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Two Characterizations of a Minimum-Information Principle for Possibilistic Reasoning

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ABSTRACT

We show that the principle of maximum U-uncertainty for ampliative possibilistic reasoning can be characterized as uniquely satisfying a small set of normative axioms. Two proofs are given—one each for Hisdal's and Dempster's definitions of conditional possibility. These results complement a similar characterization of maximum entropy for ampliative probabilistic inference, given by Paris and Vencovska.

KEYWORDS: ***possibility theory, U-uncertainty, specificity, inference process***

1. INTRODUCTION

1.1. Logical Knowledge Representation

One popular method of representing knowledge in AI is as a set of well-formed sentences of some well-understood logic, such as propositional logic or first-order predicate logic. Logic-based representations have the advantages of a precisely defined semantics and a sound and complete proof calculus (e.g. natural deduction or complete resolution) for performing deductive inference.

Most domains of interest in AI, such as medical diagnosis, involve knowledge that is merely tentative or qualified rather than definite and certain. Such inexact knowledge may suffer from being vague/ambiguous

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(involving ill-defined terms such as tall, young, red) or through being uncertain (known to hold sometimes, although exceptions do exist). Both kinds of imprecision can be modeled by assigning a numerical value in the range [0, 1] to a sentence of propositional logic. If the imprecision is due to vagueness, then the value is interpreted as a degree of truth or fuzzy truth value. When the imprecision is due to uncertainty, the value is interpreted as a degree of belief or certainty. Various fuzzy logics [1] have been proposed for manipulating logical sentences with attached fuzzy truth values, while a variety of logics have been proposed for reasoning about logical sentences with attached degrees of certainty (such as probabilistic logic [2], the Dempster-Shafer theory of evidence [3], and possibilistic logic [1, 3, 4]).

1.2. Possibilistic Logic

Possibilistic logic has emerged from both fuzzy logic [4] and the theory of evidence [3]. The possibility of a vague predicate is defined as the supremum of the membership function of the fuzzy set denoted by the predicate. In Dempster-Shafer theory, in the special case of nonconflicting evidence (consonant body of evidence, i.e. nested focal subsets), the plausibility of a propositional sentence is precisely equivalent to the possibility of that sentence regarded as a vague predicate, denoting a fuzzy subset of the underlying set of possible worlds. The membership function of this fuzzy set is the possibility distribution [5]. So we can see that possibilistic logic is applicable to both vague and uncertain reasoning.

Possibility measures, like probability measures, are decomposable [6] fuzzy measures [7], and are thus more amenable to mathematical analysis than general fuzzy measures or plausibility/credibility measures [3]. Whereas probability measures arise from dissonant evidence, possibility measures arise from imprecise but consonant evidence, and thus capture ignorance as well as uncertainty. Detailed accounts of possibility theory appear in [6] and [8].

1.3. Ampliative Reasoning and Minimum Information

In the case of certain and definite knowledge, the sound and complete deductive proof calculus is in general found to be inadequate when the knowledge base is either inconsistent or incomplete. In the latter case, various nonmonotonic logics have been proposed for performing inference.

The corresponding problem for approximate reasoning (i.e. what to do in the presence of inconsistencies or incompleteness in the knowledge base) has received relatively little attention in the past. In this context, the process of reasoning from a consistent but incomplete knowledge base K

(generally known as ampliative reasoning [8]) is often regarded as one of eliciting a complete knowledge base, Comp(K), from the available information and then reasoning in the usual way.

Various minimum-information principles are used to justify a particular choice of Comp(K), the most well-known of which is the maximum-entropy principle, which is applicable to probabilistic knowledge bases. This principle defines Comp(K) as the unique complete probabilistic knowledge base amongst those consistent with K that has the maximum entropy or minimum information content.

Though plausible, the principle of minimum information is far from compelling. For example, we might consider it fairer to choose Comp(K) as having average information content amongst those consistent with K, or we might ignore information content altogether and choose Comp(K) as a (possibly weighted) average of all those complete probabilistic knowledge bases that are consistent with K. Both of these choices can be regarded as the fairest way of eliciting knowledge from a given K, although in general they give different choices for Comp(K).

Furthermore, if K is not probabilistic, but is consistent with Dempster-Shafer theory or possibility theory, then there are several distinct minimum-information principles, one for each of the measures of information content in these frameworks. For example, in possibility theory, the information content of a possibility measure can be regarded as its specificity [9] (in the Dempster-Shafer sense), as its strife, as its discord, or as the negative fuzziness [8, 9] of the fuzzy set with membership function equal to this possibility distribution, for some measure of fuzziness. So, even if we can decide on some particular measure of fuzziness, there are at least two applicable minimum-information principles. These are the principle of minimum specificity (or maximum U-uncertainty) and the principle of maximum fuzziness. In fact, U-uncertainty is not the only measure of unspecificity of possibility measures. Alternative definitions of specificity include those given by Yager [11, 12] and Dubois and Prade [13], and minimum-information principles could be based on minimizing these measures of specificity. Other possible minimum-information principles include the principle of maximum discord and the principle of minimum strife. It appears that we need some extra-information-theoretic principles for preferring the Comp(K) given by one minimum information principle to that given by another.

1.4. Inference Processes and Principles of Ampliative Reasoning

Paris and Vencovska [14–16] have thoroughly investigated ampliative probabilistic reasoning. They view ampliative reasoning as a single inference process mapping consistent but incomplete probabilistic knowledge bases to unique probabilities for every sentence of the underlying logical language, rather than a two-stage process of elicitation followed by standard inference using conventional probabilistic logic. This subtly different viewpoint allows a shift in emphasis away from information theory to a purely logical perspective. Paris and Vencovska hypothesize a collection of compelling principles or laws of ampliative reasoning [14], which necessarily hold for any sound inference process. Under certain technical assumptions about the form of the knowledge base, Paris and Vencovska [15] demonstrate that the maximum-entropy inference process (given by the principle of maximum entropy combined with probabilistic logic) is the only inference process which is sound with respect to these principles.

1.5. Overview

Here, we apply the Paris-Vencovska approach to ampliative possibilistic reasoning. We present a purely logical justification of the principle of minimum specificity. However, there is another highly plausible principle of ampliative reasoning (the principle of atomicity) that is satisfied by the minimum-specificity (maximum-U-uncertainty) inference process, but is inconsistent with the other principles in the probabilistic case (even maximum entropy doesn't satisfy it). This highlights a weakness of probabilistic reasoning (the inability to handle ignorance properly) and shows a definite need for logics such as possibilistic logic that can better accommodate ignorance.

2. A FORMAL FRAMEWORK FOR AMPLIATIVE POSSIBILISTIC LOGIC

We now describe the details of a formalization of ampliative possibilistic reasoning, based on that of [17].

We shall take the rather unusual position of defining a possibility measure over a domain of sets of logically equivalent sentences (as in [1, 18]), rather than over a frame of discernment [3] of subsets of the set of all possible worlds (the conventional approach). These approaches are, of course, equivalent, because every logical sentence corresponds uniquely to a subset of the set of all possible worlds, i.e. all those worlds in which the sentence is true. Furthermore, two sentences are equivalent precisely if they denote the same set of possible worlds. We shall restrict consideration to propositional sentences from some finite alphabet L, so that there is a 1-1 correspondence between atoms of the underlying Lindebaum algebra of L and the set of possible worlds (valuations on L). This induces a 1-1 correspondence between the frame of discernment and the set of elements of the Lindebaum algebra (i.e. the set of equivalence classes of propositional sentences, where the equivalence relation is just logical equivalence between sentences).

DEFINITION 2.1 **R** *is the set of reals,* **N** *the set of naturals, and* **L** *the set of all finite subsets of* $\{p_n | n \in \mathbf{N}\}$, *where* $\{p_n | n \in \mathbf{N}\}$ *is a countably infinite set of propositional variables. We denote the set of all subsets of a set* S *by* pow(S). *The symbol* $\downarrow$ *denotes domain restriction.*

Suppose that L is a finite propositional language (i.e. just a finite alphabet of symbols used to denote propositions), and that SL is the set of propositional sentences constructed from L and the standard connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ by structural recursion in the usual way. We denote by $\overline{SL}$ the underlying Lindebaum algebra of L. We denote by At_L the set of atoms of $\overline{SL}$. Let $\mathrm{Aut}(\overline{SL})$ denote the set of automorphisms of $\overline{SL}$. Let $\mathbf{0} = \overline{p \wedge \neg p}$ and $\mathbf{1} = \overline{p \vee \neg p}$, where $p \in L$. Define $\leq$ on $\overline{SL}$ by $\bar{\theta} \leq \bar{\phi} \leftrightarrow \overline{\phi \rightarrow \phi} = \mathbf{1}$.

DEFINITION 2.2 *Suppose that* $\pi : \overline{SL} \rightarrow [0, 1]$. *We call* π *a* possibility measure *on L if and only if*

(π1) $\pi(\mathbf{0}) = 0$,
(π2) $\pi(\mathbf{1}) = 1$,
(π3) $\pi(\overline{\theta \vee \phi}) = \max\{\pi(\bar{\theta}), \pi(\bar{\phi})\}$ *for any* $\bar{\theta}, \bar{\phi} \in \overline{SL}$.

We note that by (π3),

$$\pi(\bar{\theta}) = \pi\left(\bigvee_{\bar{\gamma} \in \mathrm{At}_L(\bar{\theta})} \bar{\gamma}\right) = \max_{\bar{\gamma} \in \mathrm{At}_L(\bar{\theta})} \{\pi(\bar{\gamma})\} \sim (*)$$

We denote by Π_L the set of all possibility measures on L.

DEFINITION 2.3 *Suppose that* $r : \mathrm{At}_L \rightarrow [0, 1]$ *satisfies* $r(\bar{\gamma}) = 1$ *for some* $\bar{\gamma} \in \mathrm{At}_L$. *Then we call r a* possibility distribution *on L.*

Let R_L *denote the set of possibility distributions on L.*

Suppose that $\pi \in \Pi_L$. Then, by (π2) and ($*$), $\pi(\bar{\gamma}) = 1$ for some $\bar{\gamma} \in \mathrm{At}_L$. Hence, $\pi \downarrow \mathrm{At}_L \in R_L$.

Furthermore, for any $r \in R_L$, define $\pi^r : \overline{SL} \rightarrow [0, 1]$ as follows:

$$\pi^r(\mathbf{0}) = 0,$$
$$\pi^r(\bar{\theta}) = \max_{\bar{\gamma} \in \mathrm{At}_L(\bar{\theta})} \{r(\bar{\gamma})\} \quad \text{for any} \quad \bar{\theta} \neq \mathbf{0}.$$

It is readily verified that $\pi^r \in \Pi_L$, and furthermore that if $\pi \in \Pi_L$ and $\pi \downarrow \mathrm{At}_L = r$ then $\pi = \pi^r$.

In summary, every possibility measure on L is uniquely determined by its possibility distribution on L, and every possibility distribution on L can be extended to a unique possibility measure on L. Thus, possibility measures and distributions are related in the same way as probability measures and distributions.

2.1. *U*-Uncertainty

The amount of uncertainty associated with a possibility distribution r is called its *U-uncertainty*, denoted $U_L(r)$. First, we give a preliminary definition:

DEFINITION 2.4 *Let* $n = |\mathrm{At}_L|$. *We define a map* $\mu_L : R_L \to [0, 1]^n$ *as follows*: *Let* $V(r) = \{r(\bar{\gamma})|\bar{\gamma} \in \mathrm{At}_L\}$, *and let* $v = |V(r)|$. *Let* $a_1(r) = \max V(r) = 1$, *and for* $2 \le i \le v$, *let* $a_i(r) = \max[V(r) \setminus \{a_1(r), \ldots, a_{i-1}(r)\}]$. *Let* $m_i(r) = |\{\bar{\gamma} \in \mathrm{At}_L | r(\bar{\gamma}) = a_i(r)\}|$ *for* $1 \le i \le v$, *and*

$$\mu_L(r) = \Big(\underbrace{a_1(r), \ldots, a_1(r)}_{m_1(r)},\ \underbrace{a_2(r), \ldots, a_2(r)}_{m_2(r)}, \ldots,\ \underbrace{a_v(r), \ldots, a_v(r)}_{m_v(r)}\Big)$$

For any $r \in R_L$, *we write* $\mu_L(r) = (\mu_1^{(r)}, \ldots, \mu_n^{(r)})$.

DEFINITION 2.5 *We define*

$$U_L(r) = \sum_{i=1}^{n} (\mu_i^{(r)} - \mu_{i+1}^{(r)})\log_2 i \qquad (\textit{where } \mu_{n+1}^{(r)} = 0 \textit{ by convention}).$$

This measure was first suggested by Higashi and Klir [19]. They also introduced a number of information-theoretic axioms (possibilistic versions of those from Shannon's statistical information theory) and showed that the measure satisfies the axioms. This measure was later uniquely characterized by a set of normative axioms [20]. The axioms are stated and justified in [8, pp. 177–181], where a proof of the characterization is also given (see [8, Appendix A.2]). Complementary results appear in [21] and [22], the latter giving a characterization of the generalization of U-uncertainty to evidence theory (see Section 2.2 below). Here, we just note some simple monotonicity and symmetry properties of U_L.

DEFINITION 2.6 *We define a relation* $\le_L$ *on* R_L *as follows*:

$$r_1 \le_L r_2 \quad \textit{if and only if} \quad r_1(\bar{\gamma}) \le r_2(\bar{\gamma}) \quad \textit{for every } \bar{\gamma} \in \mathrm{At}_L.$$

It is readily verified that $\le_L$ is a partial order on R_L.

The following lemma is a corollary of [9, Proposition 7]. A self-contained proof of this special case is given below for completeness.

LEMMA 2.1 *If* $r_1, r_2 \in R_L$ *and* $r_1 <_L r_2$, *then* $U_L(r_1) < U_L(r_2)$.

Proof Note that $U_L(r) = \sum_{i=2}^{n} \mu_i^{(r)} [\log_2 i - \log_2(i-1)]$. It is very tedious, but straightforward, to verify that if $r_1 \leq_L r_2$, then $\mu_i^{(r_1)} \leq \mu_i^{(r_2)}$ for $1 \leq i \leq n$. Furthermore, we can show that if $r_1 <_L r_2$, then $\mu_i^{(r_1)} < \mu_i^{(r_2)}$ for some $1 < i \leq n$. So, if $r_1 <_L r_2$, then $\mu_i^{(r_1)} \leq u_i^{(r_2)}$ for every $2 \leq i \leq n$ and $\mu_j^{(r_1)} < \mu_j^{(r_2)}$ for some $2 \leq j \leq n$. Hence, $U_L(r_1) < U_L(r_2)$. ■

LEMMA 2.2 *Suppose that* $g \in \mathrm{Aut}(\overline{SL})$ *and that* $r \in R_L$. *Then* $U_L(g(r)) = U_L(r)$.

Proof Define $g(r) \in [0, 1]^{\mathrm{At}_L}$ by

$$g(r)(g(\bar{\gamma})) = r(\bar{\gamma}) \qquad \text{for any} \quad \bar{\gamma} \in \mathrm{At}_L.$$

Then $g(r) \in R_L$. Furthermore, $V(g(r)) = V(r)$ and so $a_i(r) = a_i(g(r))$ for $1 \leq i \leq v$. Also, $m_i(r) = m_i(g(r))$ for $1 \leq i \leq v$. Hence, $\mu_L(g(r)) = \mu_L(r)$, and so $U_L(g(r)) = U_L(r)$. ■

DEFINITION 2.7 *Suppose that* $r_1, r_2 \in R_L$. *Define* $\max\{r_1, r_2\} : \mathrm{At}_L \to [0, 1]$ *by*

$$(\max\{r_1, r_2\})(\bar{\gamma}) = \max\{r_1(\bar{\gamma}), r_2(\bar{\gamma})\} \qquad \text{for any} \quad \bar{\gamma} \in \mathrm{At}_L.$$

It is readily verified that $\max\{r_1, r_2\} \in R_L$.

2.2. Relation to Dempster-Shafer Theory

We may consider possibility theory as a special case of the Dempster-Shafer theory of evidence. Suppose that m is a basic probability assignment (bpa) and that its focal set $\mathcal{F}(m) = \{\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_n\} \subseteq \overline{SL}$, where $\bar{\theta}_1 < \bar{\theta}_2 < \cdots < \bar{\theta}_n$. Then we say that m is *consonant* (i.e. free of conflict in evidence), and the associated plausibility measure, Pl^m, is a possibility measure.

In fact, for every $\pi \in \Pi_L$, there is some unique bpa m such that $\pi = \mathrm{Pl}_L^m$.

We remark that U_L may be reexpressed as

$$U_L(r) = \sum_{\bar{\theta} \in \overline{SL}} m(\bar{\theta}) \log_2 |\mathrm{At}_L(\bar{\theta})|$$

(where m is a bpa such that $\mathrm{Pl}_L^m = \pi^r$). So U_L is essentially just the standard unspecificity measure of evidence theory.

2.3. Conditional Possibility

The meaning of the concept of conditional possibility is still a controversial issue. In [18], Dubois and Prade consider axioms for conditional possibility and derive two main choices:

(H) *Hisdal conditioning* [23]:

$$\pi(\bar{\theta}|\bar{\phi}) = \begin{cases} \pi(\overline{\theta \wedge \phi}) & \text{if} \quad \pi(\overline{\theta \wedge \phi}) < \pi(\bar{\phi}), \\ 1 & \text{otherwise} \end{cases}$$

for any $\bar{\theta}$, $\bar{\phi} \in \overline{SL}$, $\pi \in \Pi_L$.

(D) *Dempster conditioning* [3]:

$$\pi(\bar{\theta}|\bar{\phi}) = \begin{cases} \dfrac{\pi(\overline{\theta \wedge \phi})}{\pi(\bar{\phi})} & \text{if} \quad \pi(\bar{\phi}) > 0, \\ \text{undefined} & \text{otherwise} \end{cases}$$

for any $\bar{\theta}$, $\bar{\phi} \in \overline{SL}$, $\pi \in \Pi_L$.

Another suggestion is

(dC) *deCampos-Lamata-Moral conditioning* [24]:

$$\pi(\bar{\theta}|\bar{\phi}) = \begin{cases} \dfrac{\pi(\overline{\theta \wedge \phi})}{\pi(\overline{\theta \wedge \phi}) + (1 - \pi(\overline{\theta \vee \neg \phi})} \\ \text{if} \quad \pi(\overline{\theta \wedge \phi}) > 0 \text{ or } \pi(\overline{\theta \vee \neg \phi}) < 1, \\ \text{undefined otherwise} \end{cases}$$

for any $\bar{\theta}$, $\bar{\phi} \in \overline{SL}$, $\pi \in \Pi_L$.

Several alternative definitions have also appeared in the literature (e.g. Nguyen conditioning [25]).

2.4. Knowledge Bases and Inference Processes

A possibilistic knowledge base is a formal representation of a set of inexact rules provided by an expert.

DEFINITION 2.8 *We define a* possibilistic knowledge base on L *as a finite set of constraints*

$$\pi(\bar{\theta}_i) \in [\alpha_i, \alpha_i'], \qquad 1 \le i \le n,$$

$$\pi(\bar{\phi}_j | \bar{\psi}_j) \in [\beta_j, \beta_j'], \qquad 1 \le j \le m,$$

where $\bar{\theta}_i$, $\bar{\phi}_j$, $\bar{\psi}_j \in \overline{SL}$ *and* $0 \le \alpha_i \le \alpha_i' \le 1$, $0 \le \beta_j \le \beta_j' \le 1$ *for* $1 \le i \le n$, $1 \le j \le m$, *and* $m, n \in \mathbb{N}$.

We allow intervals of belief values, since the expert may feel too uncertain about his belief in the possibility of some statement to represent it by a precise number.

Let E denote the empty knowledge base.

DEFINITION 2.9 *Let* $\mathrm{H}\Pi_L(K) = \{r \in R_L | \pi^r$ *satisfies* K *using* (H)$\}$.

Let $\mathrm{D}\Pi_L(K) = \{r \in R_L | \pi^r$ *satisfies* K *using* (D)$\}$.

Let $\mathrm{H\Pi IK}_L = \{K | \mathrm{H}\Pi_L(K) \neq \emptyset\}$.

Let $\mathrm{H\Pi K}_L = \{K \in \mathrm{H\Pi IK}_L | \alpha_i = \alpha_i', \ \beta_j = \beta_j'$ *for* $1 \leq i \leq n, \ 1 \leq j \leq m\}$.

Let $\mathrm{D\Pi IK}_L = \{K | \mathrm{D}\Pi_L(K) \neq \emptyset\}$.

We say that K is consistent if there are possibility measures on L satisfying the constraints K. An inference process maps consistent possibilistic knowledge bases to sets of possibility measures (distributions) on the underlying logical language. Formally,

DEFINITION 2.10 *We say that* N *is a* Hisdal inference process *if* $N: \mathrm{H\Pi K} \to PR$ *where* $\mathrm{H\Pi K} = \{(L, K) | L \in \mathbf{L}, \ K \in \mathrm{H\Pi}K_L\}$, $PR = \cup_{L \in \mathbf{L}} \mathrm{pow}(R_L)$, *and* $N(L, K) \subseteq R_L$ *for any* $L \in \mathbf{L}$.

This definition is analogous to Paris and Vencovska's definition of probabilistic inference processes [15]. We write $N_L(K)$ instead of $N(L, K)$. For every $K \in \mathrm{H\Pi K}_L$ and $\bar{\theta} \in \overline{SL}$, define $N_L(K)(\bar{\theta}) = \{\pi^r(\bar{\theta}) | r \in N_L(K)\}$. Then $N_L(K)(\bar{\theta})$ is the set of possible values of $\pi(\bar{\theta})$ inferred from K.

We say that N is a *Dempster inference process* if $N: \mathrm{D\Pi IK} \to PR$, where $\mathrm{D\Pi IK} = \{(L, K) | L \in \mathbf{L}, K \in \mathrm{D\Pi IK}_L\}$, and $N(L, K) \subseteq R_L$ for any $L \in \mathbf{L}$.

2.5. Principles of Ampliative Possibilistic Inference

DEFINITION 2.11 *We define the following principles of ampliative possibilistic inference, which are analogous to those introduced by Paris and Vencovska in* [14, 15]:

Existence. *For any* $L \in \mathrm{L}$ *and any* $K \in \mathrm{H\Pi K}_L$, *one has* $N_L(K) \neq \emptyset$.
(*Justification*: *It should be possible to infer possibility values from consistent knowledge bases.*)

Compatibility. *If* $L \in \mathrm{L}$ *and* $K \in \mathrm{H\Pi K}_L$, *then* $N_L(K) \subseteq \mathrm{H}\Pi_L(K)$.
(*Justification*: *The belief values inferred from the knowledge base* K *should be consistent with* K.)

Equivalence. *If* $L \in \mathrm{L}$ *and* $K_1, \ K_2 \in \mathrm{H\Pi K}_L$ *and* $\mathrm{H}\Pi_L(K_1) = \mathrm{H}\Pi_L(K_2)$, *then* $N_L(K_1) = N_L(K_2)$.
(*Justification*: *Only the content of the knowledge base should determine* N, *not the way in which it is expressed.*)

Uniqueness. *For any* $L \in \mathrm{L}$ *and every* $K \in \mathrm{H\Pi K}_L$, $\overline{\theta} \in \overline{SL}$, *the set* $N_L(K)(\overline{\theta})$ *is a singleton.*
(*Justification*: *Inferred beliefs can be adequately represented by a single number.*)

If N satisfies uniqueness, then $N_L(K) = \{r\}$ for some $r \in R_L$. In this situation, we shall abuse notation by writing $N_L(K)$ to mean π^r rather than $\{r\}$.

Language invariance. *If* L, $L' \in \mathbf{L}$, $L \subset L'$, *and* $K \in \mathrm{H\Pi K}_L$, *then* $N_L(K)(\overline{\theta}) = N_{L'}(K)(\overline{\theta})$ *for any* $\overline{\theta} \in \overline{SL}$.
(*Justification*: *Inferred beliefs should be independent of the choice of overlying language.*)

Relevance. *If* L_1, $L_2 \in \mathrm{L}$ *and* $L_1 \cap L_2 = \emptyset$, *and* $K_1 \in \mathrm{H\Pi K}_{L_1}$ *and* $K_2 \in \mathrm{H\Pi K}_{L_2}$, *then* $N_{L \cup L_2}$ $(K_1 \cup K_2)(\theta_1) = N_{L_1}(K_1)(\overline{\theta}_1)$ *for every* $\overline{\theta_1} \in \overline{SL_1}$. (*Note that if* $K_1 \in \mathrm{H\Pi K}_{L_1}$ *and* $K_2 \in \mathrm{H\Pi K}_{L_2}$, *then* $K_1 \cup K_2 \in \mathrm{H\Pi K}_{L_1 \cup L_2}$.)
(*Justification*: *Irrelevant information should not alter inferred beliefs.*)

Obstinacy. *Suppose that* $L \in \mathbf{L}$ *and* K_1, $K_2 \in \mathrm{H\Pi K}_L$. *If* $N_L(K_1)$ *satisfies* K_2, *that is,* $N_L(K_1) \subseteq \mathrm{H\Pi}_L(K_2)$, *then* $N_L(K_1 \cup K_2) = N_L(K_1)$.
(*Justification*: *If* K_2 *is already inferred by* K_1, *then* K_2 *is providing no new information. So adding* K_2 *to the knowledge base should not alter inferred beliefs.*)

Relativization. *Suppose that* $L \in \mathbf{L}$, $\overline{\theta} \in \overline{SL}$, $\mathbf{0} < \overline{\theta} < \mathbf{1}$, *that* K_1 *is*

$$\pi\left(\overline{\theta \wedge \phi_i}\right) = \alpha_i \quad (1 \le i \le n),$$
$$\pi(\overline{\theta}) = \alpha,$$
$$\pi\left(\overline{\neg\, \theta}\right) = \beta,$$

and that K_2 *is*

$$\pi\left(\overline{\neg\, \theta \wedge \psi_j}\right) = \beta_j \qquad (1 \le j \le m),$$

where $\overline{\phi_i}$, $\overline{\psi_j} \in \overline{SL}$ *and* α_i, β_j, α, $\beta \in [0, 1]$ *for any* $1 \le i \le n$ *and* $1 \le j \le m$. *If* $K_1 \cup K_2 \in H\Pi K_L$, *then* $N_L(K_1 \cup K_2)(\overline{\theta \wedge \phi}) = N_L(K_1)(\overline{\theta \wedge \phi})$ *for any* $\overline{\phi} \in \overline{SL}$. *Justification*: *Assume that* θ *holds. Then, since* K_2 *only concerns the case when* θ *doesn't hold, it is providing no additional information to that given by the knowledge base* K_1. *Hence adding* K_2 *to* K_1 *shouldn't alter inferred beliefs about conjuncts with* θ.)

Strong symmetry. *If* $L \in \mathrm{L}$, $g \in \mathrm{Aut}(\overline{SL})$, $K \in \mathrm{H\Pi K}_L$, *and* $\overline{\theta} \in \overline{SL}$, *then* $N_L(g(K))(g(\overline{\theta})) = N_L(K)(\overline{\theta})$, *where* $g(K)$ *is just* K *with* $\overline{\theta_i}$, $\overline{\phi_j}$, $\overline{\psi_j}$ *replaced by* $g(\overline{\theta_i})$, $g(\overline{\phi_j})$, $g(\overline{\psi_j})$ *respectively.*

(*Justification*: $g(K)$ *and* $g(\bar{\theta})$ *are just renamed versions of* K *and* $\bar{\theta}$ *respectively. Inferred beliefs should be invariant under renaming.*)

Symmetry. *If* $L \in \mathrm{L}$, $g \in \mathrm{Aut}(\overline{SL})$, *and* $g(\bar{p}) \in \{\bar{q}, \overline{\neg q} | q \in L\}$ *for any* $p \in L$, *then* $N_L(g(K))(g(\bar{\theta})) = N_L(K)(\bar{\theta})$ *for every* $\bar{\theta} \in \overline{SL}$, $K \in H\Pi K_L$.
(*Justification: Renaming of literals should cause no change in inferred beliefs.*)

Atomicity. Suppose that $L, L' \in \mathrm{L}$ *and* $p \in L$. *Suppose that* $K \in \mathrm{H\Pi K}_L$ *and* K^θ *is* K *with* p *replaced everywhere by* θ, *some sentence in* L' (*where* L, L' *are disjoint and* $\mathbf{0} < \bar{\theta} < \mathbf{1}$). *Let* $L'' = L \cup L' \setminus \{p\}$. *Then, for any* $\bar{\psi} \in \overline{SL}$, $N_L(K)(\bar{\psi}) = N_{L''}(K^\theta)(\overline{\psi^\theta})$.
(*Justification*: *Reexpressing a proposition as a logical combination of simpler propositions should not alter inferred beliefs. In practice, there is a limit to the depth of analysis of a proposition, but in principle any proposition can be decomposed indefinitely.*)

(*Note that if* $K \in \mathrm{H\Pi K}_L$ *then* $K^\theta \in \mathrm{H\Pi K}_{L''}$.)

Before continuing, we need the following preliminary definitions and notation.

DEFINITION 2.12 *Suppose that* L *is a finite propositional language. Suppose that* d_L *is the Euclidean metric on* $\mathrm{R}^{\mathrm{at}_L}$. *Note that* $R_L \subseteq \mathrm{R}^{\mathrm{At}_L}$. *Now* $[0, 1]^{\mathrm{At}_L}$ *is compact, and it is readily verified that* R_L *is closed. Hence,* R_L *is compact. For any* $\delta > 0$ *and any* $S \subseteq R_L$, *define*

$$D_L^\delta(S) = \{r \in R_L | d_L(r, r') \leq \delta \text{ for some } r' \in S\}.$$

Define $\rho_L : \mathrm{pow}(R_L) \times \mathrm{pow}(R_L) \to \mathbf{R}^+$ *by*

$$\rho_L(S_1, S_2) = \inf\{\delta_1 + \delta_2 | S_1 \subseteq D_L^{\delta_1}(S_2) \text{ and } S_2 \subseteq D_L^{\delta_2}(S_1)\}.$$

For any $\delta > 0$, $S \subseteq [0, 1]$, *define*

$$D_\delta(S) = \{y \in [0, 1] | |x - y| \leq \delta \text{ for some } x \in S\}.$$

Define $\rho : \mathrm{pow}([0, 1]) \times \mathrm{pow}([0, 1]) \to \mathbf{R}^+$ *by*

$$\rho(S_1, S_2) = \inf\{\delta_1 + \delta_2 | S_1 \subseteq D_{\delta_1}(S_2) \text{ and } S_2 \subseteq D_{\delta_2}(S_1)\}.$$

DEFINITION 2.11 (Continued)

Continuity. *If* $L \in \mathrm{L}$, $K_n \in \mathrm{H\Pi K}_L$ *for every* $n \in \mathbf{N}^+$, $K \in \mathrm{H\Pi K}_L$, *and* $\rho_L(\mathrm{H\Pi}_L(K_n), \mathrm{H\Pi}_L(\mathrm{K})) \to 0$, *then* $\rho_L(N_L(K_n), N_L(K)) \to 0$.
(*Justification*: *Microscopic changes in the knowledge base should not result in macroscopic changes in inferred beliefs.*)

Open-mindedness. *If* $L \in \mathrm{L}$, $K \in \mathrm{H\Pi K}_L$, $\bar{\theta} \in \overline{SL}$, $\epsilon > 0$, *and* $K \cup \{\pi(\bar{\theta}) = \epsilon\}$ *is consistent, then* $N_L(K)(\bar{\theta}) \neq \{0\}$.

(*Justification*: *If θ is possible according to K, we shouldn't infer that θ is impossible.*)

Analogous principles may be defined for Dempster inference processes.

3. CHARACTERIZATION OF MAXIMUM U-UNCERTAINTY

3.1. The Hisdal Maximum-U-uncertainty Inference Process

DEFINITION 3.1 *We define the Hisdal maximum-U-uncertainty inference process, denoted* HMU, *as follows*:

For any $L \in \mathbf{L}$, $K \in \mathrm{H\Pi K}_L$, $\quad \mathrm{HMU}_L(K) = \arg \max_{r \in \mathrm{H\Pi}_L(K)} \{U_L(r)\}$,

i.e., $\mathrm{HMU}_L(K)$ *is the set of those* $r \in \mathrm{H\Pi}_L(K)$ *that maximize* $U_L(r)$.

LEMMA 3.1 *Suppose that* $C \subseteq R_L$ *and satisfies*:

(1) *if* $r_1, r_2 \in C$ *then* $\max\{r_1, r_2\} \in C$;

(2) *if* $(r_n)_{n \in \mathbf{N}^+} \subseteq C$ *such that* $r_n \leq_L r_{n+1}$ *for every* $n \in \mathbf{N}^+$ *and* $r_N \to r$ *as* $n \to \infty$, *then* $r \in C$.

Then there is some unique $r^* \in C$ such that $\arg \max_{r \in C}\{U_L(r)\} = \{r^*\}$.

Proof For every $\bar{\gamma} \in \mathrm{At}_L$, let $s(\bar{\gamma}) = \sup\{r(\bar{\gamma}) | r \in C\}$. Then, for each $n \in \mathbf{N}^+$, there is some $r_n^{\bar{\gamma}} \in C$ such that

$$s(\bar{\gamma}) - \frac{1}{n} < r_n^{\bar{\gamma}}(\bar{\gamma}) \leq s(\bar{\gamma}) \quad \text{for every} \quad n \in \mathbf{N}^+.$$

Let $r_n = \max\{r_n^{\bar{\gamma}} | \bar{\gamma} \in \mathrm{At}_L\}$. Then $(r_n)_{n \in \mathbf{N}^+} \subseteq C$ by (1). Now, for every $n \in \mathbf{N}^+$, let $r'_n = \max\{r_1, r_2, \ldots, r_n\}$. Then, $(r'_n)_{n \in \mathbf{N}^+} \subseteq C$ by (1). Furthermore, obviously, $r'_n \leq_L r'_{n+1}$ for every $n \in \mathbf{N}^+$.

Clearly, $r'_n \to r^*$ as $n \to \infty$, where $r^*(\bar{\gamma}) = s(\bar{\gamma})$ for every $\bar{\gamma} \in \mathrm{At}_L$. Then, by (2), $r^* \in C$. Now, for any $r \in C$, $r(\bar{\gamma}) \leq s(\bar{\gamma}) = r^*(\bar{\gamma})$ for every $\bar{\gamma} \in \mathrm{At}_L$. Hence, $r \leq_L r^*$ for any $r \in C$, and if $r \neq r^*$ then $r <_L r^*$. Hence, $U_L(r) < U_L(r^*)$ for every $r \in C \setminus \{r^*\}$, and the result follows. ■

PROPOSITION 3.1 HMU *satisfies all of the stated principles in Definition* 2.11.

Proof It is straightforward to verify that HMU satisfies equivalence and compatibility [since $\mathrm{HMU}_L(K) \subseteq H\Pi_L(K)$ by definition].

We now show that HMU satisfies uniqueness (and hence also existence). Suppose that $K \in \mathrm{H\Pi K}_L$. Then K is of the form

$$\pi(\overline{\theta_i}) = \alpha_i \qquad (1 \leq i \leq n),$$
$$\pi(\overline{\phi_j | \psi_j}) = \beta_j \qquad (1 \leq j \leq m).$$

So, by Hisdal's definition of conditional possibility, $r \in \mathrm{H\Pi}_L(K)$ if and only if π^r satisfies

$$\pi^r\left(\overline{\theta_i}\right) = \alpha_i \qquad (1 \leq i \leq n),$$
$$\pi^r\left(\overline{\phi_j \wedge \psi_j}\right) = \beta_j \qquad (j \in J),$$
$$\pi^r\left(\overline{\psi_j}\right) > \beta_j \qquad (j \in J),$$
$$\pi^r\left(\overline{\phi_j \wedge \psi_j}\right) = \pi^r\left(\overline{\psi_j}\right) \qquad (j \in J')$$

(where $J = \{j | \beta_j < 1\}$ and $J' = \{j | \beta_j = 1\}$). It is straightforward to verify that $C = \mathrm{H\Pi}_L(K)$ satisfies conditions (1) and (2) of Lemma 3.1 and hence that

$$\mathrm{HMU}_L(K) = \{r^*\}$$

for some unique $r^* \in \mathrm{H\Pi}_L(K)$. So HMU satisfies uniqueness.

It follows that $\mathrm{HMU}_L(K)(\overline{\theta}) = \pi^{r^*}(\overline{\theta}) = \max\{\pi^r(\overline{\theta}) | r \in \mathrm{H\Pi}_L(K)\}$ for any $\overline{\theta} \in \overline{SL}$.

Suppose that $L \in \mathbf{L}$ and $K_1, K_2 \in \mathrm{H\Pi K}_L$, and $\mathrm{HMU}_L(K_1)$ satisfies K_2, that is, $\emptyset \neq \mathrm{HMU}_L(K_1) \subseteq \mathrm{H\Pi}_L(K_2)$. Hence $\mathrm{HMU}_L(K_1) \subseteq \mathrm{H\Pi}_L(K_1) \cap \mathrm{H\Pi}_L(K_2) = \mathrm{H\Pi}_L(K_1 \cup K_2) \subseteq \mathrm{H\Pi}_L(K_1)$, and so,

$$\max_{r \in \mathrm{HMU}_L(K_1)} \{U_L(r)\} \leq \max_{r \in \mathrm{H\Pi}_L(K_1) \cap \mathrm{H\Pi}_L(K_2)} \{U_L(r)\} \leq \max_{r \in \mathrm{H\Pi}_L(K_1)} \{U_L(r)\}$$

Hence, $\max_{r \in \mathrm{H\Pi}_L(K_1) \cap \mathrm{H\Pi}_L(K_2)}\{U_L(r)\} = \max_{r \in \mathrm{H\Pi}_L(K_1)} \{U_L(r)\}$. So $\mathrm{HMU}_L(K_1 \cup K_2) = \mathrm{HMU}_L(K_1)$, and so HMU satisfies obstinacy.

We now show that HMU satisfies relativization. Assume the notation of that principle. Suppose that $r_2 \in \mathrm{H\Pi}_L(K_1 \cup K_2)$. If $r \in \mathrm{H\Pi}_L(K_1)$, define $r' \in R_L$ by

$$r'(\overline{\gamma}) = \begin{cases} r(\overline{\gamma}) & \text{for every } \overline{\gamma} \in \mathrm{At}_L(\overline{\theta}), \\ r_2(\overline{\gamma}) & \text{for every } \overline{\gamma} \in \mathrm{At}_L\left(\overline{\neg\theta}\right). \end{cases}$$

Then $r' \in \mathrm{H\Pi}_L(K_1 \cup K_2)$. So, for any $\overline{\gamma} \in \mathrm{At}_L(\overline{\theta})$,

$$\{r(\overline{\gamma}) | r \in \mathrm{H\Pi}_L(K_1)\} = \{r'(\overline{\gamma}) | r' \in \mathrm{H\Pi}_L(K_1 \cup K_2)\}.$$

Hence, $\mathrm{HMU}_L(K_1)(\overline{\gamma}) = \mathrm{HMU}_L(K_1 \cup K_2)(\overline{\gamma})$ for any $\overline{\gamma} \in \mathrm{At}_L(\overline{\theta})$, and the result follows.

We have shown (Lemma 2.2) that if $r \in R_L$ and $g \in \mathrm{Aut}(\overline{SL})$, then $\mu_L(g(r)) = \mu_L(r)$ and so $U_L(g(r)) = U_L(r)$. So HMU satisfies strong symmetry and hence symmetry.

We now show that HMU satisfies relevance (and hence language invariance also). Suppose that L_1, L_2 are disjoint and that $L = L_1 \cup L_2$.

Suppose that $K_1 \in \mathrm{H\Pi K}_{L_1}$, $K_2 \in \mathrm{H\Pi K}_{L_2}$. Then $K_1 \cup K_2 \in \mathrm{H\Pi K}_L$. Suppose that $r_1 \in \mathrm{H\Pi}_{L_1}(K_1)$ and $r_2 \in \mathrm{H\Pi}_{L_2}(K_2)$.

Define $r \in [0, 1]^{\mathrm{At}_L}$ as follows:

$$r(\overline{\gamma_1 \wedge \gamma_2}) = \min\{r_1(\overline{\gamma_1}), r_2(\overline{\gamma_2})\} \qquad \text{for every} \quad \overline{\gamma_1} \in \mathrm{At}_{L_1} \quad \overline{\gamma_2} \in \mathrm{At}_{L_2}.$$

It is readily verified that $r \in \mathrm{H\Pi}_L(K_1 \cup K_2)$. Also, for any $\overline{\gamma}_1 \in \mathrm{At}_{L_1}$, $\pi^r(\overline{\gamma}_1) = r_1(\overline{\gamma}_1)$. If $r \in \mathrm{H\Pi}_L(K_1 \cup K_2)$, then define $r_1(\overline{\gamma}_1) = \pi^r(\overline{\gamma}_1)$ for any $\overline{\gamma}_1 \in \mathrm{At}_{L_1}$. Then $r_1 \in \mathrm{H\Pi}_{L_1}(K_1)$. Hence, $\{\pi^r(\overline{\gamma}_1) | r \in \mathrm{H\Pi}_L(K_1 \cup \mathrm{K}_2)\} = \{r_1(\overline{\gamma}_1) | r_1 \in \mathrm{H\Pi}_{L_1}(K_1)\}$. Hence, $\mathrm{HMU}_L(K_1 \cup K_2)(\overline{\gamma}_1) = \mathrm{HMU}_{L_1}(K_1)(\overline{\gamma}_1)$ for any $\overline{\gamma}_1 \in \mathrm{At}_{L_1}$, as required.

Next we show that HMU satisfies atomicity. Assume the notation of the statement of that principle.

If $r \in \mathrm{H\Pi}_L(K)$, define $r'' \in [0, 1]^{\mathrm{At}_{L''}}$ as follows:

$$r''(\overline{\gamma''}) = r(\overline{\gamma}) \qquad \text{for every} \quad \overline{\gamma''} \in \mathrm{At}_{L''}(\overline{\gamma^\theta}) \text{ and } \overline{\gamma} \in \mathrm{At}_L.$$

Hence, $\pi^{r''}(\overline{\gamma^\theta}) = r(\overline{\gamma})$ for every $\overline{\gamma} \in \mathrm{At}_L$ and so $r'' \in \mathrm{H\Pi}_{L''}(K^\theta)$. Now, if $r'' \in \mathrm{H\Pi}_{L''}(K^\theta)$, define $r' \in [0, 1]^{\mathrm{At}_L}$ as follows:

$$r'(\overline{\gamma}) = \pi^{r''}(\overline{\gamma^\theta}) \qquad \text{for every} \quad \overline{\gamma} \in \mathrm{At}_L.$$

Then $r' \in \mathrm{H\Pi}_L(K)$. We have shown that, for every $\overline{\gamma} \in \mathrm{At}_L$,

$$\{r(\overline{\gamma}) | r \in \mathrm{H\Pi}_L(K)\} = \left\{\pi^{r''}(\overline{\gamma^\theta}) | r'' \in \mathrm{H\Pi}_{L''}(K^\theta)\right\}$$

and hence that $\mathrm{HMU}_L(K)(\overline{\gamma}) = \mathrm{HMU}_{L''}(K^\theta)(\overline{\gamma^\theta})$, as required.

Next we show that HMU satisfies continuity. Suppose that $(K_n)_{n \in \mathbb{N}^+} \subseteq \mathrm{H\Pi K}_L$ and that $K_n \to K$ as $n \to \infty$, where $K \in \mathrm{H\Pi K}_L$. So $\mathrm{H\Pi}_L(\mathrm{K}_n) \to \mathrm{H\Pi}_L(\mathrm{K})$, and hence

$$\rho(\{r(\overline{\gamma}) | r \in \mathrm{H\Pi}_L(K_n)\}, \{r(\overline{\gamma}) | r \in \mathrm{H\Pi}_L(K)\}) \to 0 \qquad \text{as } n \to \infty$$

for any $\overline{\gamma} \in \mathrm{At}_L$. Hence,

$$|\max\{r(\overline{\gamma}) | r \in \mathrm{H\Pi}_L(K_n)\} - \max\{r(\overline{\gamma}) | r \in \mathrm{H\Pi}_L(K)\}| \to 0,$$

and so $\mathrm{HMU}_L(K_n)(\overline{\gamma}) \to \mathrm{HMU}_L(K)(\overline{\gamma})$ as $n \to \infty$. So HMU satisfies continuity.

Finally, HMU satisfies open-mindedness, since if $\overline{\theta} \in \overline{SL}$, $\epsilon > 0$, $K \in \mathrm{H\Pi K}_L$, and $K \cup \{\pi(\overline{\theta}) = \epsilon\}$ is consistent, then

$$\mathrm{HMU}_L(K)(\overline{\theta}) = \max_{r \in \mathrm{H\Pi}_L(K)} \{\pi^r(\overline{\theta})\} \geq \epsilon > 0. \qquad \blacksquare$$

We now demonstrate that a small subset of these principles is sufficient to characterize HMU uniquely.

Let RK_L be the set of conditional-free possibilistic knowledge bases on L. That is, $\mathrm{RK}_L = \{K \in \mathrm{H\Pi K}_L | m = 0\}$.

LEMMA 3.2 *If N is a Hisdal inference process and satisfies compatibility, equivalence, uniqueness, obstinacy, and strong symmetry, then*

$$N_L \downarrow \mathrm{RK}_L = \mathrm{HMU}_L \downarrow \mathrm{RK}_L.$$

Proof Firstly, note that, by Proposition 3.1, HMU satisfies all of the stated principles. Suppose that $K \in \mathrm{RK}_L$. So K is a consistent set of equations

$$\pi\left(\overline{\theta_i}\right) = \alpha_i \qquad \text{for} \quad 1 \le i \le n,$$

where $\overline{\theta_i} \in \overline{SL}$, $\alpha_i \in [0, 1]$ for $1 \le i \le n$, and $n \in \mathbf{N}^+$.

We can assume, without loss of generality, that

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n.$$

Suppose that $q = |\{\alpha_1, \alpha_2, \dots, \alpha_n\}|$.

Let $a(1) = 1$. Let $a(k) = \min\{a(k-1) < i \le n | \alpha_{a(k-1)} < \alpha_i\}$ for $2 \le k \le q$. For $1 \le k \le q-1$, let $A(k) = \{a(k), \dots, a(k+1) - 1\}$, and let $A(q) = \{a(q), \dots, n\}$.

Let $\overline{\mu_k} = \bigvee_{l \in A(k)} \overline{\theta_l}$ for $1 \le k \le q$. Let $\beta_k = \alpha_{a(k)}$. So, if $i_1, i_2 \in A(k)$ then $\alpha_{i_1} = \alpha_{i_2} = \beta_k$. Let K' be the knowledge base

$$\pi\left(\overline{\mu_k}\right) = \beta_k \qquad \text{for} \quad 1 \le k \le q.$$

Note that $\beta_1 < \beta_2 < \cdots < \beta_q$.

Suppose that $r \in \mathrm{H\Pi}_L(K)$. Then

$$\begin{aligned}\pi^r\left(\overline{\mu_k}\right) &= \pi^r\left(\bigvee_{i \in A(k)} \overline{\theta_i}\right) \\ &= \max_{i \in A(k)} \left\{\pi^r\left(\overline{\theta_i}\right)\right\} = \max_{i \in A(k)} \{\alpha_i\} = \beta_k\end{aligned}$$

for every $1 \le k \le q$. Hence $r \in \mathrm{H\Pi}_L(K')$, and we have shown that $\mathrm{H\Pi}_L(K) \subseteq \mathrm{H\Pi}_L(K')$, and so K' is consistent.

Suppose that $\beta_q < 1$. For each $1 \le k \le q$, let $\overline{\psi_k} = \overline{\mu_k} \wedge \neg(\bigvee_{1 \le j < k} \overline{\mu_j})$, and let $\overline{\psi_{q+1}} = \neg(\bigvee_{1 \le j \le q} \overline{\mu_j})$. Clearly, $\bigvee_{1 \le j \le q+1} \overline{\psi_j} = \mathbf{1}$ and $\overline{\psi_k} \wedge \overline{\psi_l} = \mathbf{0}$ for $k \ne l$. Also, $\overline{\psi_k} \le \overline{\mu_k}$ for $1 \le k \le q$.

Now, let K'' be the knowledge base

$$\begin{aligned}&\pi\left(\overline{\psi_k}\right) = \beta_k \qquad \text{for} \quad 1 \le k \le q, \\ &\pi\left(\overline{\psi_{q+1}}\right) = 1.\end{aligned}$$

It is easy to verify that $H\Pi_L(K') \subseteq H\Pi_L(K'')$. Now, by compatibility and uniqueness, $N_L(K'') \in H\Pi_L(K'')$.

Fix $1 \leq k \leq q+1$. Now, $N_L(K'')(\overline{\psi_k}) = \max_{\overline{\gamma} \in \mathrm{At}_L(\overline{\psi_k})}\{N_L(K'')(\overline{\gamma})\} = \beta_k$. So there is some $\overline{\gamma} \in \mathrm{At}_L(\overline{\psi_K})$ such that $N_L(K'')(\overline{\gamma}) = \beta_k$. But if $\overline{\gamma}_1$, $\overline{\gamma}_2 \in \mathrm{At}_L(\overline{\psi_k})$, define $g : \mathrm{At}_L \to \mathrm{At}_L$ by

$$g(\overline{\gamma}_2) = \overline{\gamma}_1 \text{ and } g(\overline{\gamma}_1) = \overline{\gamma}_2 \quad \text{and} \quad g(\overline{\delta}) = \overline{\delta} \quad \text{for every} \quad \overline{\delta} \in \mathrm{At}_L \setminus \{\overline{\gamma}_2, \overline{\gamma}_1\}.$$

Let g^* be the unique automorphism of $\overline{SL}$ such that $g^* \downarrow \mathrm{At}_L = g$. Then $g^*(\overline{\psi_k}) = \overline{\psi_k}$ for $1 \leq k \leq q+1$ and hence $H\Pi_L(g^*(K'')) = H\Pi_L(K'')$. So by equivalence, $N_L(g^*(K''))(g^*(\overline{\gamma}_1)) = N_L(K'')(\overline{\gamma}_2)$. But by strong symmetry, $N_L(g^*(K''))(g^*(\overline{\gamma}_1)) = N_L(K'')(\overline{\gamma}_1)$, and so $N_L(K'')(\overline{\gamma}_1) = N_L(K'')(\overline{\gamma}_2)$. Hence, $N_L(K'')(\overline{\gamma}) = \beta_k$ for every $\overline{\gamma} \in \mathrm{At}_L(\overline{\psi_k})$, and this holds for each $1 \leq k \leq q+1$.

But for any $\overline{\gamma} \in \mathrm{At}_L$, there is a unique $1 \leq k \leq q+1$ such that $\overline{\gamma} \leq \overline{\psi_k}$. Hence, for any $r \in H\Pi_L(K'')$, we have $r(\overline{\gamma}) = \pi^r(\overline{\gamma}) \leq \pi^r(\overline{\psi_k}) = \beta_k = N_L(K'')(\overline{\gamma})$ for each $\overline{\gamma} \in \mathrm{At}_L$. Hence, $r \leq_L N_L(K'')$ for every $r \in H\Pi_L(K) \subseteq H\Pi_L(K'')$. Furthermore, $N_L(K'') \in H\Pi_L(K)$, since for $1 \leq i \leq n$, there is some unique $1 \leq k \leq q$ such that $i \in A(k)$.

Now, $N_L(K'')(\overline{\theta}_i) \leq N_L(K'')(\bigvee_{i \in A(k)}\overline{\theta}_i) = N_L(K'')(\overline{\mu}_k) = \beta_k = \alpha_i$. So $N_L(K'')(\overline{\theta}_i) \leq \alpha_i$. However, $r \leq_L N_L(K'')$ for every $r \in H\Pi_L(K)$. Hence, $\alpha_i \leq \pi^r(\overline{\theta}_i) \leq N_L(K'')(\overline{\theta}_i)$ for $1 \leq i \leq n$. Hence, $N_L(K'')(\overline{\theta}_i) = \alpha_i$ for $1 \leq i \leq n$, and so $N_L(K'') \in H\Pi_L(K)$. Since $H\Pi_L(K) \subseteq H\Pi_L(K'')$, it follows from obstinacy that $N_L(K) = N_L(K'')$ and so $r \leq_L N_L(K)$ for every $r \in H\Pi_L(K)$. So if $r \in H\Pi_L(K) \setminus \{N_L(K)\}$, then $r <_L N_L(K)$ and hence

$$U_L(r) < U_L(N_L(K)).$$

Hence, $N_L(K) = \mathrm{HMU}_L(K)$, as required.

The case when $\beta_q = 1$ is similar, except that in this case we define $\overline{\psi_q} = \neg(\bigvee_{1 \leq k < q}\overline{\mu}_k)$ and we define K'' to be the knowledge base

$$\pi\left(\overline{\psi_k}\right) = \beta_k \qquad \text{for} \quad 1 \leq k \leq q.$$

Then the proof goes through as before. ∎

Definition 3.2 *Let* $\mathrm{H\Pi K}^*_L = \{K \in \mathrm{H\Pi K}_L \mid \beta_j < 1$ *for every* $1 \leq j \leq m\}$.

Theorem 3.1 *If N is a Hisdal inference process and satisfies compatibility, equivalence, uniqueness, obstinacy, and strong symmetry, then*

$$N_L \downarrow \mathrm{H\Pi K}^*_L = \mathrm{HMU}_L \downarrow \mathrm{H\Pi K}^*_L.$$

Proof Suppose that $K \in \mathrm{H\Pi K}_L^*$. So K is consistent and of the form

$$\pi(\bar{\theta}_i) = \alpha_i \quad \text{for} \quad 1 \le i \le n,$$
$$\pi(\bar{\theta}_j | \bar{\psi}_j) = \beta_j \quad \text{for} \quad 1 \le j \le m,$$

where $\beta_j < 1$ for $1 \le j \le m$. Then, by Hisdal's definition of conditional possibility, $r \in \mathrm{H\Pi}_L(K)$ if

$$\pi^r(\bar{\theta}_i) = \alpha_i \quad \text{for} \quad 1 \le i \le n,$$
$$\pi^r(\overline{\phi_j \wedge \psi_j}) = \beta_j \quad \text{for} \quad 1 \le j \le m,$$
$$\pi^r(\bar{\psi}_j) > \beta_j \quad \text{for} \quad 1 \le j \le m.$$

Let K' be the knowledge base

$$\pi(\bar{\theta}_i) = \alpha_i \quad \text{for} \quad 1 \le i \le n,$$
$$\pi(\overline{\phi_j \wedge \psi_j}) = \beta_j \quad \text{for} \quad 1 \le j \le m.$$

Now, clearly $\mathrm{H\Pi}_L(K) \subseteq \mathrm{H\Pi}_L(K')$, and so K' is consistent. By Lemma 3.2, $N_L(K')(\theta) = \mathrm{HMU}_L(K')(\bar{\theta}) = \max\{\pi^r(\bar{\theta}) | r \in \mathrm{H\Pi}_L(K')\}$.

Fix $1 \le j \le m$. Then, since K is consistent, there is some $r \in \mathrm{H\Pi}_L(K)$ such that

$$\pi^r(\bar{\psi}_j) > \beta_j.$$

But $r \in \mathrm{H\Pi}_L(K')$ also. So $\beta_j < \pi^r(\bar{\psi}_j) \le N_L(K')(\bar{\psi}_j)$. Since this holds for each $1 \le j \le m$, it follows that $N_L(K') \in \mathrm{H\Pi}_L(K)$. So, by obstinacy, $N_L(K) = N_L(K')$.

Now, $r' \le_L N_L(K')$ for any $r \in \mathrm{H\Pi}_L(K')$, and so $r \le_L N_L(K)$ for any $r \in \mathrm{H\Pi}_L(K)$. Hence, $N_L(K) = \mathrm{HMU}_L(K)$, as required. ■

This theorem is not quite adequate, since it doesn't apply to knowledge bases containing conditionals with possibility 1. We conjecture that this theorem still holds if we replace $\mathrm{H\Pi K}_L^*$ by $\mathrm{H\Pi}K_{\mathrm{L}}$.

3.2. The Dempster Maximum-U-Uncertainty Inference Process

DEFINITION 3.3 We define the Dempster maximum-U-uncertainty inference process, denoted DMU, as follows:

$$\textit{for any } L \in \mathbf{L},\ K \in \mathrm{D\Pi IK}_L, \qquad \mathrm{DMU}_L(K) = \arg \max_{r \in \mathrm{D\Pi}_L(K)} \{U_L(r)\}.$$

LEMMA 3.3 *Given* $x_1, y_1, a, b \in [0, 1]$ *and* $x_2, y_2 \in (0, 1]$,

$$\textit{if} \quad \frac{x_1}{x_2}, \frac{y_1}{y_2} \in [a, b] \quad \textit{then} \quad \frac{\max\{x_1, y_1\}}{\max\{x_2, y_2\}} \in [a, b].$$

Proof Suppose, without loss of generality, that $x_1 < y_1$. Then, if $x_2 \le y_2$,

$$\frac{\max\{x_1, y_1\}}{\max\{x_2, y_2\}} = \frac{y_1}{y_2} \in [a, b].$$

Otherwise, $x_2 > y_2$, so $a \le x_1/x_2 < y_1/x_2 < y_1/y_2 \le b$, and so

$$\frac{\max\{x_1, y_1\}}{\max\{x_2, y_2\}} = \frac{y_1}{x_2} \in [a, b].$$ ■

PROPOSITION 3.2 DMU *satisfies all of the principles in Definition* 2.11.

Proof This is very similar to the proof for HMU (Proposition 3.1) Note that, for any $K \in \mathrm{D\Pi IK}_L$, $\mathrm{D\Pi}_L(K)$ satisfies condition (1) of Lemma 3.1 by Lemma 3.3. ■

DEFINITION 3.4 *Let* $\mathrm{RIK}_L = \{K \in \mathrm{D\Pi IK}_L | m = 0\}$.

LEMMA 3.4 *Using the notation of Theorem* 3.2 (*see below*),

$$\mathrm{D\Pi}_L(\mathscr{R}_L) = \{\mathrm{D\Pi}_L(R) | R \in \mathscr{R}_L\}$$

is compact with respect to the metric ρ_L.

Proof For any $R \in \mathscr{R}_L$, and any $\bar{\gamma} \in \mathrm{At}_L(\overline{\neg\gamma'})$, let $R(\bar{\gamma})$ be the knowledge base

$$\pi\left(\overline{\gamma'}\right) \in \left[a''\left(\overline{\gamma'}\right), b''\left(\overline{\gamma'}\right)\right], \qquad \pi(\bar{\gamma}) = 1.$$

Then $\mathrm{D\Pi}_L(R) = \bigcup_{\bar{\gamma} \in \mathrm{At}_L(\overline{\neg\gamma'})} \mathrm{D\Pi}_L(R(\bar{\gamma}))$.

Let $\mathscr{R}_L(\bar{\gamma}) = \{R(\bar{\gamma}) | R \in \mathscr{R}_L\}$. Then $\mathrm{D\Pi}_L(R(\bar{\gamma})) \subseteq R_L$ is closed, convex, and nonempty for every $R \in \mathscr{R}_L$ and $\bar{\gamma} \in \mathrm{At}_L(\overline{\neg\gamma'})$. It is easy to show that $\{\mathrm{D\Pi}_L(R(\bar{\gamma})) | R \in \mathscr{R}_L(\bar{\gamma})\}$ is closed for any $\bar{\gamma} \in \mathrm{At}_L(\overline{\neg\gamma'})$. We noted earlier that R_L is compact, so $\mathscr{R}_L(\bar{\gamma})$ is compact.

Suppose now that $(R_n)_{n \in \mathbb{N}^+} \subseteq \mathscr{R}_L$. Then, for any $\bar{\gamma} \in \mathrm{At}_L(\overline{\neg\gamma'})$ one has $(R_n(\bar{\gamma}))_{n \in \mathbb{N}^+} \subseteq \mathscr{R}_L(\bar{\gamma})$. So there is some $(R_{n_{k(\bar{\gamma})}}(\bar{\gamma}))_{k(\bar{\gamma}) \in \mathbb{N}^+} \subseteq (R_n(\bar{\gamma}))_{n \in \mathbb{N}^+} \subseteq \mathscr{R}_L(\bar{\gamma})$ and $R(\bar{\gamma}) \in \mathscr{R}_L(\bar{\gamma})$ such that $\rho_L(\mathrm{D\Pi}_L(R_{n_{k(\bar{\gamma})}}(\bar{\gamma})), \mathrm{D\Pi}_L(R(\bar{\gamma}))) \to 0$ as $k(\bar{\gamma}) \to \infty$. Hence, there is some $(R_{n_k})_{k \in \mathbb{N}^+} \subseteq (R_n)_{n \in \mathbb{N}^+}$ and $R \in \mathscr{R}_L$ such that, for every $\bar{\gamma} \in \mathrm{At}_L(\overline{\neg\gamma'})$, one has $\rho_L(\mathrm{D\Pi}_L(R_{n_k}(\bar{\gamma})), D\Pi_L(R(\bar{\gamma}))$ $\to 0$ as $k \to \infty$ and so

$$\bigcup_{\bar{\gamma} \in \mathrm{At}_L(\overline{\neg\gamma'})} \mathrm{D\Pi}_L\left(R_{n_k}(\bar{\gamma})\right) \to \bigcup_{\bar{\gamma} \in \mathrm{At}_L(\overline{\neg\gamma'})} \mathrm{D\Pi}_L(R(\bar{\gamma})) \quad \text{as} \quad k \to \infty.$$

That is, $\mathrm{D\Pi}_L(R_{n_k}) \to \mathrm{D\Pi}_L(R)$ as $k \to \infty$, and hence $\mathrm{D\Pi}_L(\mathscr{R}_L)$ is compact, as required. ■

THEOREM 3.2 *Suppose that N is a Dempster inference process. If N satisfies compatibility, equivalence, uniqueness, obstinacy, relativization, symmetry, and continuity, then $N =$ DMU.*

Proof Note that by Proposition 3.2, DMU satisfies all of the stated principles. Suppose that $L \in \mathbf{L}$. Define $R : \mathrm{D\Pi IK}_L \to \mathrm{RIK}_L$ as follows: For any $K \in \mathrm{D\Pi IK}_L$, let $R(K)$ be the knowledge base

$$\pi(\overline{\gamma}) \in [a(\overline{\gamma}), b(\overline{\gamma})] \qquad \text{for every} \quad \overline{\gamma} \in \mathrm{At}_L,$$

where $a(\overline{\gamma}) = \inf\{r(\overline{\gamma}) | r \in \mathrm{D\Pi}_L(K)\}$ and $b(\overline{\gamma}) = \max\{r(\overline{\gamma}) | r \in \mathrm{D\Pi}_L(K)\}$ for every $\overline{\gamma} \in \mathrm{At}_L$. Hence $D\Pi_L(K) \subseteq \mathrm{D\Pi}_L(R(K))$.

Given $K \in \mathrm{D\Pi IK}_L$, denote $R(K)$ by R. Since $\mathrm{D\Pi}_L(K) \subseteq R_L$, there is some $\overline{\gamma} \in \mathrm{At}_L$ such that $b(\overline{\gamma}) = 1$.

Now, choose any $\gamma' \in \mathrm{At}_L$. There are two cases (1) $b(\overline{\gamma'}) = 1$ and (2) $b(\overline{\gamma'}) < 1$.

Case (1): Let R_1 be

$$\begin{aligned} \pi(\overline{\gamma'}) &\in \left[a(\overline{\gamma'}), 1\right] \\ \pi(\overline{\neg\gamma'}) &\in \left[a(\overline{\neg\gamma'}), 1\right], \\ \pi(\overline{\gamma}) &\in [a(\overline{\gamma}), b(\overline{\gamma})] \qquad \text{for every} \quad \overline{\gamma} \neq \overline{\gamma'}, \end{aligned}$$

where $a(\overline{\neg\gamma'}) = \max_{\overline{\gamma} \neq \overline{\gamma'}}\{a(\overline{\gamma})\}$. Then $\mathrm{D\Pi}_L(R) = \mathrm{D\Pi}_L(R_1)$. So by equivalence, $N_L(R)(\overline{\gamma'}) = N_L(R_1)(\overline{\gamma'})$. If $b(\overline{\gamma}) < 1$ for every $\overline{\gamma} \neq \gamma'$, then $r(\gamma') = 1$ for every $r \in \mathrm{D\Pi}_L(R)$, and hence $N_L(R)(\overline{\gamma'}) = 1$ by compatibility and uniqueness.

If $b(\overline{\gamma}) = 1$ for some $\overline{\gamma} \neq \overline{\gamma'}$, then consider the knowledge base R_1' given by

$$\begin{aligned} \pi(\overline{\gamma'}) &\in \left[a(\overline{\gamma'}), 1\right], \\ \pi(\overline{\neg\gamma'}) &\in \left[a(\overline{\neg\gamma'}), 1\right]. \end{aligned}$$

Then, by relativization, $N_L(R_1)(\overline{\gamma'}) = N_L(R_1')(\overline{\gamma'})$.

Suppose that $L = \{p_1, p_2, \ldots, p_n\}$, $\gamma_1 = p_1^{\delta_1} \wedge \cdots \wedge p_n^{\delta_n}$, and $\gamma_2 = p_1^{\epsilon_1} \wedge \cdots \wedge p_n^{\epsilon_n}$, where $\underline{\delta}, \underline{\epsilon} \in \{0, 1\}^n$. Define g by

$$g\left(\overline{p_i^{\epsilon_i}}\right) = \overline{p_i^{\delta_i}} \text{ and } g\left(\overline{p_i^{\delta_i}}\right) = \overline{p_i^{\epsilon_i}} \qquad \text{for} \quad 1 \leq i \leq n.$$

Then g can be uniquely extended to an automorphism $g^* \in \mathrm{Aut}(\overline{SL})$ such that $\mathrm{D\Pi}_L(g^*(E)) = \mathrm{D\Pi}_L(E)$, where E is the empty knowledge base, g^* satisfies the conditions of the symmetry principle, and $g^*(\overline{\gamma}_1) = \overline{\gamma}_2$ and $g^*(\overline{\gamma}_2) = \overline{\gamma}_1$. So, by symmetry, $N_L(g^*(E))(g^*(\overline{\gamma}_2)) = N_L(E)(\overline{\gamma}_2)$. But by equivalence, $N_L(E)(\overline{\gamma}_1) = N_L(g^*(E))(g^*(\overline{\gamma}_2))$. So $N_L(E)(\overline{\gamma}_1) = N_L(E)(\overline{\gamma}_2)$ for any $\overline{\gamma}_1, \overline{\gamma}_2 \in \mathrm{At}_L$. However, since $N_L(E) \in R_L$ (by compatibility and uniqueness), there must be some $\overline{\gamma} \in \mathrm{At}_L$ such that $N_L(E)(\overline{\gamma}) = 1$. It follows that $N_L(E)(\overline{\gamma}) = 1$ for every $\overline{\gamma} \in \mathrm{At}_L$.

Now, $\mathrm{D\Pi}_L(R'_1) \subseteq \mathrm{D\Pi}_L(E) = R_L$ and $N_L(E) \in \mathrm{D\Pi}_L(R'_1)$. So, by obstinacy, $N_L(R'_1) = N_L(E)$. Hence, $N_L(R)(\overline{\gamma'}) = N_L(R'_1)(\overline{\gamma'}) = N_L(E)(\overline{\gamma'}) = 1$. So, if $b(\overline{\gamma'}) = 1$, $N_L(R)(\overline{\gamma'}) = 1$ also.

Case (2): Suppose now that $b(\overline{\gamma'}) < 1$, where $\overline{\gamma'} \in \mathrm{At}_L$. Then R is equivalent to the knowledge base, R' given by

$$\begin{aligned} \pi(\overline{\gamma'}) &\in \left[a(\overline{\gamma'}), b(\overline{\gamma'})\right], \\ \pi(\overline{\neg\gamma'}) &= 1, \\ \pi(\overline{\gamma}) &\in [a(\overline{\gamma}), b(\overline{\gamma})] \qquad \text{for every} \quad \overline{\gamma} \neq \overline{\gamma'}. \end{aligned}$$

Let R'' be the knowledge base

$$\begin{aligned} \pi(\overline{\gamma'}) &\in \left[a(\overline{\gamma'}), b(\overline{\gamma'})\right], \\ \pi(\overline{\neg\gamma'}) &= 1. \end{aligned}$$

Then, by relativization, $N_L(R')(\overline{\gamma'}) = N_L(R'')(\overline{\gamma'})$. Suppose that $N_L(R'')(\overline{\gamma'}) < b(\overline{\gamma'})$, and let $\epsilon = b(\overline{\gamma'}) - N_L(R'')(\overline{\gamma'}) > 0$. Let $\mathscr{R}_L = \{R \in \mathrm{RIK}_L | R$ is of form $\pi(\overline{\gamma'}) \in [a''(\overline{\gamma'}), b''(\overline{\gamma'})], \pi(\overline{\neg\gamma'}) = 1\}$.

By continuity and Lemma 3.4, N_L is uniformly continuous on $\mathscr{R}_L$. So there is some $\delta > 0$ such that if $\rho_L(\mathrm{D\Pi}_L(R_1), \mathrm{D\Pi}_L(R_2)) < \delta$ then

$$d_L(N_L(R_1), N_L(R_2)) < \epsilon \qquad \text{for every} \quad R_1, R_2 \in \mathscr{R}_L. \sim (*)$$

Now, choose $M \in \mathbf{N}^+$ such that $M > [1 - b(\overline{\gamma'})]/\delta$. Then, for each $i \in \mathbf{N}^+$ such that $0 \leq i \leq M$, define R''_i to be the knowledge base

$$\begin{aligned} \pi(\overline{\gamma'}) &\in \left[a(\overline{\gamma'}), b(\overline{\gamma'}) + \frac{i}{M}\left[1 - b(\overline{\gamma'})\right]\right], \\ \pi(\overline{\neg\gamma'}) &= 1. \end{aligned}$$

Then $\mathrm{D\Pi}_L(R'') = \mathrm{D\Pi}_L(R''_0) \subset \mathrm{D\Pi}_L(R''_1) \subset \cdots \subset \mathrm{D\Pi}_L(R''_M)$. For all i such that $1 \leq i \leq M$,

$$\rho_L(\mathrm{D\Pi}_L(R''_{i-1}), \mathrm{D\Pi}_L(R''_i)) = \frac{1 - b(\overline{\gamma'})}{M} < \delta$$

and so $d_L(N_L(R''_{i-1}), N_L(R''_i)) < \epsilon$ by $(*)$. Now, from $d_L(N_L(R''_0), N_L(R''_1)) < \epsilon$, we can deduce that $N_L(R''_1) \in \mathrm{D\Pi}_L(R''_0)$. This follows because if

$N_L(R''_1) \notin D\Pi_L(R''_0)$, then $N_L(R''_1)(\overline{\gamma'}) > b(\overline{\gamma'})$, and so

$$d_L(N_L(R''_1), N_L(R''_0)) > N_L(R''_1)(\overline{\gamma'}) - N_L(R''_0)(\overline{\gamma'}) > b(\overline{\gamma'}) - N_L(R''_0)(\overline{\gamma'}) = \epsilon,$$

a contradiction. Since $D\Pi_L(R''_0) \subset D\Pi_L(R''_1)$, by obstinacy we can deduce that $N_L(R''_1) = N_L(R''_0)$. We apply the above argument M times to deduce that $N_L(R''_i) = N_L(R''_0)$ for every $1 \le i \le M$. Hence, $N_L(R'') = N_L(R''_M)$. But $N_L(E) \in D\Pi_L(R''_M)$, and so $N_L(R''_M) = N_L(E)$ by obstinacy.

However, $N_L(E) \notin D\Pi_L(R'')$, since $N_L(E)(\overline{\gamma'}) = 1 > b(\overline{\gamma'})$ by assumption. So we have shown that the supposition that $N_L(R'')(\overline{\gamma'}) < b(\overline{\gamma'})$ leads to a contradiction, and it must be that $N_L(R'')(\overline{\gamma'}) = b(\overline{\gamma'})$ and hence that $N_L(R)(\overline{\gamma'}) = b(\overline{\gamma'})$. We have demonstrated that, for any $\overline{\gamma} \in \mathrm{At}_L$, $N_L(R)(\overline{\gamma}) = b(\overline{\gamma})$.

Now, by Lemma 3.1, there is some $r^* \in D\Pi_L(K)$ such that $r^*(\overline{\gamma}) = b(\overline{\gamma})$ for every $\overline{\gamma} \in \mathrm{At}_L$. So $N_L(R) \in D\Pi_L(K)$, and hence, by obstinacy, $N_L(K) = N_L(R)$.

Now, for any $r \in D\Pi_L(K)$, we have $r(\overline{\gamma}) \le b(\overline{\gamma}) = N_L(K)(\overline{\gamma})$ for any $\overline{\gamma} \in \mathrm{At}_L$, and so $N_L(K) = \mathrm{DMU}_L(K)$ for any $K \in \mathrm{D\Pi K}_L$. That is, $N_L = \mathrm{DMU}_L$, as required. ■

4. CONCLUSIONS

From purely logical considerations, we have derived the maximum-U-uncertainty inference process, thus giving a logical justification for preferring it to, say, a maximum-fuzziness inference process. This result has been shown for both Hisdal's and Dempster's definitions of conditional possibility. However, we note that the maximum-U-uncertainty inference process, deCMU, does not satisfy uniqueness when we use deCampos, Lamata, and Moral's definition of conditional possibility. A simple example is K defined as follows ($L = \{p, q\}$):

$$\pi(\bar{p}|\bar{q}) = \frac{1}{\log_2 3},$$

$$\pi(\overline{p \wedge \neg q}) = 0.$$

Then $\mathrm{deCMU}_L(K) = \{r \in \mathrm{deC\Pi}_L(K) | r(\overline{p \wedge q}) \le r(\overline{\neg p \wedge \neg q}), r(\overline{\neg p \wedge q}) = 1\}$.

Some open problems remain:

- Does Theorem 1 remain valid when $\mathrm{H\Pi K}^*_L$ is replaced by $\mathrm{H\Pi K}_L$?
- There are alternative definitions of conditional possibility (e.g. [25]). Does the characterization result still hold using these definitions?

- Are the results optimal in the sense that all the principles assumed in each proof are actually necessary?

In comparison with ampliative probabilistic reasoning, we note that the maximum-entropy inference process satisfies analogs of all the stated principles, except atomicity. It is shown in [14] that atomicity is inconsistent with uniqueness, compatibility, equivalence, relevance and strong symmetry, i.e., no probabilistic inference process satisfies all of these six principles. The problem is that in the case of complete ignorance (i.e. an empty knowledge base), we are forced by strong symmetry, uniqueness, and compatibility to allocate equal probabilities to all atomic propositions. This problem arises from the inability of probability measures to model lack of knowledge correctly. The maximum-U-uncertainty inference process, on the other hand, satisfies all of the stated principles.

We note also that computing nontrivial approximations to the maximum-entropy inference process is NP-hard, as shown in [16]. Furthermore, any probabilistic inference process satisfying uniqueness, compatibility, equivalence, and strong symmetry is NP-hard, as shown in [26]. Unfortunately, although apparently much simpler, computing a nontrivial approximations to HMU or DMU is also NP-hard and hence probably infeasible in general [27]. In fact, the theorem shown in [26] is also valid for possibilistic inference processes [27].

Note that since U-uncertainty is identical to unspecificity of Dempster-Shafer theory (see Section 2.2), it is of interest to ask if the above results hold also for the minimum-specificity inference process applied to general evidential knowledge bases. The answer is negative, since it doesn't satisfy uniqueness (see[27]). Also of interest is the problem of updating possibilistic knowledge bases with new but conflicting information. It may be possible to derive logically a unique updating process using ideas similar to those developed here.

It is also possible that, assuming a different definition of conditional possibility, an alternative inference process can be characterized by the logical axioms—e.g. maximization of strife or discord, or minimization of an alternative measure of specificity.

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