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Cutting Planes in Combinatorics

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In chapter 26 of his book, George Dantzig presented side by side (i) a number of difficult mathematical problems reducible to integer linear programming problems, and (ii) Gomory's cutting-plane method for solving integer linear programming problems. In the same spirit, we illustrate the use of cutting-plane arguments in solutions of combinatorial problems.

TO GEORGE DANTZIG FOR HIS 70th BIRTHDAY

In Chapter 26 of his book [2], George Dantzig presented side by side (i) a number of difficult mathematical problems reducible to integer linear programming problems, and (ii) Gomory's cutting-plane method for solving integer linear programming problems. In the same spirit, we are going to illustrate the use of cutting-plane arguments in the solution of combinatorial problems.

To approach the subject gently, let us first consider the following problem from recreational mathematics [8]:

How many diamonds can be packed in a Chinese checkerboard? This board consists of two order 13 triangular arrays of holes, overlapping in an order 5 hexagon, 121 holes in all. A diamond consists of four marbles that fill four adjacent holes.

Fitting 27 diamonds onto the board is quite easy (see Figure 1); showing that 28 diamonds will not fit may get a little more complicated. However, as soon as the problem is stated in integer linear programming terms, an elegant argument to establish the upper bound suggests itself.

The integer linear programming formulation is straightforward: number the holes as $1, 2, \ldots, 121$, think of each diamond D as a set of four holes, and describe each packing of diamonds in the board by setting $x_D = 1$ if D is in the packing, and $x_D = 0$ otherwise. In this notation, our problem is to

$$\begin{aligned} &\text{maximize} \sum x_D \\ &\text{subject to} \sum (x_D\colon i \in D) \leqslant 1, \quad \text{for all } i = 1, 2, \ldots, 121, \\ &x_D = 0 \text{ or } 1 \quad \text{for all } D. \end{aligned} \tag{1}$$

(Feasible solutions of (1) are in a one-to-one correspondence with packings; in this correspondence, the objective function of (1) counts the diamonds in the packing.)

As usual, one may begin to solve (1) by solving its 'LP relaxation', the linear programming problem

$$\begin{aligned} &\text{maximize} \sum x_D \\ &\text{subject to} \sum (x_D\colon i \in D) \leqslant 1, \quad \text{for all } i = 1, 2, \ldots, 121, \\ &x_D \geqslant 0 \quad \text{for all } D. \end{aligned} \tag{2}$$

Since each feasible solution of (1) is a feasible solution of (2), the optimal value of (2) provides an upper bound on the optimal value of (1). As luck would have it, the optimal value of (2) is 27·5 and so the optimal value of (1), being an integer, cannot exceed 27.

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```
                    A
                  A   A
                B   A   C
              B   B   C   C
D   D   E   E   •   B   F   C   •   G   G   H   H
  D   D   E   E   I   F   F   J   G   G   H   H
    K   K   •   I   I   F   J   J   •   L   L
      K   K   M   I   N   •   J   O   L   L
        •   M   M   N   N   P   O   O   •
      Q   Q   M   R   N   P   P   O   S   S
    Q   Q   •   R   R   T   P   •   •   S   S
  U   U   V   V   R   T   T   •   W   W   X   X
U   U   V   V   •   Y   T   Z   •   W   W   X   X
              Y   Y   Z   Z
                Y   Ω   Z
                  Ω   Ω
                    Ω
```

FIGURE 1.

Thus, the packing in Figure 1 is optimal.

So far, we have only replaced the task of showing that the optimal value of (1) is 27 by the task of showing that the optimal value of (2) is 27·5. However, establishing the best upper bounds on optimal values of linear programming problems is easy by virtue of the Duality Theorem; in our case, we only need find nonnegative numbers $y_1, y_2, \ldots, y_{121}$ such that

$$\sum (y_i\colon i \in D) \geqslant 1, \quad \text{for all } D$$

and

$$\sum_{i=1}^{121} y_i = 27{\cdot}5.$$

Such numbers are represented in Figure 2 by the values of $6y_1, 6y_2, \ldots, 6y_{121}$.

Of course, to prove that the packing in Figure 1 is optimal, one does not have to allude to the Duality Theorem at all: the conclusion follows as soon as it is verified that the total of all the 'weights' in Figure 2 is 165, and that the total weight covered by each diamond is at least six.

This example illustrates two ways of generating linear inequalities that must be satisfied by every integer solution of

$$\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \qquad i = 1, 2, \ldots, m. \tag{3}$$

First, the argument showing that every solution of (2) must satisfy $\sum x_D \leqslant 27{\cdot}5$ generalizes as follows: if $y_1, y_2, \ldots, y_m$ are nonnegative numbers then every solution of (3) must

```
                        0
                      0   0
                    0   6   0
                  0   0   0   0
0   0   0   0   0   6   0   6   0   0   0   0   0
  0   6   0   6   0   2   2   0   6   0   6   0
    0   0   0   2   2   2   2   2   0   0   0
      0   6   2   2   1   1   2   2   6   0
        0   0   2   1   3   1   2   0   0
      0   6   2   2   1   1   2   2   6   0
    0   0   0   2   2   2   2   2   0   0   0
  0   6   0   6   0   2   2   0   6   0   6   0
0   0   0   0   0   6   0   6   0   0   0   0   0
                  0   0   0   0
                    0   6   0
                      0   0
                        0
```

FIGURE 2.

satisfy the inequality

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} y_i a_{ij}\right)x_j \leq \sum_{i=1}^{m} y_i b_i. \tag{4}$$

We shall refer to (4) as a *combination* of (3). Second, the argument showing that every integer-valued solution of $\sum x_D \leq 27{\cdot}5$ must satisfy $\sum x_D \leq 27$ generalizes as follows: if, in one of the inequalities included in (3), all the left-hand side coefficients a_{ij} are integers but the right-hand side b_i is not an integer then every integer solution of this inequality (and hence every integer solution of (3)) must satisfy

$$\sum_{j=1}^{n} a_{ij}x_j \leq \lfloor b_i \rfloor, \tag{5}$$

(with $\lfloor b_i \rfloor$ standing, as usual, for b_i rounded down to the nearest integer). We shall refer to (5) as a *cutting plane* derived from (3).

Now consider a sequence of inequalities

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \qquad i=1,2,\ldots,M, \tag{6}$$

such that, for each $k=m+1, m+2, \ldots, M$, the inequality

$$\sum_{j=1}^{n} a_{kj}x_j \leq b_k$$

is either a combination of, or else a cutting plane derived from, the system

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \qquad i=1,2,\ldots,k-1.$$

We shall refer to (6) as a *cutting-plane proof* of the inequality that comes last,

$$\sum_{j=1}^{n} a_{Mj}x_j \leq b_M, \tag{7}$$

from the system (3). Clearly, if (6) is a cutting-plane proof then every integer solution of (3) must satisfy (7), and therefore it must also satisfy all the inequalities

$$\sum_{j=1}^{n} a_{Mj}x_j \leq b$$

with $b > b_M$. The following two theorems show that the converse also holds as long as (3) satisfies certain non-restrictive assumptions.

THEOREM 1. *Let the polyhedron defined by* (3) *be bounded. If every integer solution of* (3) *satisfies a linear inequality*

$$\sum_{j=1}^{n} a_j x_j \leq b$$

then there is a cutting-plane proof of some inequality

$$\sum_{j=1}^{n} a_j x_j \leq b'$$

with $b' \leq b$ *from* (3).

THEOREM 2. *Let all the numbers* a_{ij} *and* b_i *in* (3) *be rational, and let* (3) *have at least one integer solution. If every integer solution of* (3) *satisfies a linear inequality*

$$\sum_{j=1}^{n} a_j x_j \leq b$$

then there is a cutting-plane proof of some inequality

$$\sum_{j=1}^{n} a_j r_j \leq b'$$

with $b' \leq b$ *from* (3).

Theorem 1 follows from Gomory's analysis of his cutting-plane algorithms [5], [6], [7]; an alternative proof may be found in [1]. Theorem 2 is a corollary of a result of Schrijver [9, Theorem B], from which Theorem 1 can be also derived.

The analogue of Theorem 1 and 2 is false if no assumptions at all are placed on (3). As Schrijver pointed out, if a is irrational then every integer solution of

$$\begin{aligned} x_1 - ax_2 &\leq 0, \\ -x_1 + ax_2 &\leq 0, \\ -x_1 - x_2 &\leq 0 \end{aligned} \tag{8}$$

satisfies the inequality $x_1 + x_2 \leq 0$, and yet there is no cutting-plane proof of any

$$x_1 + x_2 \leq b'$$

from (8). Similarly, every integer solution of

$$\begin{aligned} 3x_1 - 3x_2 &\leq 2, \\ -3x_1 + 3x_2 &\leq -1, \\ -x_1 - x_2 &\leq 0 \end{aligned} \tag{9}$$

satisfies the inequality $x_1+x_2 \leq 0$ (actually, (9) has no integer solution), and yet there is no cutting-plane proof of any

$$x_1+x_2 \leq b'$$

from (9).

Many combinatorial results may be stated by saying that every integer solution of a specific system (3) satisfies a specific linear inequality (or inequalities). Typically, the polyhedron defined by (3) satisfies the hypotheses of both Theorems 1 and 2; whenever this is the case, either of Theorem 1 and 2 guarantees that there is a cutting-plane proof of the result. Several examples have been given in [1]; the Chinese checkerboard problem discussed here is another example; now we are going to present one more.

A family of sets $S_1, S_2, \ldots, S_m$ is called a *weak delta-system* if there are numbers n and t such that $|S_i| = n$ for all i and $|S_i \cap S_j| = t$ whenever $i \neq j$. The family is called a *strong delta-system* if there is a set T such that $S_i \cap S_j = T$ whenever $i \neq j$. Erdös and Lovász conjectured that every weak delta-system with $m \geq n^2-n+2$ must be strong. (The family of lines in a projective plane shows that the lower bound on m cannot be replaced by n^2-n+1.) This conjecture had attracted a considerable attention and remained open for some time, until it was proved by Deza [4]. In fact, Deza simply pointed out that the conjecture follows easily from an earlier result of his own [3], concerning equidistant codes. Slightly restated, this result goes as follows.

DEZA'S THEOREM. *Let $S_1, S_2, \ldots, S_m$ be a weak delta-system with $|S_i| = 2k$ for all i and with $|S_i \cap S_j| = k$ whenever $i \neq j$. If $m \geq k^2+k+2$ then $S_1, S_2, \ldots, S_m$ is a strong delta-system.*

To deduce the Erdös-Lovász conjecture from Deza's theorem, consider an arbitrary weak delta-system $S_1, S_2, \ldots, S_m$ with $|S_i| = n$ for all i and with $|S_i \cap S_j| = t$ whenever $i \neq j$. We propose to show that

$$\begin{aligned} &S_1, S_2, \ldots, S_m \text{ is a strong delta-system} \\ &\text{whenever } m \geq k^2+k+2 \text{ with } k = \max(t, n-t). \end{aligned} \tag{10}$$

It $t \leq n-t$ then take a set R of size $n-2t$ disjoint from $S_1, S_2, \ldots, S_m$ and write $S_i^* = R \cup S_i$. If $t \geq n-t$ then take sets $R_1, R_2, \ldots, R_m$ of size $2t-n$ disjoint from $S_1, S_2, \ldots, S_m$ as well from each other, and write $S_i^* = R_i \cup S_i$. In either case, Deza's theorem guarantees that $S_1^*, S_2^*, \ldots, S_m^*$ is a strong delta-system whenever $m \geq k^2+k+2$; now (10) follows at once. Finally, note that (10) is a strengthening of the original Erdös-Lovász conjecture: we have $k^2+k+2 \leq n^2-n+2$ whenever $k < n$ (which is the only nontrivial case).

Deza's theorem can be stated in terms of integer linear programming: the trick is to observe that every family of sets $S_1, S_2, \ldots, S_m$ is determined up to an isomorphism by the sizes x_A of the 2^m-1 sets

$$\bigcap_{i \in A} S_i - \bigcup_{i \notin A} S_i$$

with A running through all the nonempty subsets of $\{1, 2, \ldots, m\}$. In this notation, the requirement that $|S_i| = 2k$ for all i can be stated as

$$\sum (x_A \colon i \in A) = 2k \quad \text{for all } i, \tag{11}$$

and the requirement that $|S_i \cap S_j| = k$ whenever $i \neq j$ can be stated as

$$\sum (x_A \colon i, j \in A) = k \quad \text{whenever } i \neq j. \tag{12}$$

Conversely, every integer solution of (11), (12) and

$$x_A \geq 0 \quad \text{for all } A \tag{13}$$

defines sets $S_1, S_2, \ldots, S_m$ such that $|S_i| = 2k$ for all i and $|S_i \cap S_j| = k$ whenever $i \neq j$. These sets form a strong-delta-system if and only if

$$x_A = 0 \quad \text{whenever } 1 < |A| < m. \tag{14}$$

Thus, Deza's theorem amounts to the claim that, as long as

$$m = k^2 + k + 2,$$

every integer solution of (11), (12), (13) must satisfy (14).

Either of Theorems 1 and 2 guarantees that this claim can be justified by a cutting-plane proof; we are going to present Deza's own argument in the guise of a cutting-plane proof.

First, consider an arbitrary but fixed nonempty subset P of $\{1, 2, \ldots, m\}$, and write $p = |P|$, $q = m + 1 - p$. The sum of

the p equations (11) with $i \in P$ multiplied by $1/p^2$

the $(q-1)$ equations (11) with $i \notin P$ multiplied by $1/q^2$

the $p(p-1)$ equations (12) with $i, j \in P$ multiplied by $1/p^2$

the $(q-1)(q-2)$ equations (12) with $i, j \notin P$ multiplied by $1/q^2$

the $p(q-1)$ equations (12) with $i \in P$, $j \notin P$ multiplied by $-2/pq$

reads $\sum c_A x_A = d$ with

$$c_A = \frac{|A \cap P|}{p^2} + \frac{|A - P|}{q^2} + \frac{|A \cap P|(|A \cap P| - 1)}{p^2} + \frac{|A - P|(|A - P| - 1)}{q^2} - 2\frac{|A \cap P| \cdot |A - P|}{pq}$$
$$= \left(\frac{|A \cap P|}{p} - \frac{|A - P|}{q}\right)^2$$

and

$$d = p \cdot \frac{2k}{p^2} + (q-1) \cdot \frac{2k}{q^2} + p(p-1) \cdot \frac{k}{p^2} + (q-1)(q-2) \cdot \frac{k}{q^2} - p(q-1) \cdot \frac{2k}{pq}$$
$$= \frac{k(m+1)}{p(m+1-p)}.$$

Since $c_A \geq 0$ for all A, and $c_P = 1$, we conclude that the inequality

$$x_P \leq \frac{k(m+1)}{p(m+1-p)}$$

is a combination of (11), (12), (13). Since the right-hand side of this inequality is less than 1 whenever $k + 2 \leq p \leq k^2 + 1$, it follows that every integer solution of (11), (12), (13) has

$$x_A = 0 \quad \text{whenever } k + 2 \leq |A| \leq k^2 + 1. \tag{15}$$

Second, consider an arbitrary but fixed t such that $1 \leq t \leq m$. The sum of the equation (11) with $i = t$ multiplied by $-k/(k^2+1)$, and the $m-1$ equations (12) with $i = t$ multiplied by $1/(k^2+1)$ reads

$$\sum \left(\frac{|A| - (k+1)}{k^2+1} x_A \colon t \in A\right) = k - 1 + \frac{1}{k^2+1}.$$

Here, the coefficient at each x_A is at most 1, and it is at most zero whenever $|A| \leq k+1$. Hence, the inequality

$$\sum (x_A\colon t \in A, |A| \geq k^2+2) \geq k-1+\frac{1}{k^2+1}$$

is a combination of (11), (12), (13), (15). It follows that every integer solution of (11), (12), (13) has

$$\sum (x_A;\, t \in A, |A| \geq k^2+2) \geq k \quad \text{for all } t. \tag{16}$$

Third, let us consider families F of sets A such that

$$|A| \geq k^2+2 \quad \text{whenever } A \in F. \tag{17}$$

To begin, note that

$$\sum (x_A\colon A \in F) \leq k \quad \text{whenever } |F| \leq k+1 \text{ and (17) holds:} \tag{18}$$

in this case, the intersection of all the sets A in F includes at least two distinct points i and j, and so the left-hand side of (18) is majorized by the left-hand side of (12). To put it differently, we have just observed that each of the inequalities (18) is a combination of (12), (13). Now it is easy to see that every integer solution of (18) has

$$\sum (x_A\colon |A| \geq k^2+2) \leq k: \tag{19}$$

by (18), at most k of the integers x_A can be positive and, by (18) again, (19) follows. Of course, the same conclusion can be derived within the formal framework of cutting-plane proofs. For instance, whenever $|F| \geq k+2$ and (17) holds, a cutting-plane proof of

$$\sum (x_A\colon A \in F) \leq k \tag{20}$$

from (18) can be constructed by induction on $|F|$: having constructed cutting-plane proofs of

$$\sum (x_A\colon A \in F') \leq k \tag{21}$$

from (18) for all subfamilies F' of F such that $|F'| = |F|-1$, observe that

$$\sum (x_A\colon A \in F) \leq k+\frac{k}{|F|-1} \tag{22}$$

is a combination of (21), and that (20) is a cutting-plane derived from (22).

Now that (16) and (19) have been established, we only need observe that every solution of (12), (13), (16) and (19) satisfies (14): this is quite trivial. Formally, one might point out that, for every choice of distinct i and j, the inequality

$$\sum (x_A\colon i, j \in A, |A| < m) \leq 0$$

is a combination of (12), (13), (16), (19), with multipliers

1	at the single equation (12),
$\lvert A\rvert+1-m$	at each inequality (13) with $k^2+2 \leq \lvert A\rvert < m$,
-1	at each of the m inequalities (16),
$m-1$	at the single inequality (19).

Thus, the proof of Deza's theorem is completed.

The point of this article is that formulating combinatorial problems as integer linear programming problems is not just a self-serving exercise: the integer linear programming formulations often suggest a cutting-plane argument that is obscured by the combinatorial presentation. In particular, the Erdös–Lovász conjecture resisted efforts of several mathematicians for some time, and yet its integer linear programming formulation suggests at once an argument that goes a long way towards a proof. Here, one aims to show that every integer solution of

$$\sum (x_A\colon i \in A) = n \quad \text{for all } i, \tag{23}$$

$$\sum (x_A\colon i, j \in A) = t \quad \text{whenever } i \neq j, \tag{24}$$

$$x_A \geq 0 \quad \text{for all } A, \tag{25}$$

with A running through all the nonempty subsets of $\{1, 2, \ldots, m\}$ such that

$$m = n^2 - n + 2,$$

must satisfy

$$x_A = 0 \quad \text{whenever } 1 < |A| < m.$$

To put it differently, one aims to show that the optimal value of the problem

$$\text{maximize } x_P \text{ subject to (23), (24), (25), and } x_A = \text{integer for all } A \tag{26}$$

is zero whenever $1 < |P| < m$. (We may assume that $1 \leq t \leq n-1$, for otherwise the conclusion is trivial.) The most natural way to approach this task is to solve first the LP relaxation of (26),

$$\text{maximize } x_P \text{ subject to (23), (24), (25):} \tag{27}$$

since every feasible solution of (26) is a feasible solution of (27), the optimal value of (27) provides an upper bound on the optimal value of (26). In turn, to provide an upper bound d on the optimal value of (27), we only need construct a combination

$$\sum c_A x_A \leq d$$

of (23), (24) such that $c_A \geq 0$ for all A and such that $c_P = 1$. The symmetry of (23) and (24) allows us to restrict the range of the multipliers to only five distinct values:

y_1 at each equation (23) with $i \in P$,
y_2 at each equation (23) with $i \notin P$,
y_3 at each equation (24) with $i, j \in P$,
y_4 at each equation (24) with $i, j \notin P$,
y_5 at each equation (24) with $i \in P, j \notin P$.

Now we have

$$c_A = y_1|A \cap P| + y_2|A - P| + y_3|A \cap P|(|A \cap P| - 1) + y_4|A - P|(|A - P| - 1) + y_5|A \cap P| \cdot |A - P|$$

and

$$d = y_1 n|P| + y_2 n(m - |P|) + y_3 t|P|(|P| - 1) + y_4 t(m - |P|)(m - |P| - 1) + y_5 t|P|(m - |P|).$$

The simplest way to ensure that $c_A \geq 0$ is to make c_A a square; this requirement dictates that

$$y_1 = y_3 = v^2, \qquad y_2 = y_4 = w^2, \quad \text{and} \quad y_5 = 2vw$$

for some choice of v and w; now

$$c_A = (v|A \cap P| + w|A - P|)^2.$$

Next, the requirement that $c_P = 1$ dictates $v = 1/|P|$; d is minimized by setting

$$w = -1 \Big/ \left(m + \frac{n}{t} - 1 - |P| \right).$$

This choice of $y_1, y_2, \ldots, y_5$ yields

$$d = (n-t) \frac{m + \frac{n}{t} - 1}{|P| \cdot \left(m + \frac{n}{t} - 1 - |P| \right)}.$$

Since $d < 1$ whenever

$$(n-t+1) \leqslant |P| \leqslant \left(m + \frac{n}{t} - 1 \right) - (n-t+1),$$

we conclude that every integer solution of (23), (24), (25) has

$$x_A = 0 \quad \text{whenever } (n-t+1) \leqslant |A| \leqslant \left(m + \frac{n}{t} - 1 \right) - (n-t+1). \tag{28}$$

At this point, the integer linear programming formulation suggests no natural continuation; however, the information provided by (28) is so powerful that completing the proof in the original combinatorial terms is easy.

By (28), each point is in either at most $n-t$ or at least $m-(n-t)$ of the sets $S_1, S_2, \ldots, S_m$. Observing that $m-(n-t) > (n-t)$, we shall refer to these two kinds of points as *poor* and *rich*, respectively.

It is intuitively obvious that each S_i must include a substantial number of rich points, for else it could not intersect each of the remaining $m-1$ sets S_j in as many as t points. More precisely, we have

$$\sum_{j=1}^{m} |S_i \cap S_j| = n + (m-1)t \tag{29}$$

for each i and, when S_i includes precisely r rich points,

$$\sum_{j=1}^{m} |S_i \cap S_j| \leqslant rm + (n-r)(n-t). \tag{30}$$

Comparing (29) with (30), we find that

$$r \geqslant t - \frac{(n-t)(n-t-1)}{m-n+t};$$

since $(n-t)(n-t-1) < m-n+t$, we conclude that $r \geqslant t$. Thus,

$$\text{each } S_i \text{ includes at least } t \text{ rich points.} \tag{31}$$

On the other hand, it is also intuitively obvious that the total number of rich points cannot be too large, for else some $t+1$ rich points would belong to two different sets S_i. Pursuing this idea, we find that

$$\text{there are at most } t \text{ rich points altogether.} \tag{32}$$

To justify (32), let us assume the contrary: there is a set R of $t+1$ rich points. Now we have

$$\sum_{j=1}^{m} |S_j \cap R| \geq (t+1)(m-(n-t)) \tag{33}$$

and, since at most one S_j has $S_j \cap R = R$,

$$\sum_{j=1}^{m} |S_j \cap R| \leq (t+1)+(m-1)t. \tag{34}$$

Comparing (33) with (34), we obtain the inequality

$$m-1 \leq (t+1)(n-t)$$

which is clearly unsatisfiable: its right-hand side is at most $((n+1)/2)^2$.

Finally, (31) and (32) together imply at once that $S_1, S_2, \ldots, S_m$ is a strong delta-system.

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