Local connectivity functions on arcwise connected spaces and certain continua

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Abstract

Types of spaces are given on which every local connectivity function is a connectivity function, a connected function, or a Darboux function. A complete determination such spaces is obtained when the spaces are assumed to be arc-like continua or circle-like continua. Results provide answers to a question asked by Stallings.

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1. Introduction

If $X$ and $Y$ are topological spaces and $f$ is a function from $X$ to $Y$, then $\Gamma(f)$ denotes the graph of $f$ as a subspace of the Cartesian product space $X \times Y$; that is,

$$\Gamma(f) = \{(x, f(x)) \in X \times Y : x \in X\}.$$

We denote the restriction of a function $f$ to a subset $A$ of its domain by $f|A$. 

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The types of functions we define below have been studied extensively—see [2,12] for surveys of results and substantial bibliographies.

Let $X$ and $Y$ be topological spaces, and let $f : X \rightarrow Y$ be a function.

- $f$ is a connected function provided that $\Gamma(f)$ is connected.
- $f$ is a connectivity function provided that $\Gamma(f|C)$ is connected for all connected subsets $C$ of $X$.
- $f$ is a Darboux function provided that $f(C)$ is connected for all connected subsets $C$ of $X$.
- $f$ is a local connectivity function provided that there is an open cover $\mathcal{U}$ of $X$ such that $f|U$ is a connectivity function for each $U \in \mathcal{U}$.

Let $X$ and $Y$ be topological spaces such that $X$ is connected. Clearly, connectivity functions from $X$ to $Y$ are connected functions and are local connectivity functions; also, using the projection map of $X \times Y$ to $Y$, we see that connectivity functions from $X$ to $Y$ are Darboux functions. These are the only implications that hold in general, even for real-valued functions defined on continua; for example, there are real-valued Darboux functions defined on $[0, 1]$ that are not connected. Such a function is not of Baire class 1 [7]. We give an example: For each $x$ in the open interval $(0, 1)$, let $0.a_1(x)a_2(x)\ldots$ be the unique non-terminating binary expansion for $x$, and let $\omega$ be the Cesàro function defined on $(0, 1)$ by

$$\omega(x) = \limsup_{n} \frac{a_1(x) + a_2(x) + \cdots + a_n(x)}{n}, \quad 0 < x < 1;$$

define a function $f$ on $(0, 1)$ by (as in [5, pp. 383–384]) by

$$f(x) = \begin{cases} 0, & \text{if } x = \omega(x), \\ \omega(x), & \text{if } x \neq \omega(x) \end{cases}$$

and extend $f$ to a function $g$ on $[0, 1]$ by letting $g(0) = g(1) = \frac{1}{2}$. Then $g$ is a Darboux function that is not a connected function (since $\Gamma(g)$ is separated by the line $y = x$). We note that if we had extended $f$ by letting $g(0) = 0$ and $g(1) = 1$, then $g$ would be a connected function (see [11, p. 238]).

In general, we are concerned with classifying continua in terms of relationships among the types of functions we defined above. From this point of view, the paper is a continuation of [11], where we obtained the following result: The continua on which every connected real-valued function is a connectivity function, as well as the continua on which every connected real-valued function is a Darboux function, are precisely the dendrites $X$ each of whose arcs contains only finitely many branch points of $X$.

This paper is primarily motivated by a question of Stallings, who asked “Under what conditions is a local connectivity map $X \rightarrow Y$ a connectivity map?” [13, p. 262, #5]. Most of what is known about this question at the present time comes from combining Theorem 4 of Stallings [13, p. 253] with Theorem 1 of Hagan [4], which show the following: Every local connectivity function from $X$ to $Y$ is a connectivity function when $X$ is a connected, locally peripherally connected, unicoherent polyhedron and $Y$ is a regular Hausdorff space such that $X \times Y$ is completely normal. Also, Hagan proved that local connectivity func-
tions are connectivity functions when $X$ and $Y$ are both the unit interval [4, p. 177]. (For an update, see my comment at the end of this paper.)

Our main results provide several more answers to Stallings’ question. We obtain complete answers for dendrites, arc-like continua and circle-like continua (Corollary 6, Theorems 13 and 17). In addition, we find conditions on spaces under which other relations hold between the types of functions defined above. For example, local connectivity functions on arcwise connected metric spaces are connected functions (Theorem 2), and this relationship for real-valued functions characterizes arcwise connectivity for circle-like continua (Theorem 15). As another example, we characterize those continua on which connected real-valued functions are local connectivity functions (Theorem 7, whose other parts are the theorem in [11]).

It was somewhat surprising for us to discover that the situation for arc-like continua is different from the situation for circle-like continua, as seen by comparing Theorem 13 with Theorems 15 and 17.

2. Notation and terminology

We presented some notation and terminology and in the introduction. Here, we note only general notation and terminology that we use in most of the paper. We define some more specialized notions as they come up; the definitions of other notions that we use can be found in [10].

We denote the real line by $\mathbb{R}^1$, the Euclidean plane by $\mathbb{R}^2$, the unit circle in $\mathbb{R}^2$ by $S^1$, and the interval $[0,1]$ by $I$. We signify the closure of a subset $A$ of a space $X$ by $\overline{A}$.

A continuum is a nonempty compact connected metric space. A Peano continuum is a locally connected continuum.

A subcontinuum $T$ of a continuum $X$ is said to be terminal in $X$ provided that any subcontinuum of $X$ intersecting both $T$ and $X - T$ contains $T$.

An $\varepsilon$-map of one metric space $X$ to another is a continuous function $f$ such that $\text{diameter}(f^{-1}(f(x))) < \varepsilon$ for all $x \in X$.

A continuum $X$ is said to be arc-like provided that for each $\varepsilon > 0$, there is an $\varepsilon$-map of $X$ onto $I$ (some authors use the term chainable or snake-like instead of arc-like). A continuum $X$ is said to be circle-like (or circularly-chainable) provided that for each $\varepsilon > 0$, there is an $\varepsilon$-map of $X$ onto $S^1$.

The characteristic function $\chi_S$ for a subset $S$ of a space $X$ is defined by $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \in X - S$.

We use the term nondegenerate in referring to a space to mean that the space contains at least two points.

3. Results for arcwise connected metric spaces

We prove that every local connectivity function from an arcwise connected metric space to any topological space is a connected function (Theorem 2). We obtain two corollaries concerning when local connectivity functions are connectivity functions; then we prove a
Theorem that characterizes those continua on which connected real-valued functions are any one of the other types of functions defined in Section 1. We use Theorem 2 and Corollary 5 in Section 6.

**Lemma 1.** Let $X$ be a Peano continuum. Then every local connectivity function from $X$ to a topological space $Y$ is a connected function.

**Proof.** Let $f : X \to Y$ be a local connectivity function. Then there is an open cover $U$ of $X$ such that $f|U$ is a connectivity function for each $U \in U$. Let $\epsilon > 0$ be a Lebesgue number for the cover $U$ [6, p. 24]. By 8.10 of [10], $X = \bigcup_{j=1}^{n} X_j$, $n < \infty$, where each $X_i$ is a Peano continuum of diameter $< \epsilon$. Furthermore, by 8.13 of [10], the collection $C = \{C_1, C_2, \ldots, C_k\}$ can be indexed (allowing various members of the collection to appear more than once) as a weak chain $C_1, C_2, \ldots, C_k$; that is, $C = \{C_1, C_2, \ldots, C_k\}$ and $C_j \cap C_{j+1} \neq \emptyset$ for each $j < k$.

The cover $C$ is a refinement of the cover $U$ and each $C_j$ is connected; thus, since $f|U$ is a connectivity function for each $U \in U$, $\Gamma(f|C_j)$ is connected for each $j$. Furthermore, since $C_j \cap C_{j+1} \neq \emptyset$ for each $j < k$, $\Gamma(f|C_j) \cap \Gamma(f|C_{j+1}) \neq \emptyset$ for each $j < k$. Therefore, since $\Gamma(f) = \bigcup_{j=1}^{k} \Gamma(f|C_j)$, it follows that $\Gamma(f)$ is connected.

**Theorem 2.** Let $X$ be an arcwise connected metric space. Then every local connectivity function from $X$ to a topological space $Y$ is a connected function.

**Proof.** Let $f : X \to Y$ be a local connectivity function. Fix a point $p \in X$. For each point $x \in X$ such that $x \neq p$, let $A_x$ denote an arc in $X$ from $p$ to $x$; let $A_p = \{p\}$.

Since the restriction of a local connectivity function is a local connectivity function, $f|A_x$ is a local connectivity function for each $x \in X$. Hence, by Lemma 1, $\Gamma(f|A_x)$ is connected for each $x \in X$. Thus, since $(p, f(p)) \in \Gamma(f|A_p)$ for each $x \in X$, $\bigcup_{x \in X} \Gamma(f|A_x)$ is connected. Therefore, since $\Gamma(f) = \bigcup_{x \in X} \Gamma(f|A_x)$, $f$ is a connected function.

We show that Theorem 2 cannot be extended to the situation when $X$ has a dense arc component:

**Example 3.** Let $X$ be the $\sin(\frac{1}{x})$-continuum,

$$X = \left\{ x, \sin\left(\frac{1}{x}\right) \in \mathbb{R}^2 : 0 < x \leq 1 \right\};$$

then the characteristic function for $[0] \times [-1, 1]$ is a local connectivity function on $X$ that is not a connected function.

Our next example shows that we cannot strengthen the conclusion in Theorem 2 to say that local connectivity functions are connectivity functions even for real-valued functions. In fact, the example shows that real-valued local connectivity functions on arcwise connected continua need not even be Darboux functions.
Example 4. Let \( X \) be any arcwise connected circle-like continuum that is not a simple closed curve; for example, take \( X \) to be the quotient space of \( \{(x, \sin(\frac{1}{x})) \in \mathbb{R}^2: 0 < x \leq 1\} \) obtained by identifying \((0, -1)\) with \((1, \sin(1))\). Then there is a real-valued local connectivity function on \( X \) that is not a Darboux function and, hence, is not a connectivity function. This follows from Theorem 17 (in fact, the last part of the proof of Theorem 17 shows how to define such a function).

We give a condition that allows us to strengthen the conclusion in Theorem 2 in the way that Example 4 shows is not possible in general.

A metric space is said to be **hereditarily arcwise connected** provided that each of its connected sets is arcwise connected.

Corollary 5. Let \( X \) be a hereditarily arcwise connected metric space. Then every local connectivity function from \( X \) to a topological space \( Y \) is a connectivity function.

**Proof.** Let \( f : X \to Y \) be a local connectivity function. Let \( C \) be a connected subset of \( X \). Then \( C \) is arcwise connected and \( f|C \) is a local connectivity function. Therefore, by Theorem 2, \( f|C \) is a connected function. \( \Box \)

The converse of Corollary 5 is false since every local connectivity function from \( I \times I \) to itself is a connectivity function (by Theorem 4 of [13] and Theorem 1 of [4]).

For our next corollary, we recall that a **dendrite** is a Peano continuum that contains no simple closed curve.

Corollary 6. If \( X \) is a dendrite, then every local connectivity function from \( X \) to a topological space \( Y \) is a connectivity function.

**Proof.** Dendrites are hereditarily arcwise connected (by 10.9 of [10]). Therefore, the corollary is a special case of Corollary 5. \( \Box \)

By using Corollary 6 and making observations about two proofs in [11], we are able to characterize the continua \( X \) on which every connected real-valued function is a local connectivity function. The continua are the same as those in the theorem in [11]. For convenience in the proof, and in order to have all known equivalences in the same place, we include the theorem in [11] along with the new part of the result (part (1)). We remark that the types of dendrites ruled out in part (4) are those that contain a null comb; on the other hand, the dendrites in part (4) can have infinitely many branch points (e.g., a null sequence of simple triods with the same end point \( p \) and otherwise disjoint).

Theorem 7. For a continuum \( X \), the following four statements are equivalent:

1. Every connected real-valued function on \( X \) is a local connectivity function;
2. Every connected real-valued function on \( X \) is a connectivity function;
3. Every connected real-valued function on \( X \) is a Darboux function;
4. \( X \) is a dendrite such that each arc in \( X \) contains only finitely many branch points of \( X \).
**Proof.** The fact that (2)–(4) are equivalent to one another is Theorem 8 of [11]. Obviously, (2) implies (1). We prove that (1) implies (2) by proving that (1) implies that \( X \) is a dendrite and then applying Corollary 6.

Assume that (1) holds.

In the proof of Lemma 2 of [11], we showed that if a continuum \( X \) contains a nowhere dense nondegenerate subcontinuum \( A \), then there is a connected real-valued function \( f \) on \( X \) such that \( f \) is not a Darboux function. Furthermore, from the formula for \( f \) in the proof, we see that for any nondegenerate subcontinuum \( B \) of \( A \) such that \( B \) contains the specified point \( a \), \( f(B) = \{0, 1\} \); thus, since we can assume that \( B \) is as small as we like (by 5.6 of [10]), \( f \) is not a local connectivity function (at \( a \)). This contradicts (1). Therefore, \( X \) does not contain a nowhere dense nondegenerate subcontinuum. Hence, \( X \) is a Peano continuum (by 5.12 of [10]).

Suppose by way of contradiction that \( X \) contains a simple closed curve \( S \). By the preceding paragraph, \( S \) has nonempty interior in \( X \). Hence, we can apply the proof of Lemma 3 of [11] to obtain a connected function \( f \) that is not a local connectivity function at the special point \( a_0 \in S \). This contradicts (1). Thus, \( X \) does not contain a simple closed curve.

The previous two paragraphs show that (1) implies that \( X \) is a dendrite. Therefore, by Corollary 6, (1) implies (2).

### 4. \( \varepsilon \)-splitting sets

We introduce the following simple but very useful notion. We use the notion in the next two sections.

Let \( X \) be a metric space and let \( \varepsilon > 0 \). A subset \( S \) of \( X \) is said to be \( \varepsilon \)-splitting in \( X \) (or simply \( \varepsilon \)-splitting when there will be no ambiguity) provided that \( S \) is nondegenerate, \( S \neq X \) and there are no small connected subsets of \( X \) that intersect both \( S \) and \( X - S \); that is, there is a \( \delta > 0 \) such that for all connected subsets \( C \) of \( X \) (or, equivalently, for all subcontinua \( C \) of \( X \)) such that \( \text{diameter}(C) < \delta \), either \( C \subset S \) or \( C \subset X - S \).

For example, let \( X \) be the \( \sin(\frac{1}{x}) \)-continuum in Example 3; then the \( \varepsilon \)-splitting sets in \( X \) are \( \{0\} \times [-1, 1] \) and its complement (and there are no others).

We give two more examples of \( \varepsilon \)-splitting sets. The examples are of a general nature and will be used in proofs.

**Example 8.** Any nondegenerate proper terminal continuum \( T \) in a continuum \( X \) is \( \varepsilon \)-splitting (the definition of terminal is in Section 2). As a special case, any nondegenerate proper subcontinuum of a hereditarily indecomposable continuum is \( \varepsilon \)-splitting.

**Example 9.** Any composant \( S \) of a nondegenerate indecomposable continuum \( X \) is \( \varepsilon \)-splitting. This follows by taking \( \delta = \text{diameter}(X) \) and using the fact that \( X \) is irreducible between any two points of different composants of \( X \) (by 11.17 of [10]).

When a space \( X \) has an \( \varepsilon \)-splitting set, one can find a simple real-valued local connectivity function on \( X \) that will play a central role in the rest of the paper:
Lemma 10. Let \( X \) be a metric space that contains an \( \varepsilon \)-splitting set \( S \). Then the characteristic function \( \chi_S \) is a local connectivity function on \( X \).

Proof. Let \( \delta > 0 \) be as in the definition of \( \varepsilon \)-splitting. Then \( \chi_S \) is a local connectivity function since \( \chi_S \) is constant on any connected subset of \( X \) of diameter \( < \delta \).

Our next lemma is directly related to Example 9. We will apply the lemma to arc-like and circle-like continua. Thus, the lemma is more general than we need; however, the general lemma may be useful in other situations. First, we note some terminology.

For a given integer \( n \geq 2 \), an \( n \)-od is a continuum \( Y \) such that some subcontinuum of \( Y \) separates \( Y \) into at least \( n \) components; a continuum is atriodic provided that it contains no triod (3-od).

Lemma 11. Let \( X \) be a continuum such that \( X \) contains no \( n \)-od for some \( n \geq 3 \). If \( X \) contains a nondegenerate indecomposable continuum \( K \), then there is a composant \( S \) of \( K \) such that \( S \) is \( \varepsilon \)-splitting in \( X \).

Proof. Fix \( n \) as in the lemma, and let \( \delta = \frac{1}{n} \text{diameter}(K) \). We show that \( \delta \) satisfies the definition for some composant of \( K \) to be \( \varepsilon \)-splitting in \( X \).

We first prove the following:

(1) Any subcontinuum of \( X \) of \( \text{diameter} < n\delta \) intersects at most \( n - 1 \) composants of \( K \).

Proof of (1). Assume that some subcontinuum \( C \) of \( X \) intersects \( n \) different composants \( K_1, K_2, \ldots, K_n \) of \( X \). We prove that \( C \supset K \). Suppose by way of contradiction that \( C \nsubseteq K \). Then, since each \( K_i \) is dense in \( K \) (by 5.20(a) of [10]), it follows that \( C \nsubseteq K_i \) for any \( i = 1, 2, \ldots, n \). Hence, for each \( i = 1, 2, \ldots, n \), there is a subcontinuum \( A_i \) of \( K_i \) such that \( A_i \cap C \neq \emptyset \) and \( A_i \nsubseteq C \). Thus, since the continua \( A_1, A_2, \ldots, A_n \) are mutually disjoint (by 11.17 of [10]), it follows that \( C \cup (\bigcup_{i=1}^n A_i) \) is an \( n \)-od. This contradicts an assumption in our lemma. Therefore, we have proved that \( C \supset K \). The assertion in (1) now follows.

Next, let \( K_1, K_2, \ldots \), be countably many different composants of \( K \). Suppose by way of contradiction that our choice of \( \delta \) does not show that some \( K_i \) is \( \varepsilon \)-splitting in \( X \). Then, for each \( i \), there is a subcontinuum \( C_i \) of \( X \) such that

(2) \( C_i \cap K_i \neq \emptyset \), \( C_i \cap (X - K_i) \neq \emptyset \), and \( \text{diameter}(C_i) < \delta \).

If \( C_i \subset K \) for some \( i \), then \( C_i \) is a proper subcontinuum of \( K \) (since \( \text{diameter}(C_i) < \delta \) by (2)); thus, since \( C_i \cap K_i \neq \emptyset \) (by (2)), \( C_i \subset K_i \) (by 11.17 of [10]). This contradicts the second part of (2). Therefore, we have that

(3) \( C_i \nsubseteq K \) for each \( i \).

We prove the following:
(4) For each $j = 1, 2, \ldots$, $C_j$ can intersect at most $n - 2$ of the continua $C_i$, $i \neq j$.

**Proof of (4).** Fix $j$. Suppose by way of contradiction that $C_j \cap C_{i_k} \neq \emptyset$, where $k = 1, 2, \ldots, n - 1$, $i_k \neq j$ for each $k$, and $i_k \neq i_\ell$ when $k \neq \ell$. Let

$$C = C_j \cup \left( \bigcup_{k=1}^{n-1} C_{i_k} \right).$$

Then $C$ is a subcontinuum of $X$; furthermore, by (2), $\text{diameter}(C) < 3\delta$ and $C$ intersects each of the $n$ composants $K_j, K_{i_1}, \ldots, K_{i_{n-1}}$. This contradicts (1). Therefore, we have proved (4). \Box

Now, let $i_1 = 1$. By (4), there is an $i_2 > i_1$ such that

$$C_{i_1} \cap C_\ell = \emptyset \quad \text{for all } \ell \geq i_2.$$

Then, by (4), there is an $i_3 > i_2$ such that

$$C_{i_2} \cap C_\ell = \emptyset \quad \text{for all } \ell \geq i_3.$$

Note that the continua $C_{i_1}, C_{i_2}$ and $C_{i_3}$ are mutually disjoint. Hence, continuing in this fashion $n$ times, we arrive at $n$ mutually disjoint continua $C_{i_1}, C_{i_2}, \ldots, C_{i_n}$. Thus, it follows from (3) that $K \cup \left( \bigcup_{m=1}^{n} C_{i_m} \right)$ is an $n$-od, in contradiction to an assumption in our lemma. Therefore, (2) is false. \Box

5. The result for arc-like continua

We show that if $X$ is an arc-like continuum such that every real-valued local connectivity function on $X$ is a connected function, a connectivity function or a Darboux function, then $X$ is an arc, and conversely (Theorem 13). First, we prove a lemma.

**Lemma 12.** Any arc-like continuum other than an arc contains an $\epsilon$-splitting continuum.

**Proof.** Let $X$ be an arc-like continuum that is not an arc. Arc-like continua are atriodic (by 12.4 of [10]). Thus, if $X$ contains a nondegenerate indecomposable continuum $K$, then by Lemma 11 there is a composant $S$ of $K$ such that $S$ is $\epsilon$-splitting in $X$.

Therefore, we assume for the rest of the proof that $X$ is hereditarily decomposable. Thus, since $X$ is irreducible (by 12.5 of [10]), there is a continuous monotone function $\varphi : X \rightarrow I$ such that $\varphi^{-1}(t)$ is nowhere dense in $X$ for each $t \in I$ (by Theorem 10 of [14] or by Theorem 8 of [1]).

Since $X$ is not an arc, $\varphi^{-1}(t_0)$ is nondegenerate for some $t_0 \in I$. We take two cases:

**Case 1:** $t_0 = 0$ or $t_0 = 1$. We show that $\varphi^{-1}(0)$ is a terminal continuum in $X$ and then apply Example 8. Suppose by way of contradiction that there is a subcontinuum $A$ of $X$ such that

$$A \cap \varphi^{-1}(0) \neq \emptyset, \quad A \cap \left( X - \varphi^{-1}(0) \right) \neq \emptyset, \quad A \nsubseteq \varphi^{-1}(0).$$
Then there exists \( s \in \varphi(A) \) such that \( 0 < s < 1 \), and \( A \cup \varphi^{-1}([t_0, 1]) \) is a proper subcontinuum of \( X \) that intersects both \( \varphi^{-1}(0) \) and \( \varphi^{-1}(1) \). However, this contradicts the fact that \( X \) is irreducible between any point of \( \varphi^{-1}(0) \) and any point of \( \varphi^{-1}(1) \) [14, Theorem 8]. Therefore, \( \varphi^{-1}(0) \) is a terminal continuum in \( X \). Thus, if \( t_0 = 0 \), then \( \varphi^{-1}(t_0) \) is \( \epsilon \)-splitting in \( X \) by Example 8. Similarly, if \( t_0 = 1 \), then \( \varphi^{-1}(t_0) \) is \( \epsilon \)-splitting in \( X \).

**Case 2:** \( 0 < t_0 < 1 \). Then let

\[
L = \varphi^{-1}([0, t_0]) \cap \varphi^{-1}(t_0), \quad R = \varphi^{-1}((t_0, 1]) \cap \varphi^{-1}(t_0).
\]

Since \( \varphi^{-1}(t_0) \) is nondegenerate and nowhere dense in \( X \), \( L \) or \( R \) (possibly each) is nondegenerate. Assume first that \( L \) is nondegenerate. Let \( Y = \varphi^{-1}([0, t_0]) \) and let \( \varphi_Y = \varphi|\varphi^{-1}([0, t_0]) \). Note that \( \varphi_Y \) is a continuous monotone function of \( Y \) onto \([0, t_0]\) such that \( \varphi_Y^{-1}(t) \) is nowhere dense in \( Y \) for each \( t \in [0, t_0] \). Hence, applying case 1 to \( \varphi_Y \) with \( t_0 \) playing the role of 1, we see that \( L = \varphi_Y^{-1}(t_0) \) is \( \epsilon \)-splitting in \( Y \). It follows easily that \( \varphi^{-1}(t_0) \) is \( \epsilon \)-splitting in \( \varphi^{-1}([0, t_0]) \). Therefore, \( \varphi^{-1}([t_0, 1]) \) is \( \epsilon \)-splitting in \( X \). Similarly, if \( R \) is nondegenerate, then \( \varphi^{-1}([0, t_0]) \) is \( \epsilon \)-splitting in \( X \).

We now prove our theorem for arc-like continua.

**Theorem 13.** Let \( X \) be an arc-like continuum. Then the following four statements are equivalent:

1. every real-valued local connectivity function on \( X \) is a connected function;
2. every real-valued local connectivity function on \( X \) is a Darboux function;
3. every real-valued local connectivity function on \( X \) is a connectivity function;
4. \( X \) is an arc.

**Proof.** We divide the proof into three parts.

(4) \( \Rightarrow \) (1). This is by Lemma 1.

(1) \( \Rightarrow \) (4) and (2) \( \Rightarrow \) (4). Assume that \( X \) is not an arc. Then, by Lemma 12, \( X \) contains an \( \epsilon \)-splitting continuum \( A \). Therefore, by Lemma 10, \( \chi_A \) is a local connectivity function on \( X \); clearly, \( \chi_A \) is neither a connected function nor a Darboux function.

(4) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (2). The fact that (4) implies (3) is Theorem 2 of Hagan [4] (or apply our Corollary 5). Clearly, (3) implies (2). \( \square \)

**6. Results for circle-like continua**

One might suspect that there is a direct analogue of Theorem 13 for circle-like continua, namely, that each of the first three parts of Theorem 13 is equivalent to \( X \) being a simple closed curve. However, this is not the case: Theorems 10 and 12 determine the circle-like continua for which real-valued local connectivity functions must be connected functions or must be connectivity or Darboux functions, respectively.

As in the case of arc-like continua, we use \( \epsilon \)-splitting sets to prove our results. In particular, we use the following lemma:
Lemma 14. Any circle-like continuum that is not arcwise connected contains an $\varepsilon$-splitting set.

Proof. Assume that $X$ is a circle-like continuum that is not arcwise connected. Circle-like continua are atriodic (by 12.4 and 12.51 of [10]). Thus, if $X$ contains a nondegenerate indecomposable continuum $K$, then by Lemma 11 there is a composant $S$ of $K$ such that $S$ is $\varepsilon$-splitting in $X$.

Therefore, we assume from now on that $X$ is hereditarily decomposable. Furthermore, we can assume for the proof that $X$ is not arc-like by Lemma 12 (actually, $X$ is not arc-like by Theorem 7 of [3]). Then $X$ is not separated by any of its subcontinua by Theorem 4 of [3]. Thus, by Theorem 2 of [15, p. 74], there is a continuous monotone function $\varphi : X \rightarrow S^1$ (the unit circle in $\mathbb{R}^2$) such that $\varphi^{-1}(t)$ is nowhere dense in $X$ for each $t \in S^1$.

Since $X$ is not a simple closed curve, $\varphi^{-1}(s_0)$ is nondegenerate for some $s_0 \in S^1$. We take two cases. For use in both cases, we let $pq$ be an arc in $S^1$ such that $s_0 \in pq - \{p, q\}$; we consider $\overline{pq}$ as ordered by $<$ so that $p < s_0 < q$, and we use interval notation for connected subsets of $\overline{pq}$.

Case 1: $\varphi^{-1}(s_0)$ is not an arc. Then, since $\varphi^{-1}(s_0)$ is arc-like (by 12.51 of [10]), Lemma 12 shows that there is a subcontinuum $A$ of $\varphi^{-1}(s_0)$ such that $A$ is $\varepsilon$-splitting in $\varphi^{-1}(s_0)$. Let

$$D = A \cap \overline{\varphi^{-1}([p, s_0])}, \quad E = A \cap \overline{\varphi^{-1}([s_0, q])}.$$  

Since $\varphi^{-1}(s_0)$ is nowhere dense in $X$, at least one of $D$ and $E$ is nondegenerate. If both $D$ and $E$ are nondegenerate, then it follows that $\varphi^{-1}(s_0)$ is $\varepsilon$-splitting in $X$. If only one of $D$ and $E$ is nondegenerate, then it follows that $A$ or $X - A$ is $\varepsilon$-splitting in $X$.

Case 2: $\varphi^{-1}(s_0)$ is an arc. Let

$$G = \varphi^{-1}(s_0) \cap \overline{\varphi^{-1}([p, s_0])}, \quad H = \varphi^{-1}(s_0) \cap \overline{\varphi^{-1}([s_0, q])}.$$  

Since $\varphi^{-1}(s_0)$ is nowhere dense in $X$, at least one of $G$ and $H$ is nondegenerate. If both $G$ and $H$ are nondegenerate, then $\varphi^{-1}(s_0)$ is $\varepsilon$-splitting in $X$. So, assume that $G$ is degenerate and $H$ is nondegenerate. Assume for the moment that $\varphi$ is one-to-one on $S^1 - \{s_0\}$; then, since $\varphi^{-1}(s_0)$ is an arc and $G$ consists of only one point, we see that $X$ is arcwise connected, which contradicts our initial assumption about $X$. Therefore, there exists $s_1 \in S^1 - \{s_0\}$ such that $\varphi^{-1}(s_1)$ is nondegenerate. Let $U_p$ and $U_q$ denote the components of $S^1 - \{s_0, s_1\}$ containing $p$ and $q$, respectively. Let

$$J = \varphi^{-1}(U_p) \cap \varphi^{-1}(s_1), \quad K = \varphi^{-1}(U_q) \cap \varphi^{-1}(s_1).$$  

Since $\varphi^{-1}(s_1)$ is nowhere dense in $X$, $J$ or $K$ is nondegenerate. If $J$ is nondegenerate, then it follows that $\varphi^{-1}(U_p \cup \{s_0\})$ is $\varepsilon$-splitting in $X$. If $K$ is nondegenerate, then (since $H$ is also nondegenerate) it follows that $\varphi^{-1}(U_q)$ is $\varepsilon$-splitting in $X$. \qed

Our next theorem concerns arcwise connected circle-like continua. The structure of all arcwise connected circle-like continua is determined in Theorem 6 of [8, p. 230]. We note that even though the structure of such continua is quite simple, there are uncountably many of them [8, p. 233].
Theorem 15. Let $X$ be a circle-like continuum. Then $X$ is arcwise connected if and only if every real-valued local connectivity function on $X$ is a connected function.

Proof. The sufficiency of arcwise connectedness is due to Theorem 2. To prove the necessity, assume that $X$ is not arcwise connected. Then, by Lemma 14, $X$ contains an $\varepsilon$-splitting set $S$. Therefore, by Lemma 10, $\chi_S$ is a local connectivity function on $X$; clearly, $\chi_S$ is not a connected function. □

The only arcwise connected arc-like continuum is an arc $[10, 12.6]$. Thus, we have an analogy between Theorem 15 and the equivalence of parts (1) and (4) of Theorem 13.

Our next theorem shows how parts (2) and (3) of Theorem 13 fit into the situation of circle-like continua. We prove the theorem by using some information about continua that are a one-to-one continuous image of $[0, \infty)$.

The structure of all continua that are one-to-one continuous images of $[0, \infty)$ was determined in the theorem in [9, p. 128]; the theorem was then applied, in the corollary in [9, p. 129], to characterize all arcwise connected circle-like continua in terms of one-to-one continuous images of $[0, \infty)$. The following lemma summarizes results in [9] that are relevant for our purpose here.

Lemma 16. A continuum $X$ is an arcwise connected circle-like continuum if and only if there is a one-to-one continuous function $\psi : [0, \infty) \to X$ such that for some (unique)

$$t_0 \geq 0,$$

$$\bigcap_{r \geq 0} \psi([r, \infty)) = \psi([0, t_0]).$$

Proof. The lemma is a combination of the corollary in [9, p. 129] and Lemma 4 of [9, p. 125]. □

Theorem 17. Let $X$ be a circle-like continuum. Then the following three statements are equivalent:

(1) every real-valued local connectivity function on $X$ is a Darboux function;
(2) every real-valued local connectivity function on $X$ is a connectivity function;
(3) $X$ is a simple closed curve.

Proof. By Corollary 5, (3) implies (2). Next, (2) implies (1) since connectivity functions are Darboux (as noted in Section 1). Finally, we prove that (1) implies (3).

Assume (1). We first prove that $X$ is arcwise connected. Assume that $X$ is not arcwise connected. Then, by Lemma 14, $X$ contains an $\varepsilon$-splitting set $S$. Hence, by Lemma 10, $\chi_S$ is a local connectivity function on $X$; clearly, $\chi_S$ is not a Darboux function (since $\chi_S(X) = \{0, 1\}$). This contradicts (1). Therefore, $X$ is arcwise connected.

Now, we can let $\psi$ be as in Lemma 16. Assume that $t_0 > 0$. Let

$$Y = \psi([0, t_0]).$$
Note that $Y$ is an arc, and let $p \in Y - \{\psi(0), \psi(t_0)\}$. Let $g: Y \to I$ be a continuous (Urysohn) function such that $g(p) = 1$ and $g(\{\psi(0), \psi(t_0)\}) = 0$. Then define $f: X \to I$ as follows:

$$f(x) = \begin{cases} g(x), & \text{if } x \in Y, \\ 0, & \text{if } x \in X - Y. \end{cases}$$

It is easy to check that $f$ is a local connectivity function; however, $f$ is not a Darboux function since $\psi([t_0, \infty)) \cup \{p\} = C$ is connected but $f(C) = \{0, 1\}$. This contradicts (1). Therefore, $t_0 = 0$. It now follows easily that $X$ is a simple closed curve.

Professor Francis Jordan recently obtained a definitive answer to Stallings’ question that we mentioned in Section 1. He uses the idea of $\varepsilon$-splitting sets (introduced here in Section 4). Professor Jordan’s paper “When are local connectivity functions connectivity?” will appear in Topology Proceedings.

References

[9] S.B. Nadler Jr, Continua which are a one-to-one continuous image of $[0, \infty)$, Fund. Math. 75 (1972) 123–133.