Two-Point Inequalities, the Hermite Semigroup, and the Gauss-Weierstrass Semigroup

FRED B. WEISSLER*

Department of Mathematics. The University of Texas. Austin. Texas 78712

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Let $e^{-Ht}$, $Re z > 0$, be the Hermite semigroup on $R$ with Gauss measure $\mu$. Necessary and sufficient conditions for $e^{-zH}$ to be a bounded map from $L^p(\mu)$ into $L^q(\mu)$, $1 < p, q < \infty$, are found, and in many cases it is proved that $e^{-zH}: L^p(\mu) \rightarrow L^q(\mu)$ is in fact a contraction. Furthermore, these results and a formula relating the Hermite semigroup with the Gauss-Weierstrass semigroup $e^{-A}$ enable one to calculate the precise norm of $e^{-zA}: L^p(\mu) \rightarrow L^q(\mu)$ in a large number of cases.

INTRODUCTION

The Hermite semigroup $e^{-zH}$ has recently been the object of extensive study. Nelson [5] showed that if $e^{-2t} \leq (p - 1)/(q - 1)$, then $e^{-zH}: L^p(\mu) \rightarrow L^q(\mu)$ is a contraction. (\mu is Gauss measure on $R$.) Gross [4] simplified the proof of these “hypercontractive” estimates by showing them to be equivalent to a logarithmic Sobolev inequality. Beckner [1] then showed that Nelson’s estimates followed from the sharp form of Young’s convolution inequality. Brascamp and Lieb [2] derived the sharp convolution inequality and Nelson’s estimates from the same general result. In the same paper where he proves the sharp convolution inequality, Beckner also shows that if $1 < p \leq 2$ and $e^{-z} = i(p - 1)^{1/2}$, then $e^{-zH}: L^p(\mu) \rightarrow L^q(\mu)$ is a contraction. ($p'$ is the exponent conjugate to $p$).

This result is equivalent to the sharp form of the Hausdorff-Young inequality for the Fourier transform on $R$.

In this paper we give necessary and sufficient conditions for $e^{-zH}$ to be a bounded map from $L^p(\mu)$ into $L^q(\mu)$, where $Re z \geq 0$ and $1 \leq p, q \leq \infty$. We then investigate when, under these conditions, $e^{-zH}: L^p(\mu) \rightarrow L^q(\mu)$ is in fact a contraction. Finally, we show that these results can be used to calculate the exact norm of $e^{-zA}: L^p(\mu) \rightarrow L^q(\mu)$ for $1 < p \leq q < \infty$ and $Re z > 0$.

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I would like to mention that Coifman and Sagher [3], using interpolation techniques, have determined when $e^{-zH}: L^p(\mu) \to L^{p'}(\mu)$ is a contraction for all $\Re z \geq 0$, i.e. the special case $q = p'$. Although the present paper is independent of their result, knowledge of it was helpful in my investigations. I would like to thank R. Coifman for discussing their work with me in advance of publication. (Also, the interpolation argument at the end of Section 5 below stems from those discussions.) In addition, I would like to thank J. Gilbert, B. Palka and especially J. Vaaler for helpful remarks concerning this research.

1. STATEMENT OF RESULTS

The Hermite semigroup on $\mathbb{R}$ is given by

$$(e^{-zH}f)(x) = \left[\pi(1 - \omega^2)\right]^{-1/2} \int_{\mathbb{R}} \exp\left[-(\omega x - y)^2/(1 - \omega^2)\right] f(y) \, dy,$$  \hspace{1cm} (1.1)

where $\Re z \geq 0$ and $\omega = e^{-z}$. (Thus $|\omega| \leq 1$.) Square roots always have positive real part. If $\omega = 1$ or $\omega = -1$, then $e^{-zH}$ is respectively the identity or reflection about 0. In what follows we will always assume $\omega^2 \neq 1$.

The Gauss–Weierstrass semigroup on $\mathbb{R}$ is given by

$$(e^{\xi f})(x) = (4\pi \xi)^{-1/2} \int_{\mathbb{R}} \exp\left[-(x - y)^2/4\xi^2\right] f(y) \, dy,$$  \hspace{1cm} (1.2)

where $\Re z \geq 0$ (and $z \neq 0$). The classical Young's convolution inequality ([7] p. 178) implies that for $\Re z > 0$ and $1 \leq p \leq q < \infty$, $e^{\xi f}$ is a bounded map from $L^p(dx)$ into $L^q(dx)$. Also, it follows from the classical Hausdorff-Young Fourier transform inequality ([7], p. 178) that for $\Re z = 0$ ($z \neq 0$) and $1 \leq p \leq 2$, $e^{\xi f}$ is a bounded map from $L^p(dx)$ into $L^p(dx)$.

In order to state the first theorem a few definitions are needed. Let $\mu$ be the Gauss probability measure on $\mathbb{R}$ given by $d\mu(x) = \pi^{-1/2} \exp(-x^2) \, dx$. For $1 \leq \rho \leq \infty$, let

$$\langle I_{\rho} f(x) \rangle = \pi^{-1/2} \exp(-x^2/\rho) f(x).$$

Then $I_{\rho}$ is an isometric isomorphism of $L^\rho(\mu)$ onto $L^\rho(dx)$. For non-zero real $\gamma$, let $T_\gamma$ be the dilation operator $(T_\gamma f)(x) = f(\gamma x)$. Then $T_\gamma$ maps $L^\rho(dx)$ onto itself and $\| T_\gamma f \|_{\rho} = |\gamma|^{-1/\rho} \| f \|_{\rho}$. Also, for all complex $\alpha$, let $M_{\alpha}$ be the multiplication operator $(M_{\alpha} f)(x) = e^{i\alpha x} f(x)$.

Finally, we denote by $\| e^{-zH} \|_{p,q}$ the norm of $e^{-zH}$ as a map from $L^p(\mu)$ into $L^q(\mu)$. More precisely, $e^{-zH}$ is a contraction on $L^2(\mu)$ whenever $\Re z \geq 0$. If, for a particular value of $z$, $e^{-zH}$ extends or restricts to a bounded map from $L^p(\mu)$ into $L^q(\mu)$, $\| e^{-zH} \|_{p,q}$ denotes the norm of that map. Otherwise, $\| e^{-zH} \|_{p,q}$
is taken to be infinity. Note that since $e^{-zH}$ takes the constant function $1$ into itself, we always have $\| e^{-zH} \|_{p,q} \geq 1$.

**Theorem 1.** Let $1 \leq p, q \leq \infty$ and $\Re z \geq 0$ (with $\omega = e^{-z} \neq \pm 1$). Then for any non-zero real $\gamma$ such that $\Re(\gamma/\omega) \geq 0$,

$$e^{-zH} = (\gamma/\omega)^{1/2} e^{1/2} \gamma^{1/2} M_t^{T_p} M_q^{T_p} e^{(\gamma(1-\omega^2)/4\omega^2)} M_s^{I_p},$$

where

$$\alpha = 1/(1 - \omega^2) - 1/p - \omega/\gamma(1 - \omega^3),$$

$$\beta = 1/(1 - \omega^2) - 1/q - \gamma\omega/(1 - \omega^2).$$

Furthermore, in the case $1 \leq p \leq q \leq \infty$, $\| e^{-zH} \|_{p,q} < \infty$ if and only if

$$\Re 1/(1 - \omega^2) \geq 1/p, \quad \Re 1/(1 - \omega^2) \geq 1/q', \quad [\Re 1/(1 - \omega^2) - 1/p][\Re 1/(1 - \omega^2) - 1/q'] \geq [\Re \omega/(1 - \omega^2)]^2. \quad (1.5)$$

In the case $1 \leq q < p \leq \infty$, if $\Re \omega/(1 - \omega^2) = 0$, then (1.4) is necessary and sufficient for $\| e^{-zH} \|_{p,q}$ to be finite. If $\Re \omega/(1 - \omega^2) \neq 0$, the necessary and sufficient conditions are (1.4) and

$$[\Re 1/(1 - \omega^2) - 1/p][\Re 1/(1 - \omega^2) - 1/q'] > [\Re \omega/(1 - \omega^2)]^2. \quad (1.6)$$

Moreover, if $p, q < \infty$, conditions (1.4) and (1.5) are together equivalent to

$$|p - 2 - \omega^2(q - 2)| \leq p - |\omega|^2 q. \quad (1.7)$$

The proof of Theorem 1 is elementary. (See Section 2.) The deepest facts used are the mapping properties of $e^{az}$ described above, and these are consequences of the classical convolution and Fourier transform inequalities. Note that the values of $x = 1/p$ and $y = 1/q'$ allowed by (1.4) and (1.5) make up the area in the first quadrant bounded by the lower branch of a hyperbola (or two perpendicular rays if $\Re \omega/(1 - \omega^2) = 0$). This area is always contained in the square $0 \leq x \leq 1, 0 \leq y \leq 1$.

One can easily check that in special cases Theorem 1 reduces to known results. For example, if $1 \leq q \leq 2 \leq p \leq \infty$, Theorem 1 says that $\| e^{-zH} \|_{p,q}$ is always finite. Indeed, in this case it is known that $\| e^{-zH} \|_{p,q} = 1$ (since $\| e^{-zH} \|_{2,2} = 1$ for all $\Re z \geq 0$). Also, if $z = t > 0$, (1.4) is automatic and (1.5) becomes Nelson's criterion for hypercontractivity. And if $\omega$ is pure imaginary and $q' = p$, we recover Beckner's condition. In both of these cases we also know that $\| e^{-zH} \|_{p,q} = 1$.

It is natural to ask if $e^{-zH} : L^p(\mu) \rightarrow L^q(\mu)$ is always a contraction whenever
it is bounded. This is definitely not the case if $1 < q < p < 2$ or $2 < q < p < \infty$. Indeed, fix $w$ with $\Re \omega/(1 - w^2) \neq 0$ and suppose that $\| e^{-zH}f \|_q \leq \|f\|_p$ for all such $p$ and $q$ satisfying (1.4) and (1.6). Then $\| e^{-zH}f \|_q \leq \|f\|_p$ for all such $p$ and $q$ satisfying (1.4) and (1.5), contradicting Theorem 1.

On the other hand, it is reasonable to expect that if $1 < p < q < \infty$, then $e^{-zH}: L^p(\mu) \to L^q(\mu)$ is a contraction whenever it is bounded. The next theorem establishes this for a slightly smaller range of $p$ and $q$.

**Theorem 2.** (a) If either $\Im \omega = 0$, $\Re \omega = 0$, or $|\omega| = 1$, then in every case where $e^{-zH}: L^p(\mu) \to L^q(\mu)$ is bounded, $1 < p, q < \infty$, it is a contraction.

(b) Let $\Re z > 0$ and $1 \leq p \leq q < \infty$, but exclude the values $2 < p < q < 3$ and $3/2 < p < q < 2$. If $\| e^{-zH} \|_{p,q} < \infty$, i.e. if (1.4) and (1.5) hold, then $\| e^{-zH} \|_{p,q} = 1$.

The first statement follows easily from Nelson's and Beckner's results and Theorem 1. The approach taken in proving the second statement is, following Beckner, to prove the corresponding two-point inequality. (See Sections 3 and 4.) The author believes that part (b) of Theorem 2 is true without the exclusion and therefore that the proof can be extended or improved upon.

Theorems 1 and 2 together enable one to calculate the precise norm of $e^{sz}$, $\Re s > 0$, as map from $L^p(dx)$ into $L^q(dx)$. Indeed, if $\Re \alpha = \Re \beta = 0$ in (1.3) and $\| e^{-zH} \|_{p,q} = 1$, then $\| e^{\{\Re(1-\omega^2)/4\omega}d \|_{p,q}$ can literally be read off. (For the Gauss–Weierstrass semigroup $\| \|_{p,q}$ is taken with respect to Lebesgue measure.) This procedure was carried out for real $s$ and $z$ in Theorem 1 of [8]. Here we have the following result.

**Theorem 3.** (a) Let $s = r e^{i\theta}$ with $r > 0$ and $\phi \in (-\pi/2, \pi/2)$; and let $1 < p < q < \infty$, but exclude the values $2 < p < q < 3$ and $3/2 < p < q < 2$. Then there exist $\omega$ and $\gamma$, with $|\omega| < 1$ and $\gamma > 0$, such that $\arg(1 - \omega^2)/\omega = \phi$, (1.7) holds with equality, and $\Re \alpha = \Re \beta = 0$ in (1.3). For such $\omega$ and $\gamma$,

$$
\| e^{sz} \|_{p,q} = |\gamma(1 - \omega^2)/4\pi r \omega |^{1/2} |\omega/\gamma |^{1/2} \gamma^{1/q}.
$$

Moreover, for some Gaussian function $g(x) = e^{-\sigma x^2}$ with $\Re \sigma > 0$, $\| e^{sg} \|_q = \| e^{sz} \|_{p,q} \| g \|_p$.

In the special case $p = q'$, $1 < p < 2$, (1.8) becomes

$$
\| e^{sz} \|_{p,p'} = \left[ \frac{(1 - x)(p - 1 - x)}{4\pi r(2 - p) \cos \phi} \right]^{(9 - p)/4p} x^{(3p - 4)/4p}
$$

where $x$ is the solution to the cubic equation

$$(1 + x)^2(x - (p - 1)^2) = (1 - x)(p - 1 - x^2 \tan^2 \phi)
$$

such that $(p - 1)^2 \leq x \leq p - 1$. 
(b) Let $s = re^{i\phi}$ with $r > 0$ and $\phi \in (-\pi/2, \pi/2)$; and let $3 \leq p < \infty$. If $(p - 2)/p \leq \cos \phi \leq 1$, then $e^{s\phi}$ is a contraction on $L^p(dx)$. If $0 < \cos \phi < (p - 2)/p$, then

$$\|e^{s\phi}\|_{p,p} = \frac{(\cos \phi)^{1/2 - 1/p} p |\sin \phi| - ((p - 2)^2 - p^2 \cos^2 \phi)^{1/2}}{2^{1/(p - 1)^{1/2} p} ((p - 2)^2 - p^2 \cos^2 \phi)^{1/2 - 1/p} }.$$

(1.10)

Moreover, in the range $0 < \cos \phi < (p - 2)/p$, the norm given by (1.10) is achieved for some Gaussian function $g(x) = e^{-\sigma x^2}$ with $\Re \sigma > 0$.

As the proof will show, $\omega$ and $\gamma$ in part (a) can be computed explicitly. Furthermore, if Theorem 2 is true for the excluded values of $p$ and $q$, the same is true for Theorem 3.

2. Proof of Theorem 1

It is straightforward to verify that formula (1.3) is correct. We will show that formula implies that $\|e^{-zH}\|_{\mu, q}$ is finite under the conditions stated in Theorem 1.

Suppose first that $\Re \omega/\Re(1 - \omega^2) = 0$. Then $|\omega| = 1$ or $\Re \omega = 0$, and so $1 \leq \Re 1/(1 - \omega^2) \leq 1$. Thus, if we let $\Re 1/(1 - \omega^2) = 1/p = 1/q'$, then $1 < p < 2$; and it follows from the Hausdorff–Young inequality, as noted above, that $e^{\nu(1 - \omega^2)/4\omega} A$ is a bounded map from $L^p(dx)$ into $L^q(dx)$. Since $\Re \alpha = \Re \beta = 0$, it follows from (1.3) that $e^{-zH}$ is bounded from $L^p(\mu)$ into $L^q(\mu)$. Certainly then if $r \geq p$ and $s \leq q$, $e^{-zH}$ is bounded from $L^r(\mu)$ to $L^s(\mu)$. Thus for $1 < p, q < \infty$, condition (1.4) implies that $\|e^{-zH}\|_{\mu, q} < \infty$.

Now suppose that $\Re \omega/\Re(1 - \omega^2) \neq 0$. Then $\Re \gamma/\Re(1 - \omega^2) > 0$, and so $\Re \gamma(1 - \omega^2)/4\alpha > 0$. Consequently, for $1 \leq p \leq q < \infty$, $e^{\nu(1 - \omega^2)/4\alpha} A$ is a bounded map from $L^p(dx)$ into $L^q(dx)$. If $p$ and $q$ satisfy (1.4) and (1.5), then $\gamma$ can be chosen so that $\Re \alpha > 0$ and $\Re \beta > 0$. It follows from (1.3) that $e^{-zH}$ is bounded from $L^p(\mu)$ into $L^q(\mu)$. If $1 < q < p < \infty$ and (1.4) and (1.6) are satisfied, then $\gamma$ can be chosen so that $\Re \alpha > 0$ and $\Re \beta > 0$. Thus $M_\beta$ is a bounded map from $L^p(dx)$ into $L^q(dx)$; and since $e^{\nu(1 - \omega^2)/4\omega} A$ is bounded on $L^p(dx)$, (1.3) again implies that $e^{-zH}$ is bounded from $L^p(\mu)$ into $L^q(\mu)$.

For the converse suppose that $\|e^{-zH}\|_{\mu, q} < \infty$. We will prove that (1.4) and (1.5) must hold, and that (1.6) holds if $q < p$ and $\Re \omega/(1 - \omega^2) \neq 0$. To do this we need to calculate the action of $e^{-zH}$ on an arbitrary Gaussian function $g_s(x) = e^{sx^2}$. If $Re s < Re 1/(1 - \omega^2)$, then $e^{-zH} g_s$ can be computed formally (1.1), yielding

$$(e^{-zH} g_s)(x) = [1 - s(1 - \omega^2)]^{-1/2} \exp \left[ \frac{s \omega^2 x^2}{1 - s(1 - \omega^2)} \right].$$

(2.1)

If in addition $g_s \in L^p(\mu)$ and the right hand side of (2.1) is in $L^q(\mu)$, then (2.1) gives $e^{-zH} g_s$ correctly.
In other words, if $L(s)$ is the linear fractional transformation given by

$$L(s) = \frac{s\omega^2}{1 - s(1 - \omega^2)},$$

then (2.1) is valid provided

$$\text{Re } s < \text{Re } 1/(1 - \omega^2), \quad \text{Re } s < 1/p, \quad \text{Re } L(s) < 1/q.$$  

Now $L$ takes $1/(1 - \omega^2)$ into $\infty$, and therefore takes some circle passing through $1/(1 - \omega^2)$ onto the line with constant real part $1/q$. Consequently, if $\text{Re } 1/(1 - \omega^2) < 1/p$, we can let $s \to s_0$ in such a way that $\text{Re } s \leq (1/p) - \epsilon$, $\text{Re } L(s) < 1/q$ and $\text{Re } L(s_0) = 1/q$. Thus $g_s$ remains bounded in $L^q(\mu)$ as $s \to s_0$, but $e^{-zH}g_s$ blows up in $L^q(\mu)$. Since $\|e^{-zH}\|_{p,q} < \infty$, we conclude that $\text{Re } 1/(1 - \omega^2) \geq 1/p$.

By a similar argument, it now follows that if $\text{Re } s < 1/p$, then $\text{Re } L(s) < 1/q$. Let

$$L_1(s) = L(s) + \omega^2/(1 - \omega^2) = \frac{\omega^2(1 - \omega^2)^2}{1/(1 - \omega^2) - s}. \quad (2.2)$$

Thus, if $\text{Re } s = 1/p$, then

$$\text{Re } L_1(s) \leq \text{Re } \omega^2/(1 - \omega^2) + 1/q = \frac{1}{1/(1 - \omega^2) - 1/p}. \quad (2.3)$$

In particular, letting $s$ tend to infinity along the line $\text{Re } s = 1/p$, we see that $\text{Re } 1/(1 - \omega^2)) \geq 1/q'$. Thus (1.4) is verified.

Suppose that $\text{Re } 1/(1 - \omega^2) = 1/p$. Then the image under $L_1$ of the line $\text{Re } s = 1/p$ is some line passing through 0 and, by (2.3), having bounded real part. Thus $\text{Re } L_1(s) = 0$ whenever $\text{Re } s = 1/p$. Therefore, (2.2) implies that $\omega^2/(1 - \omega^2)^2$ must be real. Now $\omega$ can not be real since $\text{Re } 1/(1 - \omega^2) = 1/p$; and so $\omega/(1 - \omega^2)$ is pure imaginary. Certainly then (1.5) must hold.

Now suppose that $\text{Re } 1/(1 - \omega^2) > 1/p$. In this case $L_1$ takes the line $\text{Re } s = 1/p$ into a circle passing through 0. To prove (1.5), it suffices by (2.3) to show that some point on that circle has real part

$$\text{[Re } \omega/(1 - \omega^2)]^2[\text{Re } 1/(1 - \omega^2) - 1/p]^{-1}.$$  

But that is easy. The center of the circle is $\frac{1}{2}L_1(s_0)$, where $s_0$ minimizes $\| 1/(1 - \omega^2) - s \|$ subject to $\text{Re } s = 1/p$. One simply checks that $\frac{1}{2}L_1(s_0) + |\frac{1}{2}L_1(s_0)|$ has the desired real part.

Finally, if $q < p$ and $\text{Re } \omega/(1 - \omega^2) \neq 0$, we must show that equality cannot hold in (1.5). Indeed, if it did, then $\gamma$ could be chosen so that $\text{Re } \alpha = \text{Re } \beta = 0$ in (1.3). Then (1.3) would imply that $e^{\gamma(1 - \omega^2/4\omega^4)}d$ is bounded from $L^p(dx)$...
into $L^q(dx)$, which is false. Thus we have shown that the conditions in Theorem 1 are necessary and sufficient for $\| e^{-zH} \|_{p,q}$ to be finite.

To complete the proof of Theorem 1, it remains to show that for $p, q < \infty$, conditions (1.4) and (1.5) are together equivalent to (1.7). If one squares (1.7) and divides by $pq$, the result is

$$\left| \frac{1}{pq'} \right| |\omega|^2 - \frac{1}{2}(1 - 2/p)(1 - 2/q') \text{Re}(\omega^2) + (1 - 1/p)(1 - 1/q') \geq \frac{1}{2} |\omega|^2,$$

and this is easily seen to be equivalent to (1.5). Furthermore, $p - |\omega|^2 q > 0$ is the same as $1/q' + |\omega|^2/p < 1$. So if both $1/p$ and $1/q'$ are bigger than $\text{Re} 1/(1 - \omega^2)$, (1.7) implies that $(1 + |\omega|^2) \text{Re} 1/(1 - \omega^2) < 1$, which is false. Thus, if (1.7) holds, one of $1/p$ and $1/q'$ is less than or equal to $\text{Re} 1/(1 - \omega^2)$; and by (1.5) so is the other. On the other hand, it is straightforward to show that (1.4) and (1.5) imply $1/q' + |\omega|^2/p \leq 1$. Thus (1.7) is equivalent to (1.4) and (1.5).

3. THEOREM 2 AND TWO-POINT INEQUALITIES

Let us first dispense with part (a) of Theorem 2. If $\omega$ is real, this is precisely Nelson's result. If $\text{Re} \omega = 0$ or $|\omega| = 1$, then $\text{Re} \omega/(1 - \omega^2) = 0$; and so by Theorem 1 $\| e^{-zH} \|_{p,q} < \infty$ if and only if $p > p_0$ and $q < q_0$, where $1/p_0 = 1/q_0' = \text{Re} 1/(1 - \omega^2)$. The result now follows from Beckner's theorem.

In proving part (b) of Theorem 2 we use a method developed by Gross and Beckner. Let $\nu$ be the probability measure on $R$ with mass $1/2$ at the points 1 and $-1$. Then every function in $L^p(\nu)$ is equivalent to a first degree polynomial $a + bx$, and

$$\| a + bx \|_p = \frac{1}{2} |a - b|^p + \frac{1}{2} |a + b|^p, \quad p < \infty.$$

Let $B$ be the orthogonal projection in $L^p(\nu)$ onto the orthogonal complement of the constant functions, i.e. $B(a + bx) = bx$. Then $e^{-zB}(a + bx) = a + \omega bx$, where again $\omega = e^{-z}$. The following theorem, although never explicitly stated, is proved in Beckner [1], pp. 163–166.

**THEOREM.** Let $1 < p \leq q < \infty$ and $\text{Re} z \geq 0$. Suppose that $e^{-zB}: L^p(\nu) \to L^q(\nu)$ is a contraction. Then $e^{-zB}: L^p(\mu) \to L^q(\mu)$ is a contraction.

Note that if (1.4) and (1.5) are satisfied with either $p = 1$ or $q = \infty$, then $\omega$ is real; and in this case statement (b) of Theorem 2 is already known. Thus, it suffices to restrict attention to the case $1 < p \leq q < \infty$. We will therefore prove statement (b) of Theorem 2 by establishing the following result.

**THEOREM 2'.** Let $\text{Re} z \geq 0$, $\omega = e^{-z}$, and $1 < p \leq q < \infty$, but exclude
the values $2 < p \leq q < 3$ and $3/2 < p \leq q < 2$. Then $e^{-zB}$ is a contraction from $L^p(v)$ into $L^q(v)$ if and only if (1.7) holds.

The proof will occupy the rest of this section and the following section. Let $\omega = |\omega| e^{-i\theta}$. Then $0 < |\omega| \leq 1$. It is clear that $\|e^{-zB}\|_{p,q} \leq 1$ is equivalent to the following statement: For all $x, y \geq 0$ and $\alpha \in [0, 2\pi]$, if

$$|xe^{i\alpha} - y|^p + |xe^{i\alpha} + y|^p = 2,$$

then

$$|x| |\omega| e^{i(\alpha-\theta)} - y|^q + |x| |\omega| e^{i(\alpha+\theta)} + y|^q < 2.$$  

Now for all $x, y \geq 0$, $\alpha \in R$, and $p \in (1, \infty)$, define

$$F(x, y, \alpha, p) = |xe^{i\alpha} - y|^p + |xe^{i\alpha} + y|^p = (x^2 + y^2 - 2xy \cos \alpha)^{p/2} + (x^2 + y^2 + 2xy \cos \alpha)^{p/2}. \quad (3.1)$$

For $x, y > 0$ one can check that $\partial_x F(x, y, \alpha, p) > 0$ and $\partial_y F(x, y, \alpha, p) > 0$. (If $p < 2$, this is a bit tricky; and it is perhaps more conveniently done using the first expression for $F$ given above.) Thus, one can define $f(x, \alpha, p)$ implicitly by

$$F(x, f(x, \alpha, p), \alpha, p) = 2. \quad (3.2)$$

To show that $e^{-zB}: L^p(v) \rightarrow L^q(v)$ is a contraction is now evidently equivalent to showing

$$f(x, \alpha, p) \leq f(x |\omega|, \alpha - \theta, q), \quad x \in [0, 1], \quad \alpha \in [0, \pi]. \quad (3.3)$$

These restrictions for $x$ and $\alpha$ are justified by the following proposition. Its proof is elementary.

**Proposition 1.** Let $f$ be defined by (3.2). Then:

(a) $f(x, \alpha, p)$ is defined for $x \in [0, 1]$ and for no other values of $x$. $f(0, \alpha, p) = 1$ and $f(1, \alpha, p) = 0$.

(b) The function $x \mapsto f(x, \alpha, p)$ is strictly decreasing and is its own inverse function.

(c) If $p = 2$ or $\alpha = \pi/2$, then $f(x, \alpha, p) = (1 - x^2)^{1/2}$.

(d) If $p < q$, then $f(x, \alpha, p) \geq f(x, \alpha, q)$. In particular if $p \leq 2 \leq q$, then

$$f(x, \alpha, q) \leq (1 - x^2)^{1/2} \leq f(x, \alpha, p).$$
We now begin serious work on the proof of Theorem 2'.

**Proposition 2.** \( e^{-\epsilon B}: L^p(\nu) \to L^q(\nu) \) is a contraction if and only if

\[
|\omega| \leq \inf_{\alpha \in [0,\pi]} \sup_{x \in [0,1]} \frac{f(x, \alpha, \omega)}{f(x, \alpha + \theta, \omega)}.
\] (3.4)

*Proof.* It suffices to show that (3.3) and (3.4) are equivalent. Let \( x \mapsto g(x, \alpha, \omega) \) be the inverse function of \( x \mapsto f(x, \alpha, \omega) \), ignoring for the moment that \( g = f \). Then the inverse function of \( x \mapsto f(x \mid \omega \mid, \alpha - \theta, \omega) \) is \( x \mapsto |\omega|^{-1} g(x, \alpha - \theta, \omega) \). Therefore, for a fixed \( \alpha \),

\[
f(x, \alpha, \omega) \leq f(x \mid \omega \mid, \alpha - \theta, \omega), \quad x \in [0,1]
\]

is equivalent to

\[
g(x, \alpha, \omega) \leq |\omega|^{-1} g(x, \alpha - \theta, \omega), \quad x \in [0,1].
\]

The proposition follows since \( g = f \).

**Proposition 3.** For all \( q \in (1, \infty) \) and \( \alpha \in \mathbb{R} \),

\[
\lim_{x \to 1} \frac{1 - x^2}{f(x, \alpha, q)} = 1 + (q - 2) \cos^2 \alpha.
\] (3.5)

Thus

\[
\inf_{\alpha \in [0,\pi]} \inf_{x \in [0,1]} \frac{f(x, \alpha, q)}{f(x, \alpha + \theta, q)} \leq \inf_{\alpha \in [0,\pi]} \left[ \frac{1 + (q - 2) \cos^2(\alpha + \theta)}{1 + (q - 2) \cos^2 \alpha} \right]^{1/2}. \] (3.6)

*Proof.* The limit is more conveniently computed after a change to polar coordinates. Fix \( \alpha \) and \( q \). If \( x = r \cos \phi \) and \( y = r \sin \phi \), \( \phi \in [0, \pi/2] \), then

\[
F(x, y, \alpha, q) = r^q[(1 - \sin 2\phi \cos \alpha)^{q/2} + (1 + \sin 2\phi \cos \alpha)^{q/2}].
\]

Therefore, in polar coordinates \( f(x, \alpha, q) \) is given by \( r(\phi) \), where

\[
r(\phi)^{-q} = \left[ \frac{1}{2}(1 - \sin 2\phi \cos \alpha)^{q/2} + \frac{1}{2}(1 + \sin 2\phi \cos \alpha)^{q/2} \right]^{2/q}. \] (3.7)

Consequently,

\[
\lim_{x \to 1} \frac{1 - x^2}{f(x, \alpha, q)} = \lim_{\phi \to 0} \frac{1 - r(\phi)^2 \cos^2 \phi}{r(\phi)^2 \sin^2 \phi} = \lim_{\phi \to 0} \frac{r(\phi)^{-2} - \cos^2 \phi}{\sin^2 \phi}.
\]
This last limit can be evaluated by two applications of l'Hôpital's rule, thereby confirming (3.5).

**PROPOSITION 4.** For $1 < p, q < \infty$, formula (1.7) is equivalent to

$$|\omega|^2 \leq \inf_{\alpha \in [0,\pi]} \left[ \frac{1 + (p - 2) \cos^2(\alpha + \theta)}{1 + (q - 2) \cos^2 \alpha} \right]^{1/2}.$$  \hspace{1cm} (3.8)

**Proof.** Since $\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$, (3.8) is equivalent to

$$|\omega|^2 \leq \frac{p + (p - 2) \cos(\beta + 2\theta)}{q + (q - 2) \cos \beta}$$

or

$$[(q - 2) |\omega|^2 - (p - 2) \cos 2\theta] \cos \beta + (p - 2) \sin 2\theta \sin \beta \leq p - |\omega|^2 q$$

for all $\beta \in [0, 2\pi]$. Maximizing with respect to $\beta$ yields (1.7).

It follows that (1.7) is a necessary condition for $e^{-Bt}:L^p(\nu) \to L^q(\nu)$ to be a contraction, for all $p$ and $q$ with $1 < p, q < \infty$. Moreover, if for a specific $p, q$ and $\theta$, equality holds in (3.6), then (3.4) is equivalent to (3.8), and hence (1.7), for those values of $p, q, \text{ and } \theta$. In other words, to prove Theorem 2, it now suffices to show

$$\inf_{\alpha \in [0,\pi]} \inf_{\alpha \in [0,1]} \frac{f(x, \alpha, q)}{f(x, \alpha + \theta, p)} \geq \inf_{\alpha \in [0,\pi]} \left[ \frac{1 + (p - 2) \cos^2(\alpha + \theta)}{1 + (q - 2) \cos^2 \alpha} \right]^{1/2}$$  \hspace{1cm} (3.9)

for all $\theta$ and all values of $p$ and $q$ allowed in Theorem 2'.

4. **VERIFICATION OF (3.9)**

In this section we verify inequality (3.9) for the appropriate values of $p$ and $q$, thereby completing the proof of Theorem 2'. The following proposition accomplishes this in the case $1 < p \leq 2 \leq q < \infty$.

**PROPOSITION 5.** Let $1 < p \leq 2 \leq q < \infty$. Then

$$\frac{1 - x^2}{f(x, \alpha, p)^2} \geq 1 + (p - 2) \cos^2 \alpha.$$  \hspace{1cm} (4.1)

$$\frac{f(x, \alpha, q)^2}{1 - x^2} \geq \frac{1}{1 + (q - 2) \cos^2 \alpha}$$  \hspace{1cm} (4.2)

for all $x \in [0, 1]$ and all $\alpha$. In particular, (3.9) holds.
Proof. We show (4.2), the proof of (4.1) being entirely analogous. As in the proof of Proposition 3, we change to polar coordinates. (4.2) then becomes

\[
\frac{r(\phi)^2 \sin^2 \phi}{1 - r(\phi)^2 \cos^2 \phi} \geq \frac{1}{1 + (q - 2) \cos^2 \alpha}
\]

or

\[
1 + (q - 2) \cos^2 \alpha \sin^2 \phi \geq r(\phi)^{-2}
\]

(4.3)

for all \( \phi \in (0, \pi/2] \), where \( r(\phi) \) is given by (3.7).

Now let

\[
t = \frac{1 - (1 - \cos^2 \alpha \sin^2 \phi)^{1/2}}{\cos \alpha \sin 2\phi}.
\]

(If the denominator is zero, (4.2) is trivial.) Then \( |\cos \alpha | \sin 2\phi = 2t/(1 + t^2) \), and so

\[
r(\phi)^{-2} = (1 + t^2)^{-1} \left[ \frac{1}{2} (1 - t)^2 + \frac{1}{2} (1 + t) \right]^{1/2}.
\]

At this point we use the two-point analogue of Nelson's inequality, [1] p. 180, namely

\[
\left[ \frac{1}{2} |1 - t^2| \frac{1}{2} |1 + t^2| \right]^{2/3} \leq \frac{1}{2} (1 + (q - 1)^{1/2} t^2) + \frac{1}{2} (1 - (q - 1)^{1/2} t^2)
\]

\[
= 1 + (q - 1) t^2.
\]

Thus, to establish (4.3) it suffices to verify

\[
1 + (q - 2) \cos^2 \alpha \sin^2 \phi \geq \frac{1 + (q - 1) t^2}{1 + t^2} = 1 + \frac{(q - 2) t^2}{1 + t^2}.
\]

(4.4)

But \( t^2/(1 + t^2) = \frac{1}{2} [1 - (1 - \cos^2 \alpha \sin^2 2\phi)^{1/2}] \), and so (4.4) is just

\[
2 \cos^2 \alpha \sin^2 \phi \geq 1 - (1 - \cos^2 \alpha \sin^2 2\phi)^{1/2},
\]

which can be easily verified.

It remains now to consider the case \( 2 < p \leq q < \infty \). (Theorem 2' in the case \( 1 < p \leq q < 2 \) will then follow by duality.) In the previous proposition we were aided by the fact that \( f(x, \alpha, q) \leq (1 - x^2)^{1/2} \leq f(x, \alpha + \theta, p) \). If \( 2 < p \leq q < \infty \), then \( f(x, \alpha, q)/f(x, \alpha + \theta, p) \) can be both greater and less than 1, and so this case is fundamentally different.

**Proposition 6.** Let \( 2 < p \leq q < \infty \). To prove (3.9) it suffices to consider \( \theta \in [0, \pi/2] \). Also, the infimum over \( \alpha \in [0, \pi] \) on the left hand side of (3.9) need only be taken over \( 0 \leq \alpha \leq \alpha + \theta \leq \pi/2 \).
Proof. Proposition 1(e) implies that the left hand side of (3.9) is invariant under \( \theta \mapsto \theta + \pi \) and \( \theta \mapsto -\theta \). The same is clearly true for the right hand side. Thus \( \theta \in [0, \pi/2] \) is sufficient.

Next we claim that it is sufficient to take the infimum on the left hand side of (3.9) over those \( \alpha \) for which \( \cos^2(\alpha + \theta) \leq \cos^2 \alpha \). Indeed, suppose \( \cos^2(\alpha + \theta) > \cos^2 \alpha \). Then by Proposition 1(e), (f)

\[
\frac{f(x, \alpha, q)}{f(x, \alpha + \theta, p)} > \frac{f(x, \pi - \alpha - \theta, q)}{f(x, \pi - \alpha, p)} = \frac{f(x, \beta, q)}{f(x, \beta + \theta, p)},
\]

where \( \beta = \pi - \alpha - \theta \). Since \( \cos^2(\beta + \theta) \leq \cos^2 \beta \), this proves the claim.

Now let \( \alpha \in [0, \pi] \), \( \theta \in [0, \pi/2] \) and \( \cos^2(\alpha + \theta) \leq \cos^2 \alpha \). Other than \( 0 \leq \alpha \leq \alpha + \theta \leq \pi/2 \), there are two possibilities:

If \( A \) is true then \( 0 \leq \pi/2 - \theta \leq \alpha \leq \pi/2 \), and so

\[
\frac{f(x, \alpha, q)}{f(x, \alpha + \theta, p)} \geq \frac{f(x, \pi/2 - \theta, q)}{f(x, \pi/2, p)}.
\]

If \( B \) is true then \( 0 \leq \alpha + \theta - \pi \leq \theta \leq \pi/2 \), and so

\[
\frac{f(x, \alpha, q)}{f(x, \alpha + \theta, p)} \geq \frac{f(x, 0, q)}{f(x, \alpha, p)}.
\]

This proves the proposition.

Since

\[
\frac{f(x, \alpha, q)}{f(x, \alpha + \theta, p)} = \frac{f(x, \alpha, q)}{f(x, \alpha + \theta, q)} \cdot \frac{f(x, \alpha + \theta, q)}{f(x, \alpha + \theta, p)},
\]

the next two propositions prove (3.9) in the case \( 2 < p \leq q < \infty \) with \( q \geq 3 \), and hence complete the proof of Theorem 2'.

**Proposition 7.** Let \( q \geq 3 \) and \( 0 \leq \alpha \leq \alpha + \theta \leq \pi/2 \). Then for all \( x \in [0, 1) \)

\[
\frac{f(x, \alpha, q)}{f(x, \alpha + \theta, q)} \geq \left[ \frac{1 + (q - 2) \cos^2(\alpha + \theta)}{1 + (q - 2) \cos^2 \alpha} \right]^{1/2}.
\]  

(4.5)
Proposition 8. Let $2 < p < q < \infty$ with $q \geq 3$, and let $\alpha \in [0, \pi/2]$. Then for all $x \in [0, 1)$

$$
\frac{f(x, \alpha, q)}{f(x, \alpha, p)} \geq \left[ \frac{1 + (p - 2) \cos^2 \alpha}{1 + (q - 2) \cos^2 \alpha} \right]^{1/2}.
$$

(4.6)

Proof of Proposition 7. We must show that

$$
\alpha \mapsto (1 + (q - 2) \cos^2 \alpha)^{1/2} f(x, \alpha, q)
$$

is decreasing on $[0, \pi/2]$. This is most conveniently done after a change of variables. Let $s = \cos \alpha$ and $k = \frac{1}{2}(q - 2)$. For all $y > 0, s \in [0, 1]$, and $k > 0$ define

$$
G(y, s, k) = \left[ 1 + \frac{y^2}{1 + 2ks^2} - \frac{2ys}{(1 + 2ks^2)^{1/2}} \right]^{k+1}
$$

and then define $g(s, k)$ implicitly by

$$
G(g(s, k), s, k) = 2x^{-\frac{2k+2}{2}}
$$

for any fixed $x \in (0, 1)$. (In proving (4.5) we may clearly assume $x > 0$.) Then by (3.1) and (3.2),

$$
(1 + (q - 2) \cos^2 \alpha)^{1/2} f(x, \alpha, q) = xg(s, k).
$$

Consequently, we need to show that \( \partial_s g(s, k) \geq 0 \) for $s \in [0, 1]$ and $k \geq \frac{1}{2}$; and this is the same as showing \( \partial_s G(y, s, k) \leq 0 \) for all $s \in [0, 1]$, $k \geq \frac{1}{2}$, and $y > 0$.

A straightforward calculation shows that \( \partial_s G(y, s, k) \leq 0 \) precisely when

$$
(1 + \nu^2 - 2\nu s)(1 + 2k\nu s) \geq (1 + \nu^2 + 2\nu s)(1 - 2k\nu),
$$

(4.7)

where $\nu = y/(1 + 2ks^2)^{1/2}$. Thus we need to prove (4.7) for all $\nu > 0, s \in [0, 1]$, and $k \geq \frac{1}{2}$. (Note that if $0 < k < \frac{1}{2}$ and $s = \nu = 1$, then (4.7) is false.) Now if $2k\nu > 1$, then (4.7) is immediate. If $2k\nu < 1$, then (4.7) becomes

$$
\frac{1 + \nu^2 - 2\nu s}{1 + \nu^2 + 2\nu s} \geq \left[ \frac{1 - 2k\nu}{1 + 2k\nu} \right]^{1/k}.
$$

(4.8)

If $k = \frac{1}{2}$, one readily verifies that (4.8) holds. Moreover, the right hand side of (4.8) is a decreasing function of $k \in [0, 1/2\nu)$. This establishes (4.8) for all $k \geq \frac{1}{2}$ and therefore proves the proposition.
Proof of Proposition 8. For \( y > 0, s \in [0, 1], \) and \( p > 2, \) let

\[
H(y, s, p) = \left\{ \begin{array}{l}
\left( \frac{1}{2} \left[ 1 + \frac{y^2}{1 + (p - 2) s^2} - \frac{2ys}{1 + (p - 2) s^{2}\,\sqrt{1 + (p - 2) s^{2}}} \right] \right)^{p/2}
+ \left( \frac{1}{2} \left[ 1 + \frac{y^2}{1 + (p - 2) s^2} + \frac{2ys}{1 + (p - 2) s^{2}\,\sqrt{1 + (p - 2) s^{2}}} \right] \right)^{p/2} \end{array} \right\}^{2/p}
\]

and define \( h(s, p) \) implicitly by

\[
H(h(s, p), s, p) = x^{-2}
\]

for any fixed \( x \in (0, 1). \) (In proving (4.6) we may clearly assume \( x > 0. \) Then

\[
(1 + (p - 2) \cos^2 \alpha)^{1/2} f(x, \alpha, p) = xh(s, p)
\]

where, as in the previous proof, \( s = \cos \alpha. \) Consequently, to establish (4.6), we must show that \( h(s, p) \leq h(s, q) \) for all \( s \in [0, 1] \) and \( 2 < p < q < \infty \) with \( q > 3. \) Since \( \partial_{p}H \geq 0, \) that is the same as showing

\[
H(y, s, p) \geq H(y, s, q)
\]

for \( y > 0, s \in [0, 1], \) and \( 2 < p \leq q < \infty \) with \( q > 3. \)

If we substitute \( v = y/(1 + (p - 2) s^2)^{1/2} \), divide by \( (1 + v^2) \), and then substitute \( u = 2v/((1 + v^2)) \), (4.9) becomes

\[
2\left[ \frac{1}{2}(1 - u)^{p/2} + \frac{1}{2}(1 + u)^{p/2} \right]^{q/p}
\]

\[
\geq \left[ \frac{1}{2}[1 + (1 - (u/s)^2)] + \frac{1}{2}[1 - (1 - (u/s)^2)] \right] \left( \frac{1 + (p - 2) s^2}{1 + (q - 2) s^2} \right)^{1/2} \left( \frac{1 + (p - 2) s^2}{1 + (q - 2) s^2} \right)^{1/2} u^{q/p}
\]

and this must be shown whenever \( 0 < u \leq s \leq 1. \) We remark that the choice of signs in front of \( (1 - (u/s)^2)^{1/2} \) corresponds to \( v < 1. \) Reversing those signs, and thus allowing \( v \geq 1, \) decreases the right hand side of (4.10). Thus it suffices to consider (4.10) with the signs as they are.

Now if \( s = 1, \) then (4.9) is the two-point analogue of Nelson's inequality, [1] p. 180. Consequently, we know (4.10) to be correct if \( s = 1. \) Furthermore,
the left hand side of (4.10) is independent of \( s \); and so it suffices to show that the right hand side, as a function of \( s \in [u, 1] \), obtains its maximum at \( s = 1 \). Let \( t = (u/s)^3 \). Then we need to show that

\[
g(t) = \left[ \frac{1}{2} [1 + (1 - t)^{1/2}] + \frac{1}{2} [1 - (1 - t)^{1/2}] \right] \frac{t + (p - 2) u^2}{t + (q - 2) u^2}
- u \left( \frac{t + (p - 2) u^2}{t + (q - 2) u^2} \right)^{1/2} \frac{q/2}{1/2}
+ \left[ \frac{1}{2} [1 + (1 - t)^{1/2}] + \frac{1}{2} [1 - (1 - t)^{1/2}] \right] \frac{t + (p - 2) u^2}{t + (q - 2) u^2}
+ u \left( \frac{t + (p - 2) u^2}{t + (q - 2) u^2} \right)^{1/2} \frac{q/2}{1/2}
\]

obtains its maximum over the interval \([u^2, 1]\) at \( t = u^2 \). We will do this by showing \( g'(t) \leq 0 \) for \( u^2 < t < 1 \).

A tedious but straightforward calculation shows that \( g'(t) < 0 \) precisely when

\[
[\frac{1}{2}(1 + (1 - t)^{1/2}) + \frac{1}{2}(1 - (1 - t)^{1/2}) \beta^2 - \beta u]^k [1 + \beta r(u, t)]
\geq [\frac{1}{2}(1 + (1 - t)^{1/2}) + \frac{1}{2}(1 - (1 - t)^{1/2}) \beta^2 + \beta u]^k [1 - \beta r(u, t)], \quad (4.11)
\]

where \( k = \frac{1}{2}(q - 2) \).

\[
\beta = \left[ \frac{t + (p - 2) u^2}{t + (q - 2) u^2} \right]^{1/2}
\]

and

\[
r = r(u, t) = \frac{ku^2 + 1 - t/2 - (1 - t)^{1/2}}{u(1 - t)^{1/2}}.
\]

We will verify (4.11) for the following values: \( t \in (u^2, 1) \), \( \beta > 0 \), \( k \geq \frac{1}{2} \). This will certainly guarantee that \( g'(t) \leq 0 \) in the specified interval. Note that since \( 1 - t/2 > (1 - t)^{1/2} \), \( r \) is always positive.

Although (4.11) is similar to (4.7), its proof requires a more intricate argument. If \( \beta r \geq 1 \), (4.11) is trivial; and thus we may assume that \( \beta \in (0, 1/r) \). Moreover, if \( \beta = 1/r \), strict inequality holds in (4.11); and so (4.11) holds for all \( \beta \) sufficiently close to and less than \( 1/r \). Now, raising both sides of (4.11), to the \( 1/k \) power, expanding \((1 \pm \beta r)^{1/k}\) with the binomial theorem, collecting powers of \( \beta \) together, and dividing by \( \beta \), we get

\[
rl(1 + (1 - t)^{1/2}) - 2u
+ \sum_{n \geq 1, \text{even}} \left[ r(1 + (1 - t)^{1/2}) \frac{l - n}{n + 1} + \frac{n(1 - (1 - t)^{1/2})}{r(l - n + 1)} - 2u \right](\frac{l}{n}) (r\beta)^n \geq 0,
\quad (4.12)
\]
where $l = 1/k$ and we assume $l$ is not an integer. One can readily check that
\[ r(l + (1 - t)^{1/2}) - 2u > 0. \]
If $0 < l < 1$, all the coefficients in the above power series are positive and so (4.12) holds for all $\beta \in (0, 1/r)$. If $1 < l < 2$, all the coefficients for $n \geq 4$ are negative. Thus if we divide the left hand side of (4.12) by $\beta^2$, the result is a decreasing function of $\beta$ on the interval $(0, 1/r)$. Moreover, we have already noted that (4.12) holds for $\beta$ close enough to $1/r$
Thus (4.12) holds for all $\beta \in (0, 1/r)$. We have therefore verified (4.11) for all $k > \frac{1}{4}$ except $k = 1$. Clearly then (4.11) must hold at $k = \frac{1}{2}$ and $k = 1$.

This completes the proof that $g'(t) \leq 0$ for $u^2 < t < 1$ and thereby completes the proof of the proposition.

Remarks. The proof of Theorem 2', although somewhat tedious, is fairly natural. Formula (3.8) presents itself as a necessary condition for $\| e^{-s\beta} \|_{p, q} = 1$ without too much work, and so (1.7) arises in a natural way. The hard work is concentrated in verifying (3.9). It is in that verification that the two-point analogue of Nelson's theorem was invoked, in the proofs of Propositions 5 and 8.

One can not help but ask why the condition $q \geq 3$ was needed. In the proof of Proposition 7 it was noted that (4.7) is not always true if $q < 3$. Thus, for every $q$ strictly between 2 and 3, inequality (4.5) fails for some $\alpha$ and $\theta$ in the appropriate range. A power series argument similar to the one used in the proof of Proposition 8 can be used to show that (4.7) holds for all $\nu > 0$ if and only if $4s^2(1 - k^2) \leq 3$. It follows that for all $q > 2$, (4.5) holds whenever $\pi/6 \leq \alpha \leq \alpha + \theta \leq \pi/2$.

Even though Proposition 7 is false without the condition $q \geq 3$, one should not give up hope for (3.9). Observe that Propositions 7 and 8 prove something stronger than (3.9), namely that
\[
\frac{f(x, \alpha, q)}{f(x, \alpha + \theta, p)} \geq \left[ \frac{1 + (p - 2) \cos^2(\alpha + \theta)}{1 + (q - 2) \cos^2 \alpha} \right]^{1/2}
\]
for all $\alpha$ and $\theta$ with $0 \leq \alpha \leq \alpha + \theta \leq \pi/2$. Conceivably, (3.9) could be true even though the above inequality fails for some values of $\alpha$ and $\theta$.

5. The Gauss-Weierstrass Semigroup

In this section we prove Theorem 3. For the moment we let $1 < p \leq q < \infty$, and will later distinguish the cases $p < q$ and $p = q$.

Note that for $\delta > 0$ and $\text{Re} \, s > 0$,
\[
e^{s^2 \delta s} = T_{1/\delta} e^{s^2 \delta T_0},
\]
(where $T_8$ is the dilation operator defined in Section 1). Thus, if $\delta^2 s = \gamma(1 - \omega^2)/4\omega$, (1.3) implies that

$$e^s d = (\omega/\gamma)^{1/2} \pi^{1/2q-1/2p} T_8^{1/\gamma} T_p^{1/2} M^{-1} \tilde{M}^{-1} T_8^{1/\gamma},$$

where the notation of Theorem 1 is being used. Consequently, if $\|e^{-zH}\|_{p,q} = 1$ and $\text{Re} \alpha = \text{Re} \beta = 0$, then

$$\|e^s d\|_{p,q} = (\delta^2/\pi)^{1/2p-1/2q} |\omega/\gamma|^{1/2} |\gamma|^{1/2}.$$

Moreover, since $e^{-zH}$ preserves the constant functions, the norm in (5.2) is achieved by that Gaussian function $g(x) = e^{-\sigma x^2}$ for which $\tilde{g} = g$ constant.

Let $s = r e^{i\phi}$ with $r > 0$ and $\phi \in (\pi/2, \pi/2)$. In order for $\delta^2 s$ to equal $\gamma(1 - \omega^2)/4\omega$, restricting ourselves to $\gamma > 0$, we need

$$\arg(1 - \omega^2)/\omega = \phi,$$

$$\delta^2 = \left|\gamma(1 - \omega^2)/4\omega\right|.$$  

Note that since $|\omega| \leq 1$ and $|\phi| < \pi/2$, (5.3) implies that $\text{Re} \omega > 0$ and $|\omega| < 1$. Let $\omega = r \exp(-i\phi)$. Then (5.3) is equivalent to each of the following three statements (all of which we shall use).

$$\frac{1 + |\omega|^2}{1 - |\omega|^2} \tan \theta \equiv \tan \phi$$  

$$\frac{1 + |\omega|^2}{1 - |\omega|^2} \sin \theta = \sin \phi$$  

$$\frac{1 - |\omega|^2}{1 - |\omega|^2} \cos \theta = \cos \phi.$$  

Furthermore, in order that $\|e^{-zH}\|_{p,q} = 1$, we need (1.4) and (1.5) to hold; and if we wish to choose $\gamma$ so that $\text{Re} \alpha = \text{Re} \beta = 0$, we need equality in (1.5). In other words, we need equality in (1.7):

$$|p - 2 - \omega^2(q - 2)| = p - |\omega|^2 q.$$  

This equation also has several equivalent forms, of which we shall need the following.

$$|\omega|^4 (q - 1) - [(p - 2)(q - 2) \sin^2 \theta + p + q - 2] |\omega|^2 + (p - 1) = 0,$$

$$|\omega|^4 \leq p/q.$$  

(5.5a)
2(q - 1) | \omega|^2 = p + q - 2 + (p - 2)(q - 2) \sin^2 \theta
- [(p + q - 2 + (p - 2)(q - 2) \sin^2 \theta)^2 - 4(p - 1)(q - 1)]^{1/2}. \quad (5.5b)

If \( p \neq 2 \) and \( q \neq 2 \), the following is also equivalent to (5.5).

\[
\tan^2 \theta = \frac{(1 - |\omega|^2)(p - 1 - (q - 1) |\omega|^2)}{pq |\omega|^2 - (1 + |\omega|^2)(p - 1 + (q - 1) |\omega|^2)} , \quad |\omega| \leq p/q.
\]  

(5.5c)

To summarize, if \( \omega \) satisfies (5.3) and (5.5), or any of their equivalent forms, then there exists a \( \gamma > 0 \) such that (5.1) and (5.2) hold, with \( \delta > 0 \) given by (5.4).

Suppose that \( p < q \). For every \( \theta \) there exists a unique value of \( |\omega| \) such that (5.5b) holds, and \( |\omega| \) depends continuously on \( \theta \). Furthermore, for such \( |\omega| \), (5.5) implies \( |\omega|^2 \leq p/q < 1 \); and so \( |\omega| \) is uniformly bounded away from 1. Consequently, if \( \omega = |\omega| e^{i\theta} \) satisfies (5.5) and \( \theta \) ranges over \((-\pi/2, \pi/2)\), then the left hand side of (5.3a) ranges over all of \( \mathbb{R} \). Thus for any \( \phi \in (-\pi/2, \pi/2) \), there is an \( \omega \) satisfying both (5.3a) and (5.5b). Substituting the value of \( \delta^2 \) given by (5.4) into (5.2), we get (1.8).

In fact \( \omega \) can be found explicitly. If either \( p = 2 \) or \( q = 2 \), (5.5) easily gives \( |\omega|^2 \) and (5.3a) determines \( \theta \). Otherwise we may substitute (5.5c) into (5.3a); and this yields a cubic equation in \( |\omega|^2 \) with coefficients in terms of \( p, q, \) and \( \tan^2 \phi \). Any root in the interval \((0, p/q]\) is acceptable, and at least one such root exists. Once \( |\omega|^2 \) has been computed, the appropriate value of \( \theta \in (-\pi/2, \pi/2) \) is determined by (5.5c).

As for \( \gamma \), we simply let

\[
\gamma = [\text{Re } \omega/(1 - \omega^2)][\text{Re } 1/(1 - \omega^2) - 1/p]^{-1} = [\text{Re } 1/(1 - \omega^2) - 1/q'][\text{Re } \omega/(1 - \omega^2)]^{-1}.
\]

The two expressions are equal since equality holds in (1.5).

In the special case \( q' = p, 1 < p < 2 \), (1.4) and equality in (1.5) imply \( \text{Re } 1/(1 + \omega) = 1/p \), and so

\[
\cos \theta = \frac{p - 1 - |\omega|^2}{(2 - p) |\omega|}.
\]

Substituting this into (5.3c), we get

\[
|1 - \omega^2| = \frac{(1 - |\omega|^2)(p - 1 - |\omega|^2)}{(2 - p) |\omega| \cos \phi}.
\]

(5.6)

Furthermore, (5.5c) becomes

\[
\tan^2 \theta = \frac{(1 - |\omega|^2)(|\omega|^2 - (p - 1)^2)}{(p - 1 - |\omega|^2)^2}.
\]
and so by (5.3a)
\[
\frac{(1 + |\omega|^2)(|\omega|^2 - (p - 1)^2)}{(1 - |\omega|^2)(p - 1 - |\omega|^2)^2} = \tan^2 \phi.
\] (5.7)

Finally, \( \gamma = 1 \) guarantees that \( \Re \alpha = \Re \beta = 0 \). Putting the values of \( |1 - \omega^2| \) and \( |\omega| \) given by (5.6) and (5.7) into (1.8) with \( \gamma = 1 \) and \( q = \rho' \), we get (1.9).

Now suppose that \( \rho = q \geq 3 \). In this case (5.5) and (5.5a) become
\[
\frac{p - 2}{p} = \frac{1 - |\omega|^2}{1 - \omega^2},
\]
(5.5d)

\[
|\sin \theta| = \frac{1 - |\omega|^2}{|\omega|} \cdot \frac{(p - 1)^{1/2}}{p - 2}.
\] (5.5e)

If both (5.3c) and (5.5d) are satisfied, then \( \cos \theta = \rho \cos \phi/(p - 2) \); and so we get the necessary condition that \( \cos \phi \leq (p - 2)/p \). Also, (5.3b), (5.5d), and (5.5e) together yield
\[
1 + |\omega|^2 = |\omega| \rho (p - 1)^{-1/2} |\sin \phi|, \quad |\omega| < 1.
\] (5.8)

We now must exclude the case \( \cos \phi = (p - 2)/p \), for at that value of \( \phi \), (5.8) implies \( |\omega| = 1 \), which is impossible if (5.5d) is to hold.

If \( 0 < \cos \phi < (p - 2)/p \), let
\[
|\omega| = \frac{p |\sin \phi| - \sqrt{((p - 2)^2 - p^2 \cos^2 \phi)^{1/2}}}{2(p - 1)^{1/2}}
\] (5.9)

and \( \theta \) be given by (5.5e), with \( \cos \theta > 0 \) and \( \sin \theta \) having the same sign as \( \sin \phi \). Then \( \omega = |\omega| e^{i\theta} \) satisfies (5.3b) and (5.5e). Consequently, for some \( \gamma > 0 \) formula (5.2) holds with this \( \omega \) (and \( p = q \)).

To compute \( \gamma \), it suffices to find \( \gamma \) so that \( \Re \alpha = \Re \beta \). Since equality holds in (1.5), it then follows that \( \Re \alpha = \Re \beta = 0 \). Thus we want
\[
0 - \Re(\alpha - \beta) - (p - 2)/p + (\gamma - \gamma^{-1}) \Re \omega/(1 - \omega^2).
\]

But \( \arg \omega/(1 - \omega^2) = -\phi \) and so
\[
\Re \omega/(1 - \omega^2) = |\omega/(1 - \omega^2)| \cos \phi;
\]

and by (5.5d)
\[
|(1 - \omega^2)/\omega| = \frac{p(1 - |\omega|^2)}{(p - 2)|\omega|} = \frac{p}{p - 2} \left[ \frac{(p - 2)^2 - p^2 \cos^2 \phi}{p - 1} \right]^{1/2},
\]
since $|\omega|^{-1} - |\omega|$ is the difference between the two roots of (5.8). Putting all this together we get

$$\gamma - \gamma^{-1} = -\left[ \frac{(p-2)^2 - p^2 \cos^2 \phi}{(p-1) \cos^2 \phi} \right]^{1/2}, \quad \gamma > 0,$$

and so

$$\gamma = \frac{(p-2) \sin \phi - [(p-2)^2 - p^2 \cos^2 \phi]^{1/2}}{2(p-1)^{1/2} \cos \phi}.$$  \tag{5.10}

Substituting (5.9) and (5.10) into (5.2) with $p = q$, we get (1.10).

If $\cos \phi$ approaches $(p-2)/p$ from below, the right hand side of (1.10) approaches 1. Thus $\|e^{x f}\|_{p,a} \leq 1$ for $\cos \phi = (p-2)/p$. We extend this to all $\cos \phi \geq (p-2)/p$ by an interpolation argument. For simple functions $f$ and $g$ with finite Lebesgue measure support on $R$,

$$h(s) = \int_R (e^{x f}) g \, dx$$

is an analytic function on the open sector $\cos \phi > (p-2)/p$ and continuous on its closure. On the boundary, $\cos \phi = (p-2)/p$, we have

$$h(s) \leq \|f\|_p \|g\|_p.$$  \tag{5.11}

Moreover, $h(s)$ is bounded on the closed sector since $|h(s)| \leq \|f\|_a \|g\|_a$ for all $Re s \geq 0$. Thus by the Phragmen–Lindelöf Theorem (see Theorem 12.9 in [6] and conformally map the sector onto the strip), it follows that (5.11) holds on the interior of the sector. This completes the proof of Theorem 3.

REFERENCES

3. R. Coifman, personal communication.

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