself in his introduction, saying that he studied mathematics when young and had dedicated part of his life to that science by dictating many treatises containing original ideas. He complains that his work was made well known to students, some of whom plagiarized his work, claiming it as their own.

It is understandable that al-Zanjānī dedicated only a part of his life to the study of mathematics. Biographical sketches by al-Khwānsārī, al-Baghdādī, and others describe ^{c}Izz al-Dīn ^{c}Abd al-Wanhhāb ibn Ibrāhīm ibn Muḥammad al-Jurjānī al-Zanjānī al-Khazrajī (d. 660 A.H./1262 A.D.) as a *faqīh* [jurisprudent], grammarian, linguist, etymologist, and prosodist who also had a good knowledge of other sciences. Not much is known about his life except that he settled in Tabriz, lived in Mosul, and died in Baghdad. He wrote many books on language, poetry, and grammar.

Al-Zanjānī proved himself a competent mathematician in his *Qustās al-Mu ᶜādala*. In addition to this text, a second manuscript on mathematics entitled *ᶜUmdat al- Ḥisāb* (No. 3145) is extant in the same library in Istanbul [Karatay 1962, 737]. Al-Zanjānī makes references to other mathematical works such as *Kitāb al-Burhān,* of which no information is presently available.

This unique text, *Qusṭas al-Mu ᶜadala,* was found in the Ahmad III library (No. 3457) in Topkapu Saray in Istanbul [Karatay 1962, 737] by the late Dr. Martin Levey [1]. It is in large handwriting with, for the most part, twenty-one lines on a page. It is divided into ten chapters dealing with addition, subtraction, multiplication, division, fractions, powers, etc. The total number of folios is 232.

As was customary with scientific works of the period, the text has no mathematical notation, and is completely verbal and rhetorical. There are no numerals , no vowel signs, and no punctuation whatsoever. In the fashion characteristic of his predecessors, al-Khwārazmī and Abū Kāmil, al-Zanjānī called an unknown a "thing." In the text, the square of the thing is called *māl*. Cube is referred to as *kaᶜb*. x^4 is called *māl māl*. In the same way, *mal kaᶜb* is x^5, *kaᶜb kaᶜb* is x^6, and *mal mal kaᶜb* is x^7. For convenience, I shall use the modern terminology.

The book exhibits a systematic approach to the subjects treated: i.e., the author uses definitions, theorems, and generalizations. Yet there are no formal proofs of any of the problems treated. Al-Zanjānī acknowledges this at the beginning of the book, requesting students not to be concerned with this point. He maintains that the place of proof is in the science of geometry. He prays that God Almighty will make it possible for him to write another book, in which he will furnish both geometrical and algebraic proofs of his problems.

Al-Zanjānī's discussion of the binomial theorem begins at the end of Chapter 6. At first glance, it looks as though al-Zanjānī were trying to explain how to find the roots of algebraic numbers. However, upon closer examination it is clear that the author is using the binomial theorem inversely to extract the square root of a polynomial, a method described originally by al-Karajī [Rashed 1970, 241].

A comparative study of al-Zanjānī's Chapter 6 with that of *al-Bāhir* (pp. 63-71 of the Arabic text, pp, 28-36 of the French translation) shows clearly that al-Zanjānī was well aware of the works of his predecessors: al-Karajī and al-Samaw'al. In some instances, al-Zanjānī used exactly the same words and examples.

The methods for finding the square root of a polynomial are those reported in *al-Bāhir* (pp. 63-64 of the Arabic text, p. 29 of the French translation) and therefore will not be repeated here.

Al-Zanjānī begins on line 4 of folio 25b to give a detailed exposition of the binomial theorem. He writes, "We shall end this chapter by discussing the procedure for the expansion of algebraic expressions. This will enable us to factor such expressions in case we need to." Then he proceeds with the expansion of $(a+b)^n$ up to $n = 7$. For comparison with al-Karajī's work, a translation of al-Zanjānī's expansion $(a+b)^7$ is provided as follows:

> [Line 14, fol. 25b] *The seventh power of the sum is equal to the seventh power of each of the terms plus the product of each one by the sixth power of the other taken seven times plus the product of the square of each term by the fifth power of the other taken twenty-one times plus the product of the cube of each term by the fourth power of the other taken thirty-five times* [2].

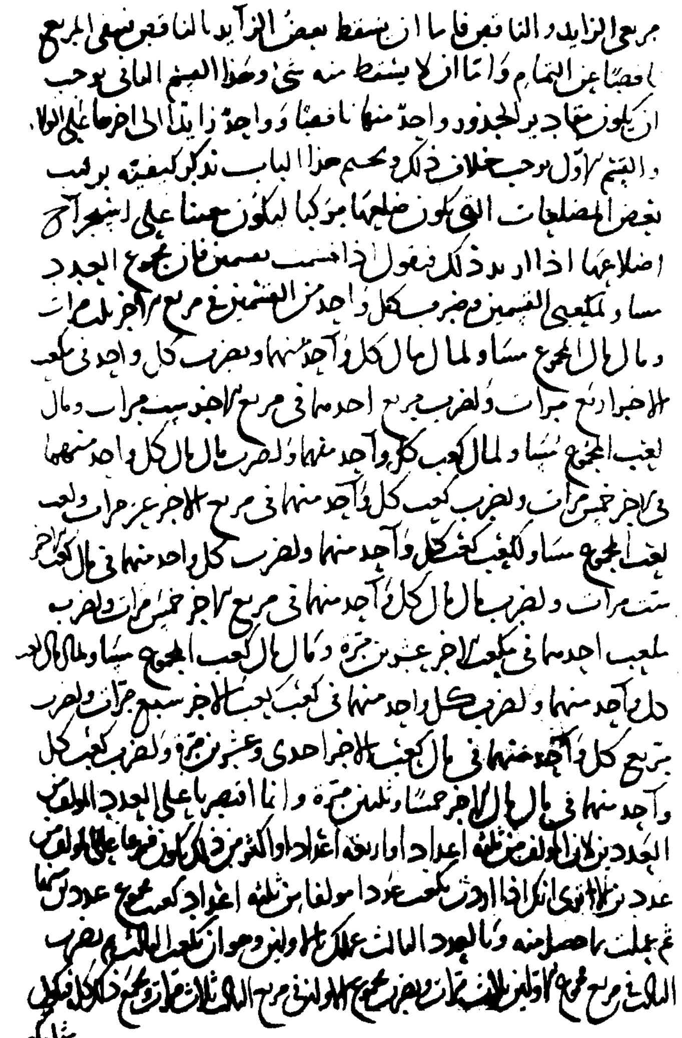
Al-Zanjānī concludes the chapter by generalizing the theorem for polynomials, writing:

> [Line 17, fol. 25b; see the facsimile of part of this passage reproduced on the facing page] *We have concerned ourselves with the expression consisting of two terms because those which consist of three, four, or more terms are nothing more than special cases of the two terms. Don't you see that if you want to find the cube of a three term expression you combine two of them into one? That is, combine the first two and raise it to the third power. Also raise the third term itself into a cube. Multiply the third term by the square of the sum of the first two thrice. Then multiply the sum of the first two by the first two by the third thrice. The sum is the final answer to the original one. Follow the procedure for all the other powers.*

Al-Zanjānī's treatment of this subject is similar in style and content to that of al-Karajī. As an example, Rashed's French translation of al-Karajī's expansion for $(a+b)^5$ is given below:

> *... son quadrato-cube est égal à la somme des quadrato-cubes de chacune de ses parties, cinq fois le produit de chacune des parties par le carré-carré d l'autre et dix fois le produit du carré de chacune d'alles par le cube de l'autre* [Rashed 1972, 5].

While al-Zanjānī provides no proof for the expansion of the binomial theorem, al-Karajī uses a "slightly old-fashioned form of mathematical induction" to demonstrate the expansion of $(a+b)^3$ and $(a+b)^4$ [Rashed 1970, 243]. Al-Samaw'al presents al-Karajī's description of what is commonly known as Pascal's triangle. While al-Zanjānī makes no reference to such a triangle at all, his treatment of the binomial theorem suggests that he was well aware of it.



Facsimile of the translated passage (fol. 25b) from al-Zanjānī's *Qusṭās al-Muᶜādala fī ᶜIlm al-Jabr wa al-Muqābala*. Original text can be found in Ahmad III library (no. 3457), Topkapu Saray, Istanbul. Reproduced by permission of the Topkapu Saray Museum.

NOTES

1. A photographic copy of the book was given to me by the late Dr. Martin Levey. However, except for the name of the transcriber and the date of the manuscript, no other information was available to me concerning the author. It was finally Dr. Fuat Sezgin who identified the author of this work, and thus I owe him a great deal of appreciation and thanks. I must also thank Dr. George Saliba of Columbia University under whose guidance part of this research was carried out.

2. $(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.

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