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Perspective

Bus interconnection networks

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Abstract

In bus interconnection networks every bus provides a communication medium between a set of processors. These networks are modeled by hypergraphs where vertices represent the processors and edges represent the buses. We survey the results obtained on the construction methods that connect a large number of processors in a bus network with given maximum processor degree Δ, maximum bus size r, and network diameter D. (In hypergraph terminology this problem is known as the (Δ, D, r)-hypergraph problem.)

The problem for point-to-point networks (the case $r = 2$) has been extensively studied in the literature. As a result, several families of networks have been proposed. Some of these point-to-point networks can be used in the construction of bus networks. One approach is to consider the dual of the network. We survey some families of bus networks obtained in this manner. Another approach is to view the point-to-point networks as a special case of the bus networks and to generalize the known constructions to bus networks. We provide a summary of the tools developed in the theory of hypergraphs and directed hypergraphs to handle this approach.

1. Introduction

A bus interconnection network is a collection of processing elements (processors) and communication elements (buses). The processors produce and/or consume messages and the buses provide communication channels to exchange messages among the processors. Every bus provides a communication link between two or more processors.

For practical reasons, a processor may only be connected to a limited number of buses (this number is known as the processor degree) and a bus may only connect a limited number of processors (this number is known as the bus size). Therefore, messages may have to be relayed by a number of intermediate processors before arriving at their destinations, and thus the message transmission time becomes a function of the distance (measured in terms of the number of buses traversed by a message) between

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processors. The maximum distance over all pairs of processors is the network diameter. Fig. 1 depicts a bus network of degree 3, bus size 3, and diameter 1. For some other examples see Fig. 2.

For given upper bounds on the processor degree Δ, bus size r, and network diameter D, the construction of bus networks with maximal number of processors is an important problem in the design of interconnection networks. Our aim is to survey the results obtained on this problem with an emphasis on the tools used in the construction. Other design parameters such as network reliability, symmetry properties, ease of message routing, balanced message traffic throughout the network, implementation issues (algorithms and architecture) should also be taken into consideration.

In the case of traditional point-to-point networks, where a link can connect only two processors (these networks are modeled by graphs) the aforementioned problem has been extensively studied in the literature. As a result, different families of networks with large number of processors for given degree and diameter have been proposed. (Surveys on this topic can be found in [6, 11, 23] and also in the special issues [3, 41].)

Although considerably less studied, the construction of bus networks ($r > 2$) is receiving more interest due to technological advances (see e.g. [43]). In this paper, we survey the results obtained on the construction methods that connect a large number of processors in a bus network, given Δ, D, and r.

This paper is organized as follows. Section 2 deals with the undirected bus networks. In Section 2.1 we give the terminology of hypergraphs and define the (Δ, D, r)-hypergraph problem. In Section 2.2, an upper bound on the number of vertices in (Δ, D, r)-hypergraphs (known as the Moore bound) is introduced and some general results concerning this bound are given. Section 2.3 is devoted to the case of diameter 1 in which there are infinitely many (Δ, D, r)-hypergraphs attaining the Moore bound. This case is a subject of study in Combinatorial Design Theory. In Section 2.4 we survey the results in the case of degree 2, where the duality tools are helpful. In Section 2.5, we describe compound techniques to obtain large (Δ, D, r)-hypergraphs. In Section 2.6, we survey various other families of bus networks proposed in the literature.

Section 3 deals with directed bus networks. We give the terminology in Section 3.1. The Moore bound and the directed hypergraphs that attain it are the subject of Section 3.2. In Section 3.3 we give two infinite families of directed hypergraphs that approach the Moore bound asymptotically, and generalize the well-known de Bruijn and Kautz networks.

2. Design of bus networks

We use hypergraphs to represent the underlying topology of the bus interconnection networks. The vertices of the hypergraph correspond to the processors and the edges correspond to the buses.

2.1. (Δ, D, r)-hypergraph problem

An (*undirected*) *hypergraph* H is a pair $H = (\mathcal{V}(H), \mathcal{E}(H))$, where $\mathcal{V}(H)$ is a non-empty set of elements, called *vertices*, and $\mathcal{E}(H)$ is a finite set of subsets of $\mathcal{V}(H)$ called *edges*. The number of vertices in the hypergraph is $n(H) = |\mathcal{V}(H)|$ and the number of edges is $m(H) = |\mathcal{E}(H)|$ where the vertical bars denote the cardinality of the set. The *degree* of a vertex v is the number of edges containing it and is denoted by $\Delta_H(v)$. The *maximum degree* over all of the vertices in H is denoted by $\Delta(H)$. The *size* of an edge $E \in \mathcal{E}(H)$ is its cardinality, and is denoted by $|E|$. The *rank* of H is the size of its largest edge, and is denoted by $r(H)$. A *path* in H from vertex u to vertex v is an alternating sequence of vertices and edges $u = v_0, E_1, v_1, \ldots, E_k, v_k = v$ such that $\{v_{i-1}, v_i\} \subseteq E_i$ for all $1 \leqslant i \leqslant k$. The *length* of a path is the number of edges in it. The *distance* between two vertices u and v is the length of a shortest path between them. The *diameter* of H is the maximum of the distances over all pairs of vertices, and is denoted by $D(H)$.

We call a hypergraph with maximum degree Δ, diameter D, and rank r, a (Δ, D, r)-*hypergraph*. The problem on bus networks we considered in the introduction is known as the (Δ, D, r)-*hypergraph problem* and consists of finding (Δ, D, r)-hypergraphs with the maximum number of vertices or finding large (Δ, D, r)-hypergraphs. The maximum number of vertices in any (Δ, D, r)-hypergraph is denoted by $n(\Delta, D, r)$.

In the case $r = 2$ (graph case), this problem has been extensively studied and is known as the (Δ, D)-graph problem (see e.g. [11, 12]), and the maximum number of vertices in any (Δ, D)-graph is denoted by $n(\Delta, D)$.

Note that parts of this problem have been studied in other contexts with different notation. For example d or r is used for maximum degree, k or d is used for diameter, and b or k is used for rank. (In the notation of Design Theory r and k are used for maximum degree and rank, respectively.) We follow the notation of Hypergraph Theory [2].

Finally, let us mention that the drawing of hypergraphs can be very complex and therefore it is useful to represent a hypergraph H with a bipartite graph,

$$R(H) = (\mathcal{V}_1(R) \cup \mathcal{V}_2(R), \mathcal{E}(R))$$

called the *bipartite representation graph*. Every vertex v_i in $\mathcal{V}(H)$ is represented by a vertex v_i in $\mathcal{V}_1(R)$ and every edge E_j in $\mathcal{E}(H)$ is represented by a vertex e_j in $\mathcal{V}_2(R)$. We draw an edge between $v_i \in \mathcal{V}_1(R)$ and $e_j \in \mathcal{V}_2(R)$ if and only if $v_i \in E_j$ in H.

If H is a (Δ, D, r)-hypergraph and $R(H)$ is its bipartite representation graph, then the maximum degrees in $\mathcal{V}_1(R)$ and in $\mathcal{V}_2(R)$ are Δ and r, respectively. The distance between two vertices of $\mathcal{V}_1(R)$ is at most $2D$, but the diameter of $R(H)$ can be $2D$, $2D + 1$ or $2D + 2$ as the vertices of $\mathcal{V}_1(R)$ and $\mathcal{V}_2(R)$ do not play the same role. So, the (Δ, D, r)-hypergraph problem is partly related but different from the $(\Delta_1, \Delta_2; D')$-bipartite graph problem, i.e. finding large bipartite graphs with maximum vertex degrees Δ_1, Δ_2 and diameter D' (for details of this problem see [19]). Nevertheless, this bipartite representation can be helpful.

2.2. Moore bound and Moore geometries

A bound on the maximum number of vertices in a (Δ, D, r)-hypergraph (analogous to the the classical Moore bound [40]) can easily be calculated: Each vertex belongs to at most Δ edges and each edge contains at most r vertices. Thus there can be at most $\Delta(r-1)$ vertices at distance one from any vertex. In general, the maximum number of vertices at distance i from any vertex can be at most $\Delta(\Delta-1)^{i-1}(r-1)^i$. Therefore,

Proposition 1. $n(\Delta, D, r) \leqslant 1 + \Delta(r-1)\sum_{i=0}^{D-1}(\Delta-1)^i(r-1)^i$.

This bound is known as the *Moore bound for undirected hypergraphs*, and the hypergraphs that attain it are known as *Moore geometries*.

Combined results of Fuglister [37, 38], Damerell and Georgiacodis [27], Damerell [26], Kuich and Sauer [46], Bose and Dowling [20], and Kantor [44] show that, for $D > 2$, Moore geometries cannot exist, with the exception of the cycles of length $2D+1$ (the case $\Delta = 2$ and $r = 2$). For a comprehensive survey on these results see [6].

For $D = 2$ and $r \neq 5$, Moore geometries can exist only for a finite number of cases. For $r = 3$, Moore geometries do not exist; and for $r = 4$ a Moore geometry with $\Delta = 7$ may exist (with 400 vertices, and 700 edges). However, there are no known Moore geometries with $D = 2$ and $r > 2$.

For $D = 2$ and $r = 2$ (graph case), only four Moore graphs can exist. Three of them are the pentagon ($\Delta = 2$), the Petersen graph ($\Delta = 3$) and the Hoffman–Singleton graph ($\Delta = 7$) [40]. A fourth Moore graph with $\Delta = 57$ may exist. This graph, if exists, cannot be vertex-transitive (see [22], p. 102; [1]).

2.3. Case $D = 1$

In a hypergraph of diameter 1 every pair of vertices belongs to at least one common edge. The reader might see the similarity with Design Theory. (For more information on Design Theory see e.g. [42], and for the use of Design Theory in Computer Science see the survey [25].) Recall that an (n, r, λ) *design* on a set of n objects (called "points") is a collection of subsets (called "blocks") such that every block contains exactly r points and every pair of points belongs to exactly λ blocks. In fact, Moore geometries of diameter 1 are the $(n, r, 1)$ designs:

Proposition 2. $n(\Delta, 1, r) \leqslant 1 + \Delta(r-1)$, *and the equality is attained if and only if there exists an* $(n, r, 1)$ *design.*

Simple counting arguments show that $n\Delta = mr$ in an $(n, r, 1)$ design, where m is the number of blocks (edges). Fisher's inequality (cf. [42, p. 34]) states that $\Delta \geqslant r$ in any (n, r, λ) design with $n > r$ points. If $\Delta = r$, the existence of a $(\Delta, 1, r)$-hypergraph depends on the existence of a $(q^2+q+1,\ q+1, 1)$ design, known as a projective plane

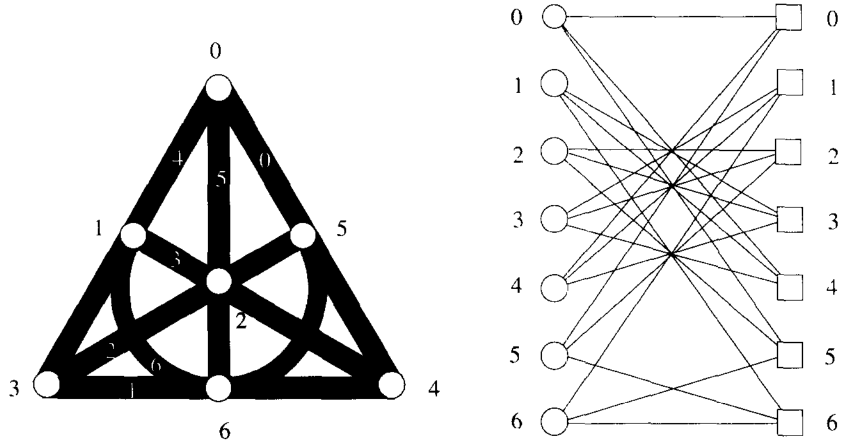

Fig. 1. Fano plane: (3, 1, 3)-hypergraph and its bipartite representation graph.

of order q, with $q = r - 1$. This was already pointed out by Mickunas [48]. It is well known that a projective plane exists when q is a prime power. It is also known from the Bruck–Ryser–Chowla theorem (cf. [42, p. 56]) that a projective plane does not exist when $q \equiv 1, 2 \pmod 4$, and q is not the sum of two integer squares (for example $q = 6$ or $q = 14$). Fig. 1 depicts the symmetric (7,3,1) design (also known as the *Fano plane*). In this figure the circles represent the points (vertices) and the (thick) lines represent the blocks (edges).

When the Moore bound cannot be attained, tight upper bounds on $n(\Delta, 1, r)$ are established in [8, 9, 4]. For $\Delta \geqslant r$, the results on coverings were used. We give the following theorem as an example.

Theorem 3 (Bermond et al. [8]). *If* $\Delta \geqslant r$, *then*

$$n(\Delta, 1, 3) = \begin{cases} 2\Delta + 1 & \text{if } \Delta \equiv 0 \text{ or } 1 \pmod 3, \\ 2\Delta & \text{if } \Delta \equiv 2 \pmod 3). \end{cases}$$

In the case $D = 1$ and $\Delta < r$, Bermond et al. [8] showed that a $(\Delta, 1, r)$-hypergraph with $\Delta r - (\Delta - 1)\lceil r/\Delta \rceil$ vertices can be constructed if there exists a projective plane of order $\Delta - 1$, namely by splitting each vertex into roughly r/Δ vertices. Füredi [39] proved that this bound is asymptotically optimal.

2.4. Duality tools

The *dual* of a hypergraph $H = (\mathcal{V}(H), \mathcal{E}(H))$ is the hypergraph $H^* = (\mathcal{V}(H^*), \mathcal{E}(H^*))$ where the vertices of H^* correspond to the edges of H, and the edges of H^* correspond to the vertices of H. A vertex e_j^* is a member of an edge V_i^* in H^* if and only if the vertex v_i is a member of E_j in H. Fig. 2 shows some graphs and their dual hypergraphs.

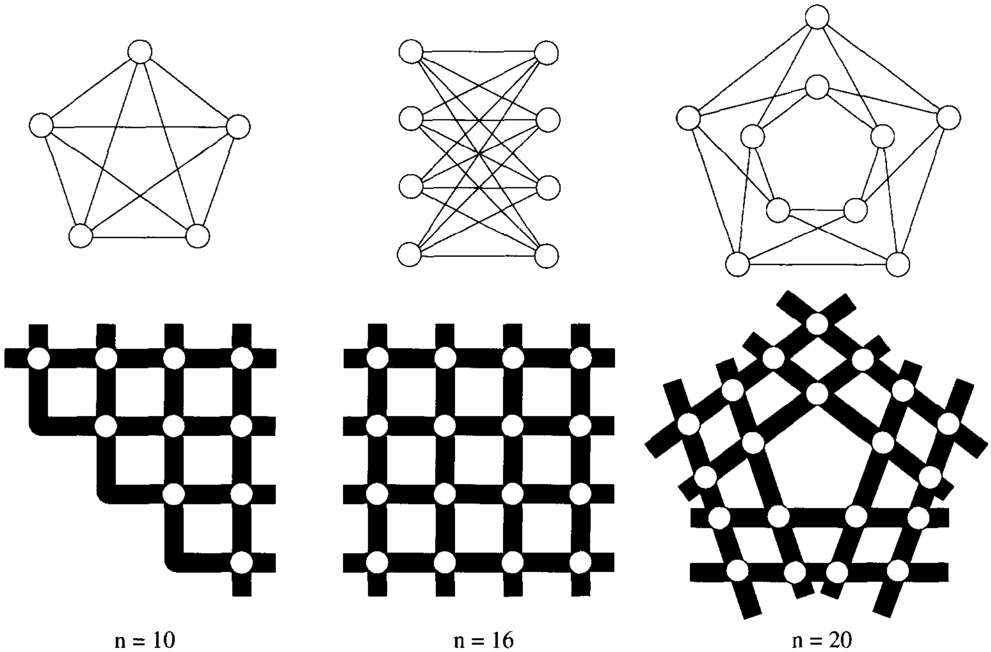

Fig. 2. Some (2, 2, 4)-hypergraphs obtained by dual hypergraph operation.

Bermond et al. [7] observed the following relationship between a hypergraph and its dual.

Proposition 4 (Bermond et al. [7]). *If H is a (Δ, D, r)-hypergraph then its dual hypergraph H^* is a (r, D^*, Δ)-hypergraph, where $D - 1 \leqslant D^* \leqslant D + 1$.*

Note that, if G is a graph of maximum degree Δ and diameter D then its dual is a $(2, D^*, \Delta)$-hypergraph. Furthermore,

Proposition 5 (Bermond et al. [7]). *If G is a bipartite (Δ, D)-graph then its dual hypergraph H^* is a $(2, D^*, \Delta)$-hypergraph, where $D^* \leqslant D$.*

With the help of Propositions 4 and 5, it is possible to construct large $(2, D, r)$-hypergraphs by using the existing large (Δ, D)-graphs. Below, we give some examples. (For more examples, see [7].)

The dual hypergraph of the binary hypercube was conceived as a bus network [55], where every edge of the hypercube represents a processor and every vertex represents a bus. Thus every processor is connected to two buses and if the hypercube is d-dimensional then every bus connects d processors. Using the same technique the dual hypergraph of the generalized hypercube has also been proposed as a bus network [17].

In [52], Pradhan proposed a generalization of the shuffle-exchange network. The dual of this network gives a hypergraph of degree 2, rank r, diameter $2k - 1$ (k is a positive integer). It has $r^{k+1}/2$ vertices.

Kautz graphs are obtained from Kautz digraphs [15] by replacing the directed edges with undirected ones. Kautz graphs of maximum degree r (r is even) and diameter D have $(r/2)^D + (r/2)^{D-1}$ vertices. The dual hypergraph of the Kautz graph of diameter $D-1$, and maximum degree r, is a $(2,D,r)$-hypergraph with $(r/2)^D + (r/2)^{D-1}$ vertices.

The *bipartite double* $\tilde{G}$ of a graph G is constructed as follows [21]: For every vertex $v \in \mathcal{V}(G)$ there are two vertices v^+ and $v^- \in \mathcal{V}(\tilde{G})$. The vertices v_i^+ and v_j^- are adjacent in $\tilde{G}$, if and only if the vertices v_i and v_j are adjacent in G. Let $\tilde{G}$ be the bipartite double of the de Bruijn graph (see [15], or [53]) of maximum degree r (r is even), and diameter $D-1$. Then $\tilde{G}$ is regular of degree r, has $2(r/2)^{D-1}$ vertices and diameter D. The dual hypergraph of $\tilde{G}$ is a $(2,D,r)$-hypergraph with $2(r/2)^D$ vertices.

An extension of the bipartite double of de Bruijn graphs, is the "C_s graphs" of Delorme and Farhi [28]. The vertices of "C_s graphs" are labeled by $(i; a_1,\dots,a_k)$, where i belongs to Z_q (set of integers modulo q), and a_j $(1 \leqslant j \leqslant k)$, belongs to an alphabet A of d letters. The vertex $(i; a_1,\dots,a_k)$ is joined to the vertices $(i+1; a_2,\dots,a_k,\alpha)$ and $(i-1; \alpha, a_1,\dots,a_{k-1})$ where α is any letter from the alphabet A. These graphs have qd^k vertices and maximum degree $2d$. Their diameter depends on the values of q and k. For example, if $q=3$, the diameter is always $k+1$. If $q=5$, the diameter is $k+1$ for $k \equiv 1$ or $4 \bmod 5$), and $k+2$, otherwise.

Bermond et al. [7] used the dual hypergraphs of "C_s graphs" to obtain large $(2,D,r)$-hypergraphs. A particular case of the "C_s graphs" ($q=5$ and $k=1$) is the graph G obtained from the pentagon by replacing each vertex with $r/2$ vertices and every edge with the edges of a complete bipartite graph $K_{r/2,r/2}$ (see Fig. 2). Let H be the dual hypergraph of G. Then H has diameter 2, rank r, and $\frac{5}{4}r^2$ vertices.

The result for the cases $q=3$ and $q=5$ are given in the following theorem (the case $q=3$ was pointed out by Rote). Although these results are for even values of r, they can be extended to the odd values as well.

Theorem 6 (Bermond et al. [7]). *If r is even, then*

$$n(2,D,r) \geqslant 3\left(\frac{r}{2}\right)^D.$$

$$n(2,D,r) \geqslant 5\left(\frac{r}{2}\right)^D \quad \textit{if } D \equiv 0 \textit{ or } 2 \pmod 5.$$

Furthermore, from the work of Kleitman (unpublished) and from [24] and [51], it follows that

Theorem 7. *If r is even, then $n(2,2,r) = \frac{5}{4}r^2$.*

Finally, we note that the duals of the asymmetric block designs give hypergraphs of diameter two [16, 50].

2.5. Compound techniques

One of the techniques used to construct large (Δ,D,r)-hypergraphs is to start from good ones for small values of Δ, D and r, then to combine them to build larger ones.

Bond [18] constructs (Δ, D, r)-hypergraphs by taking r copies of a $(\Delta - 1, D - 1, r)$-hypergraph and joining the vertices with the same labels through a common edge. Thus,

Proposition 8 (Bond [18]). $n(\Delta, D, r) \geqslant r \cdot n(\Delta - 1, D - 1, r)$.

For example, from Theorem 6, $n(2, 1, r) = 3r/2$ (for r even), thus we obtain $n(3, 2, r) \geqslant 3r^2/2$.

A somewhat more sophisticated construction is the following: Consider a $(p(r - 1) + 1, r, 1)$ design, that is a $(p, 1, r)$-hypergraph H. Replace each vertex x of H by a copy of a (Δ, D, r)-hypergraph H' on p vertices, in such a way that each of the p edges of H containing x contain exactly one vertex of the copy H'_x. The hypergraph thus obtained is denoted by $H[H']$, and has degree $\Delta + 1$, diameter $2D + 1$, and edge size r. Therefore,

Proposition 9 (Bermond et al. [6]). $n(\Delta + 1, 2D + 1, r) \geqslant n(\Delta, D, r) \cdot (1 + (r - 1) \cdot n(\Delta, D, r))$.

For example, we have $n(1, 1, r) = r$. If $q = r - 1$ is a prime power, there exists a $(r(r - 1) + 1, r, 1)$ design and in that case $n(2, 3, r) \geqslant r^3 - r^2 + r$. For these values of r, the result is better than the lower bound $n(2, 3, r) \geqslant \frac{3}{8}r^3$ obtained in Theorem 6. Slight improvements can still be obtained (see [29]). For instance, $n(2, 1, r) = 3r/2$ (for r even), and thus $n(3, 3, r) \geqslant \frac{1}{4}(9r^3 - 9r^2 + 6r)$ if there exists a $(\frac{3}{2}r(r - 1) + 1, r, 1)$ design.

Bermond et al. (see [5, 29]) have used more elaborate techniques mixing transversal designs (or orthogonal latin squares) and large bipartite regular graphs of diameter five. (For the construction of large bipartite graphs see [19].) These techniques give the best-known large hypergraphs of diameter two. For example they proved that $n(3, 2, r) \geqslant \frac{28}{9}r^2$ if $r \equiv 0 \pmod 3$ and $n(4, 2, r) \geqslant \frac{9}{2}r^2$ if $r \equiv 0 \pmod 4$, and $r \neq 8,\ 24$.

2.6. Various families of bus networks

In this section we survey various direct constructions of bus networks. Some of these constructions apply only in particular cases such as maximum degree 2, or degree equal to the diameter. For $\Delta = 2$, Finkel and Solomon [34] have proposed two networks, called *snowflake bus network* and *dense snowflake bus network*. These networks have $r^{\log_2 D}$ processors.

A construction method based on hypercubes to obtain bus networks of varying degree and bus size is the *spanning bus hypercube* [56]. In an r-ary, d-dimensional spanning bus hypercube each bus connects processors in one direction (i.e. processors sharing a bus spanning the hypercube in the ith dimension have identical coordinates except in the ith position). Therefore, each processor is connected to d buses. Recall that in this network the diameter is equal to the degree.

The *Dual-bus hypercube* [56] is derived from the spanning bus hypercube by removing some spanning buses so that every processor is connected only to two spanning buses. (Note that the term "dual" in this construction does not refer to the dual hypergraph of any graph, but it simply states the fact that every processor is connected to two buses.) The number of processors in this network is relatively small (of order $r^{(D+1)/2}$, to be compared with $3(\frac{r}{2})^D$ obtained in Theorem 6).

An iterative method to construct bus networks of degree Δ and the bus size r is the "*lens interconnection strategy*" [35]. In this method, at level-1 all of the $r-1$ processors are on all of the $\Delta-1$ buses. At this level every processor is deficient, i.e. it is connected to only $\Delta-1$ of the Δ buses allowed. The buses are also deficient, since each bus has only $r-1$ processors. At level-k, $\Delta-1$ copies of the level-$(k-1)$ lens network are taken and connected together as follows: For each deficient processor a new bus is introduced and is connected to the corresponding processor. A number of new processors are introduced such that each new processor is on the same bus in each copy, and each new bus has r processors on it. The number of processors at level-k is $n=\sum_{i=1}^{k}(\Delta-1)^{k-i}(r-1)^i$. The diameter of the level-k lens interconnection network is $2k$. If $\Delta=r$, this scheme produces symmetric networks, that is every processor's view of the network is the same. Furthermore, in this special case, the dual of this network is also a lens network. By connecting the deficient processors to the deficient buses at the last level, the *completed lens* is obtained which has diameter $\lfloor\frac{3}{2}k\rfloor$.

Another method to construct bus networks is to partition the set of links in a point-to-point network into the buses. A particular case of this method is the spanning bus hypercube (see above). Ferreira et al. [32] considered such a generalization of grids (called hypergrids) and studied communication problems on them. See also the works of Stout [54] and Prasanna and Raghavendra [47] where buses are used on top of a classical grid to speed up the algorithms. (Also see [49] for meshes with reconfigurable buses.)

Doty [30] partitioned the links of the point-to-point de Bruijn networks to buses, and obtained bus networks with large number of vertices. Bermond et al. [10] gave formal methods for this partitioning. Similarly, Kautz bus networks are obtained from the point-to-point Kautz networks by grouping certain links into buses [10]. The number of processors in de Bruijn bus networks is $(\frac{\Delta r}{4})^D$, and the number of processors in Kautz bus networks is $(\frac{\Delta r}{4})^D+(\frac{\Delta r}{4})^{D-1}$, where D is the diameter, and Δ and r always assume even values.

In Table 1 we list some properties of some of the networks discussed above, for comparison.

3. Directed bus networks

De Bruijn and Kautz bus networks are in fact obtained by extending the definition of de Bruijn and Kautz digraphs to directed bus networks. In the directed bus networks the processors on a bus are divided into two, not necessarily disjoint, sets. The processors

Table 1
Comparison of some bus networks

Network	Max. Deg.	Diam.	No of nodes	No of buses	Node conn.	Bus conn.
Snowflake	2	$2^k - 1$	r^k	$\frac{r^k-1}{r-1}$	1	1
Dual of binary hypercube	2	r	$r2^{r-1}$	2^r	$2(r-1)$	2
Dual-bus hypercube	2	$2k-1$	r^k	$2r^{k-1}$	$2(r-1)$	2
Dual of Kautz graphs	2	D	$(\frac{r}{2})^D + (\frac{r}{2})^{D-1}$	$(\frac{r}{2})^{D-1} + (\frac{r}{2})^{D-2}$	$2(r-1)$	2
Dual of C_s graphs	2	D	$3(\frac{r}{2})^D$	$3(\frac{r}{2})^{D-1}$		
Spanning bus hypercube	Δ	Δ	r^Δ	$\Delta r^{\Delta-1}$	$\Delta(r-1)$	Δ
Completed lens (if $\Delta = r$)	r	$\lfloor \frac{3}{2}k \rfloor$	$k(r-1)^k$	$k(r-1)^k$		
de Bruijn bus networks	Δ	D	$(\frac{\Delta r}{4})^D$	$\frac{\Delta^2}{4}(\frac{\Delta r}{4})^{D-1}$	$\frac{1}{2}\Delta r - 2$	$\Delta - 1$; if $r \geqslant 8$ $\geqslant \Delta - 2$; if $r \leqslant 6$
Kautz bus networks	Δ	$D > 2$	$(\frac{\Delta r}{4})^D + (\frac{\Delta r}{4})^{D-1}$	$\frac{\Delta^2}{4}(\frac{\Delta r}{4} + 1)(\frac{\Delta r}{4})^{D-2}$		

Notes:
1. The bus size is r in all of the networks.
2. All bus networks in this table are undirected.
3. In the Dual of C_s graphs r assumes even values.
4. In de Bruijn and Kautz bus networks Δ and r assume even values.

in one set can use the bus only to send messages while the processors in the other set can use the bus only to receive messages. Formally, we represent the directed bus networks with directed hypergraphs.

3.1. Terminology and notation

A *directed hypergraph* H, is a pair $(\mathcal{V}(H), \mathcal{E}(H))$, where $\mathcal{V}(H)$ is a non-empty set of elements (called *vertices*) and $\mathcal{E}(H)$ is a set of ordered pairs of non-empty subsets of $\mathcal{V}(H)$ (called *hyperarcs*). If $E = (E^-, E^+)$ is a hyperarc in $\mathcal{E}(H)$, then the non-empty vertex sets E^- and E^+ are called the *in-set* and the *out-set* of the hyperarc E, respectively. The sets E^- and E^+ need not be disjoint. $|E^-|$ is the *in-size*, and $|E^+|$ is the *out-size* of hyperarc E. The *maximum in-size* and the *maximum out-size* of a directed hypergraph H are, respectively,

$$s^-(H) = \max_{E\in\mathcal{E}(H)} |E^-| \quad \text{and} \quad s^+(H) = \max_{E\in\mathcal{E}(H)} |E^+|.$$

If $s^- = s^+ = 1$, a directed hypergraph is nothing more than a digraph.

Let v be a vertex in $\mathcal{V}(H)$. The *in-degree* of v is the number of hyperarcs that contain v in their out-set, and is denoted by $d_H^-(v)$. Similarly, the *out-degree* of vertex v is the number of hyperarcs that contain v in their in-set, and is denoted by $d_H^+(v)$. The *maximum in-degree* and the *maximum out-degree* of H are, respectively,

$$d^-(H) = \max_{v\in\mathcal{V}(H)} d_H^-(v) \quad \text{and} \quad d^+(H) = \max_{v\in\mathcal{V}(H)} d_H^+(v).$$

A *walk* in H from vertex u to vertex v is an alternating sequence of vertices and hyperarcs $u = v_0, E_1, v_1, E_2, v_2, \dots, E_k, v_k = v$ such that $v_{i-1} \in E_i^-$ and $v_i \in E_i^+$ for each $1 \leqslant i \leqslant k$. The *length* of a walk is equal to the number of hyperarcs on it. The *distance* and the *diameter* are defined analogously to those in the undirected case.

We can represent the incidence relations between the vertices and hyperarcs in a directed hypergraph H using a bipartite digraph,

$$R(H) = (\mathcal{V}_1(R) \cup \mathcal{V}_2(R), \mathcal{E}(R))$$

called the *bipartite representation digraph*. Every vertex v_i in $\mathcal{V}(H)$ is represented by a vertex v_i in $\mathcal{V}_1(R)$ and every hyperarc E_j in $\mathcal{E}(H)$ is represented by a vertex e_j in $\mathcal{V}_2(R)$. We draw an arc from $v_i \in \mathcal{V}_1(R)$ to $e_j \in \mathcal{V}_2(R)$ if and only if $v_i \in E_j^-$ in H, and we draw an arc from $e_j \in \mathcal{V}_2(R)$ to $v_i \in \mathcal{V}_1(R)$ if and only if $v_i \in E_j^+$ in H.

If only the adjacency relations between the vertices in a directed hypergraph H are considered, we can use the underlying multi-digraph $\widehat{H}$ (also called associated multi-digraph and denoted by $A(H)$). The vertex set of $\widehat{H}$ is the same as that of H. There are as many arcs from u to v in $\widehat{H}$, as there are hyperarcs E in H such that $u \in E^-$ and $v \in E^+$. Then a hyperarc of H corresponds to a "bipartite complete digraph" (shortly *diclique*), and a directed hypergraph corresponds to a multi-digraph with a partitioning of its arc set into dicliques.

3.2. Directed Moore hypergraphs

We call a directed hypergraph with maximum out-degree d, maximum out-size s, and diameter D, a (d,D,s)-directed hypergraph. The (d,D,s)-directed hypergraph problem is the directed analogue of the (Δ,D,r)-hypergraph problem: Find directed hypergraphs of maximum out-degree (resp. in-degree) d, diameter D, and maximum out-size (resp. in-size) s, such that the number of vertices in the hypergraph is maximized. The maximum number of vertices in any (d,D,s)-directed hypergraph can be at most

$$1 + ds + (ds)^2 + \cdots + (ds)^D = \sum_{i=0}^{D} (ds)^i.$$

We call this upper bound the *Moore bound for directed hypergraphs,* and we call the hypergraphs attaining it the *directed Moore hypergraphs.*

Ergincan and Gregory [31] showed that Moore bound for directed hypergraphs cannot be attained if $ds > 1$ or $D > 1$. If $ds = 1$ then the directed Moore hypergraph is nothing more than a directed cycle of length $D+1$. Finding directed Moore hypergraphs of diameter 1 is equivalent to finding matrix factorizations $J - I = XY$, where J is the all-ones matrix, I is the identity matrix, X is an $n \times m$ (0, 1)-matrix and Y is an $m \times n$ (0, 1)-matrix; both X and Y have constant row sums [31]. There may exist several such matrix factorizations even if some extra conditions (for example $m = n$) are introduced. This problem is also equivalent to the partitioning of the arc set of a complete symmetric digraph into dicliques.

Since the directed Moore hypergraphs exist only in a few cases, it is of interest to construct directed hypergraphs with a large number of vertices. In the following section we survey two families of directed hypergraphs that approach the Moore bound asymptotically.

3.3. De Bruijn and Kautz hypergraphs

De Bruijn and Kautz hypergraphs are the generalizations of de Bruijn digraphs and Kautz digraphs to directed hypergraphs. De Bruijn and Kautz digraphs can be defined in at least three different ways (see [15]). These definitions are based on (1) alphabets, (2) line digraph iterations on complete digraphs, and (3) arithmetical congruences.

In the same manner, de Bruijn hypergraphs can be defined using three different definitions as mentioned above, and Kautz hypergraphs can be defined using the last two definitions. Details of these definitions can be found in [10]; here we will only mention some techniques used in the generalization.

To generalize the second definition, a new notion was introduced [14]: The *directed line hypergraph*, $L(H)$, of a directed hypergraph H has the following vertex set and hyperarc set:

$$\mathcal{V}(L(H)) = \bigcup_{E\in\mathcal{E}(H)} \{(uEv) \mid u \in E^-,\ v \in E^+\},$$

$$\mathcal{E}(L(H)) = \bigcup_{v\in\mathcal{V}(H)} \{(EvF) \mid E^+ \ni v,\ F^- \ni v\}$$

The in-set and the out-set of a hyperarc (EvF) are, respectively,

$$(EvF)^- = \{(uEv) \mid u \in E^-\} \quad \text{and} \quad (EvF)^+ = \{(vFu) \mid u \in F^+\}.$$

Note that the vertices and the hyperarcs of $L(H)$ correspond to the paths of length one in H and its *directed dual* H^*, respectively. (Also note that $L(H)$ as defined above does not denote the *line graph of a hypergraph* defined in [2, p. 31.]).

For maximum out-degree d, maximum out-size s and diameter D, de Bruijn hypergraphs have $(ds)^D$ vertices and Kautz hypergraphs have $(ds)^D+(ds)^{D-1}$ vertices. They also have other good properties such as optimum connectivity [13].

In the arithmetical definition of de Bruijn and Kautz hypergraphs the vertices are numbered from 0 to $n-1$ and the hyperarcs are numbered from 0 to $m-1$ where n is the number of vertices and m is the number of hyperarcs. The incidence relations between the vertices and hyperarcs are given using arithmetic congruences. For example, in Kautz hypergraphs vertex v is incident to the hyperarcs

$$E_j \equiv dv + \alpha \pmod m \quad 0 \leqslant \alpha \leqslant d-1,$$

and the out-set of hyperarc E is

$$v_i \equiv -sE - \beta \pmod n \quad 1 \leqslant \beta \leqslant s.$$

The arithmetical definition lets us define hypergraphs with properties similar to those of de Bruijn and Kautz hypergraphs, but on any number of vertices n and hyperarcs m, so long as the following two conditions hold:

$$dn \equiv 0 \pmod m \quad \textit{and} \quad sm \equiv 0 \pmod n.$$

Nice properties, such as optimal connectivity are obtained when $dn = sm$ (see [13]). Furthermore, the bipartite representation digraphs of Kautz hypergraphs give large bipartite digraphs. (In the case $d = s$ these digraphs were already found by Fiol and Yebra [36].)

4. Conclusion

We hope to have shown to the reader how different tools of the theory of hypergraphs and directed hypergraphs can be helpful in the design of the large bus interconnection networks. There remains a lot to do on this topic in different ways. There are still studies on the tools developed, in particular for directed hypergraphs. One can also consider finding new large bus interconnection networks. However, one of the promising areas will be to study the properties such as routing, communication, bus load, algorithm construction, and implementation issues for the existing networks.

An important problem in the implementation of the bus networks is the communication method used on the buses to resolve the conflicts. There is a rich literature on this subject with performance evaluation of different models. We did not include this topic in the survey.

The nature of the data exchanges and the technology to be used in the implementation of the buses are very important issues in the design of bus interconnection networks. If the data exchanges are limited to certain permutations among the processors, solutions that do not cause conflicts have been proposed in [45] and [33]. Recently, an implementation using fiber optics, which realizes simultaneous broadcasting without conflicts on the buses, has been proposed [43].

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