Completeness results for intuitionistic and modal logic in a categorical setting

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Abstract

Versions and extensions of intuitionistic and modal logic involving biHeyting and bimodal operators, the axiom of constant domains and Barcan's formula, are formulated as structured categories. Representation theorems for the resulting concepts are proved. Essentially stronger versions, requiring new methods of proof, of known completeness theorems are consequences. A new type of completeness result, with a topos theoretic character, is given for theories satisfying a condition considered by Lawvere (1992). The completeness theorems are used to conclude results asserting that certain logics are conservatively interpretable in others.

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1. Introduction

The great insight of F.W. Lawvere's that has created categorical logic was to observe that "all" the significant logical operations arise as adjoints to naturally given functors. For example, binary products in a category $C$ are given as the right adjoint to the diagonal functor $C \to C \times C$; and further, for any fixed $A \in C$, exponentiation $(\ )^A$ is given as the right adjoint to $A \times (\ ) : C \to C$. The pair of examples mentioned shows that the process of creating new operations is hierarchical (exponentiation relies on product), and that the functors to which we seek adjoints may be defined by parameters (in our case, the object $A$). The last sentence contains the more essential half of the idea of a categorical doctrine for logic. The specification of any categorical doctrine contains a choice of certain specific "adjoint operations"; these come in an ordered sequence so that each operation is defined in terms of the earlier ones as the adjoint to a specific functor explicitly definable, possibly with parameters, in terms of the earlier operations. The other half of the specification is the requirement that certain "exactness properties" involving the operations should universally hold.

Incidentally, the sense of "doctrine" here is essentially disjoint from that of "hyper-doctrine" (sometimes also abbreviated to "doctrine"). Both uses originate with Lawvere, the first in [20]. The concept of doctrine in categorical logic in the sense indicated here is discussed in [18] entitled "Doctrines in categorical logic".

The appeal of Lawvere's approach is the generation, via the general abstract principle of adjunction, of the logical operations from the mere notions of abstract set, function and functional composition. It is to be emphasized that each adjoint is essentially uniquely determined by being an adjoint to a definite functor; the requirement inherent in the specification of the doctrine is that the adjoint should exist. Moreover, the existence of any particular adjoint functor is a local condition; for any one object of the domain of the desired adjoint, an object with a certain universal (extremal) property in the codomain category is required to exist.

There is no more convincing argument for the fundamentally objective nature of logic than Lawvere's way of introducing its operations.

The point left vague in the above description is the starting point for building up the operations. Usually, an object of a doctrine is based on a category $C$; the "first-level" operations are adjoints to functors of the form $C^H \to C^G$, induced from a map $G \to H$ of finite graphs, etc. In this case, the doctrine itself as a category, with morphisms the functors preserving the operations, has a forgetful functor to $\text{CAT}$, the category of categories; we say the doctrine is based on $\text{CAT}$. In this paper, we would like to advocate the adoption and study of doctrines that are based not on single categories, but on structures on the next level of complexity, namely a pair of categories connected with a functor. In this case, the doctrine has a forgetful functor to $\text{CAT}^\to$, the category of functors. For example, our proposal for a categorical formulation of Lewis's S4 predicate logic is a doctrine based on $\text{CAT}^\to$. There will also be doctrines with objects based on three categories, with two connecting functors.
Usually, even before a particular doctrine has been specified, we have an idea of certain standard objects of it. For example, we may start with Set, the category of small sets, single out the operations of finite limits and finite coproducts (to give one specific choice of many possible ones), and ask for the doctrine that has all categories having the operations mentioned, and “having all the exactness properties possessed by Set with respect to the operations singled out”. In the particular case, we obtain the doctrine of distributive categories. The point is that something very compact suffices to nail down the exactness properties: the finite coproducts have to be finite disjoint sums stable under pullback. The reason why we know that this specification answers the desideratum in quotes is a representation theorem: every small distributive category has a conservative operation-preserving functor into Set, a Cartesian power of Set. By the nature of exactness properties, each such is inherited by the domain of a conservative operation-preserving functor from the codomain, much as a subalgebra inherits all identities (and all universal Horn properties) from the algebra it is a subalgebra of. Likewise, exactness properties are inherited by (Cartesian) products.

The example described shows how the inquiry into the logical properties of a single category, or possibly of a class of categories, can give rise to a doctrine. Fixing Set as our category of interest does not by itself define the doctrine; we have to single out the adjoint operations of interest on Set. In fact, there are a number of doctrines all centered on Set; in each of them, the corresponding representation theorem expresses the central position of Set.

What we outlined above is the categorical framework for completeness theorems in logic. Let us hasten to add that this is not a universally applicable framework; it is in fact quite selective, but also, quite widely applicable. H.J. Keisler’s completeness theorem on the quantifier “there exist uncountably many” does not fit the framework; the mentioned quantifier does not arise as an adjoint operation. On the other hand, classical logic, and important fragments of it that without category theory would not be noticed so easily, various infinitary logics, higher-order logic, non-classical logics such as intuitionistic and modal logics, do fit the framework. Categorical logic is not to be restricted to the categorical doctrines as given here; there are operations in categories that are fundamental after all and that do not arise as adjoint operations. However, it remains the case that the categorical doctrines in the specific sense described here form an interestingly distinctive field of study.

The idea that interesting versions of modal logic are obtained by natural choices of categorical structures with respect to which the version can be proved to be complete is due to Ghilardi and Meloni [8, 9] and independently, to the second author [35] and Lavendhomme, Lucas and the second author [19]. However, in the papers mentioned our doctrinal frameworks are not developed.

For early discussions of the idea of a categorical doctrine for logic, see [20, 18].

In this paper, we establish doctrinal frameworks for several completeness theorems for strengthened versions of intuitionistic and modal logic. It is basic to our approach that the doctrines arise from natural “standard” objects. For instance, the doctrine S4
arises from considering the objects of the form \( i^* : \text{Set}^K \to \text{Set}^{K_1} \), with \( K \) any category, and \( i: |K| \to K \) the "identity" map from the discrete category on the objects of \( K \) to \( K \) itself. Selecting for study the coherent operations (see below) in both domain and codomain, as well as the right adjoints to the induced maps \( S(X) \to S(i^* \chi) \), with \( X \) any object in the domain (\( S(Y) \) the lattice of subobjects of \( Y) \), we get our version of S4 predicate logic with equality.

When compared with the traditional version of the same logic, ours turns out to be a proper extension; every S4 theory in the usual sense sits, via a "Lindenbaum–Tarski category", in our doctrine, but not every object of the doctrine corresponds to such a theory. For one thing, our "theories" are many sorted. More importantly, in our "theories", the necessity operator is applicable only to predicates whose free variables are of certain distinguished sorts; the classical setup does not envisage such a distinction among sorts.

The representation theorem for the S4 doctrine, already enunciated by specifying what we take the standard objects to be, thus turns out to be an extension of the classical Kripke completeness theorem. The situation with S4 just described repeats itself in broad outline involving other known logics and new doctrines.

From the conceptual point of view, the central section of the paper is Section 4. Here, we clarify the close connection between, on the one hand, the invariance under substitution of coimplication and the past-possibility operator, duals of Heyting implication and (S4) necessity, and on the other hand, the axiom of constant domains and Barcan's formula. Our doctrinal framework is especially revealing in this respect; in fact, the technically simple discussion of these topics in the doctrinal framework is, in our minds, the main contribution of this paper. The substitutivity of the necessity operator is a consequence of the definition of the S4 doctrine, which definition does not have any other elements than a basic requirement that the categories and the functor involved be coherent, and the definition of necessity as an adjoint; no other exactness condition is explicitly required. The requirement of the connecting functor being conservative makes the domain-category of the S4 category Heyting, and defines implication in the well-known manner. This gives rise to the possibility (in the form of a left adjoint to the forgetful functor from S4 categories to Heyting categories) of a "free" interpretation of intuitionistic logic in S4 logic; the fact that this is a conservative interpretation is our version of the Gödel interpretation of intuitionistic logic in S4 modal (predicate) logic. Further, because of the fact that the coimplication is generated by the past-possibility operation in a similar manner as implication by necessity, there is an immediate connection between the (non-automatic) substitutivity of coimplication, and that of past-possibility. Adding to this the fact that the substitutivity of the past-possibility is in a natural connection with Barcan's formula, we have outlined the main concerns of the paper. Our main results, proved in Sections 6 and 8, are all answers to questions immediately arising out of this situation. For example, intuitionistic logic with selected sorts demanded to satisfy the axioms of constant domains, has a conservative free interpretation in modal logic with the same domains satisfying Barcan's formula, and even more strongly, in bi-modal logic in
which the past-possibility operator is substitutive with respect to selected substitutions.

Several of these facts have familiar versions in the literature. However, categorical logic gives a framework in which they appear in a conceptually satisfying manner. Also, as we point out below, our specific results are stronger than the corresponding classical results (when such corresponding results exist), and they require new methods with respect to those usually employed in the context of non-classical logics.

In Section 6, we give Kripke-type completeness results both for intuitionistic and modal logic, with and without the additional operations of coHeyting implication and the “past-necessity” operator, in the context of the axiom of constant domains and Barcan’s formula. In contrast to the literature, our concepts provide a single context for the situations with or without the constant-domains axiom (resp. Barcan’s formula), by allowing a parameter we called $B$, the set of constant sorts. The classical contexts are the two extremes when $B$ is empty (the additional axioms are not present), and when $B$ is the set of all sorts (the axioms are required for all sorts; in fact, in the literature, only one sort is considered, but the many sorted case, with all of them constant, is a mathematically inessential generalization of the classical case).

Allowing arbitrary $B$’s is natural in the categorical context. From the work in Section 6, in the brief Section 7 we obtain an axiomatization of the “logic of constant sets among variable sets”, in the form of a very natural categorical doctrine.

Also, allowing arbitrary $B$’s results in an essential generalization, in the sense that the known methods are not sufficient to deal with the general case. In Section 6, we use the so-called special (derived from saturated) models from model theory; see [4]. In Section 5, we review the definitions in the context of many-sorted logic (that requires no essential change with respect to the usual setting), and prove a result, to be used in Section 6, that is of a familiar kind in model theory; it is a version of a preservation theorem, expressed in terms of the existence of mappings of a certain kind between special models.

For the completeness theorems of Section 6 in their full generality, specifically for possibly uncountable theories, we do not know of any other method of proof, except in the extreme cases of the pure intuitionistic and the pure S4 doctrines, without constant sorts and without coimplication and without past-necessity. Ghilardi [7] has also used special models in modal logic. His results are quite different from ours, but the mathematics in Section 6 is related to Ghilardi’s. Our work was independent of Ghilardi’s.

In the last section, which is in the way of an appendix, we present alternative methods, notably omitting types and other Henkin-type arguments for Kripke-type completeness results. These methods give slightly different specific results; they apply only in the case of countable theories, but in that case, they allow the use of “all” countable models, without saturation conditions. We reproduce a surprising Henkin-type argument due to Ghilardi and Meloni [8] that works in our more general context too. If one is only interested in the arithmetical (syntactical) consequences of the various completeness theorems (e.g., the conservative enrichment
results (see below)), there is no loss in restricting oneself to countable theories; this is a good reason why in the literature on non-classical logics there is practically no mention of uncountable theories. There is still one case, notably bi-intuitionistic logic, for which we have not been able to prove the requisite completeness theorem even in the countable case without the use of some saturativity condition on models.

Section 9 is a suitable context to compare the usual methods used in Kripke-type completeness proofs with or without constant domains, which are all Henkin-type arguments, with the Henkin-type arguments used for our doctrines. Rather than the fact of having several sorts instead of just one, the essential new element in our doctrines is the classification of sorts into two classes as indicated above in the case of the S4 doctrine, or in the case of modal and intuitionistic logic having certain domains constant, and others not. The reader will be able to compare the new methods required when constant sorts are mixed with non-constant ones (e.g., omitting types), with the "classical" methods in the non-mixed case; the latter will be reproduced in outline.

The mixing of kinds of sorts is a new element in this work also with respect to recent work such as [9, 19], and it is the result of our specific choice of the categorical doctrines for the various logics. In [19], there is no attempt at defining doctrines, although the standard structures of several of our doctrines do appear in that paper. The completeness theorems dealt with in [19] are for theories formulated in the symbolic-logical framework, with target categories the structures just referred to. In [9], there are hyper-doctrine-type doctrines that, in essence, are close to usual symbolic-logical theories.

Section 8 gives completeness results that use Grothendieck toposes more general than presheaf toposes. From our point of view, the use of presheaf toposes as target categories for representation theorems is essentially the same as Kripke's semantics, although [6] successfully makes the point that the use of general presheaf toposes gives results that cannot be achieved by using only presheaves over preorders, which, strictly speaking, is the original context for Kripke's semantics. We were pleased to find that our earlier work in [29, 26], originally motivated by independent considerations, was instrumental in arriving at the results in Section 8.

We systematically use the idea of conservative enrichment to isolate proof-theoretical consequences of the representation theorems, showing that the latter have arithmetical (syntactical, proof-theoretical) content. We say that one doctrine is a conservative enrichment of another, if there is a "forgetful" functor from the first to the second whose left adjoint has a unit with conservative components. This general categorical concept encompasses Gödel's interpretation of intuitionistic logic in S4 modal logic, and the several analogs of this result that are the main subject of this paper. The conservative enrichment results will be immediate consequences of corresponding completeness (representation) theorems. We are not aware of a clear statement in the literature of this state of affairs; that is, e.g., of the fact that the conservativeness of Gödel's interpretation is a corollary to Kripke's completeness theorem for intuitionistic logic. However, as the referee has pointed out, the use of
semantic arguments in showing conservativeness is "folklore" in non-classical logic. In the preface of [17], we read "we prefer ... to develop the semantics of intuitionistic logic independently of that of S4; this procedure will enable us, we believe, to obtain ... the mapping into S4 as a consequence ...". Let us add that the proof-theoretical results we obtain seem to be stronger than their "usual" versions with "unmixed" sorts.

Let us mention two further specific features of the present paper. One is the emphasis on the fact that categorical logic is a natural extension of propositional logic done algebraically. To bring out this very important, because constantly motivating, fact, we first give a treatment in propositional logic of some of our themes, where these themes are in fact quite well-known; consequently, there is little (or nothing?) mathematically new in Section 2. The reader may not have realized, however, that e.g. the Kripke completeness theorem for S4 propositional logic is the same as the statement that a certain very canonically defined morphism is an S4 homomorphism. In Section 3, we guide the reader through some facts, some of which are still well known, although lesser than previously, that show that the propositional logic of Section 2, with all its canonical constructions, has a striking lifting to predicate logic, resulting from generalizing preorders to categories, and, most importantly, from replacing the 2-element order by the category of sets.

Another conclusion to be drawn from this paper is that "everything in non-classical logics is based on coherent logic". We believe that this somewhat exaggerated claim will become reasonably convincing in the course of the paper. This circumstance is one of the general conclusions of categorical logic not realized by traditional symbolic logic (although, by hindsight, elements of it may be found e.g. in the usual treatments of canonical models; see [13]). The main instance appearing in this paper of the claim is Joyal's theorem that presents the canonical "Kripke" model of an intuitionistic theory as a canonical construction based on the category of ordinary models of the same theory-category as a coherent theory.

We have made an effort to make the paper self-contained, and easy-to-read for the reader with a certain background in model theory and non-classical logics. With the exception of some topos-theoretical terminology used in Section 8 (and at other places where it can be ignored without losing the continuity of the exposition), the paper relies only on the most basic categorical concepts without explanation. The standard reference for category theory is [25].

2. Algebras for intuitionistic and modal propositional logic

In this section we discuss the Kripke semantics of intuitionistic and S4 modal propositional logic, together with their extensions with operations dual to implication and necessity. We show how the category of distributive lattices forms a basis on which the Kripke completeness theorems appear as properties of canonical constructions, rather than existence theorems. In the next section, we will lift the ideas of this section into the context of predicate logic, by replacing 2 by Set.
DI denotes the category of distributive lattices (with least and greatest elements 0 and 1, resp.) with lattice homomorphisms as morphisms. We call monomorphisms in DI and other related categories of structured posets conservative to emphasize that the important thing about them is that they reflect the order: \( f: A \to B \in \text{DI} \) is a monomorphism iff \( f(x) \leq_B f(y) \) implies \( x \leq_A y \) iff \( f \) is a one-to-one function. This usage will also accord with the notion of "conservative extension" in logic. We abbreviate "distributive lattice" as "d.l.". 2 denotes the two-element d.l.

Let \( A \in \text{DI} \). A morphism \( h: A \to 2 \) is identified with a prime filter \( p \subseteq A \) on \( A \); \( x \in p \iff h(x) = 1 \) (\( x \in A \)). We write \( A^* \) for the poset of all prime filters on \( A \), with ordering the set-theoretic containment relation (with the elements of \( A^* \) understood as 2-valued homomorphism, \( A^* = \text{hom}(A, 2) \), the ordering on \( A^* \) is the pointwise ordering inherited from 2). Now, for any poset (or even quasi-ordering) \( P \), and any d.l. \( B \), the poset \( B^p \) of all order-preserving maps \( P \to B \) ordered pointwise (using the order on \( B \)) is a d.l. again; in fact, the lattice operations are computed pointwise. The elements \( f \in 2^p \) can be identified with the upward closed subsets \( X \) of \( P \); the identification is given by the relation \( p \in X \iff f(p) = 1 \). In this way, \( 2^p \) gets identified with \( \mathcal{P}_1(A^*) \), the poset (ordered by set-inclusion) of upward closed subsets of \( P \). Thus, we have the d.l. \( 2A^* \) (or \( \mathcal{P}_1(A^*) \)) that we denote by \( A^{**} \). Moreover, we have the canonical "evaluation map"

\[
e_A: A \longrightarrow A^{**}
\]

\[
x \longmapsto [f \in A^* \mapsto f(x) \in 2] \quad \text{("function" formulation)}
\]

\[
p \in e_A(x) \iff x \in p \quad \text{("set" formulation)};
\]

it is immediate that \( e_A \) is a morphism of d.l.'s. The Stone representation theorem says that

\[
e_A \text{ is a conservative morphism in } \text{DI}.
\]

Let us emphasize the easily seen fact that (2.1) is equivalent to

\[
\text{the evaluation } A \to 2\text{|}A^* \text{ is conservative;}
\]

(2.1')

here, the power is an ordinary Cartesian power.

For \( x, y \in A \), the relative pseudo-complement of \( x \) with respect to \( y \), or the (Heyting) implication of \( x \) and \( y \), denoted \( x \to y \), is the element determined by the property

\[
\text{for all } z \in A, \ z \leq x \to y \iff x \land z \leq y
\]

provided such an element exists. A Heyting algebra is a d.l. in which all implications exist (actually, distributivity is a consequence of the latter property). A morphism of Heyting algebras is a DI-morphism also preserving implications. In this paper, \( H_0 \) will stand for the category of Heyting algebras (the use of the subscript 0 is because a more prominent role will be played by \( H \), the bicategory of Heyting categories; see the next section).
For any preorder $P$, $2^P$ is a Heyting algebra; for $X, Y \in \mathcal{P}_1(P)$, we have

$$x \in X \to Y \iff \forall y \geq x. \ y \in X \Rightarrow y \in Y;$$

in particular, $A^{**}$ is a Heyting algebra.

A conditionally Heyting morphism $h: A \to B$ is a d.l. morphism that preserves all existing implications: if $x, y \in A$ and $x \to y$ exists, then $h(x) \to h(y)$ exists and is equal to $h(x \to y)$. A result that in [11] was attributed to Joyal is that

$$e_A: A \to A^{**} \text{ is conditionally Heyting.} \quad (2.2)$$

For the proof (which the reader knowing the "prime filter existence theorem" will find without difficulty), see e.g. [11]. In particular, with (2.1), we have

any Heyting algebra $H$ has a (canonical) Heyting embedding into a Heyting algebra of the form $2^P$; in fact we can take $P = H^*$. \quad (2.3)

Joyal's theorem can be seen as algebraic formulation of the Kripke completeness theorem [17] for intuitionistic propositional logic; for a discussion, see e.g. [11].

The (Heyting) coimplication, or difference $y \setminus x$ of $x, y \in A$ is defined, if it exists, as the Heyting implication $x \to y$ in the opposite $A^\circ$ of $A$. In other words,

for all $z \in A$ $y \setminus x \leq z \iff y \leq x \lor z$.

A coHeyting algebra is a d.l. in which all coimplications exist; $A$ is a coHeyting algebra iff $A^\circ$ is a Heyting algebra. A biHeyting algebra is Heyting algebra which is also a coHeyting algebra. For these notions, see [21, 22, 37]. biHo will stand for the category of biHeyting algebras.

For any preorder $P$, $2^P$ is a biHeyting algebra; for $X, Y \in 2^P$, we have

$$x \in Y \setminus X \iff \exists y \leq x. \ y \in Y \land y \notin X.$$

As we show below, the last observation with (2.3) gives

bi-intuitionistic propositional logic (biIPL) is a conservative extension of intuitionistic propositional logic (IPL), \quad (2.4)

a result due to Rauszer [34]. biIPL is obtained by adding the binary connective $\setminus$ and the following rules to IPL:

$$\frac{\psi \setminus \phi \vdash \theta}{\psi \vdash \phi \lor \theta} \quad \frac{\psi \vdash \phi \lor \theta}{\psi \setminus \phi \vdash \theta}$$

To say that biIPL is a conservative extension of IPL means that if an entailment $\phi \vdash \psi$ with formulas $\phi, \psi$ of IPL is deducible in biIPL, then it is already deducible in IPL. We will give an algebraic formulation of a strengthened form of (2.4), and show that it follows from (2.3). Since we will have several conclusions of the same general kind in the paper, we give a general setup to unify the discussion.
Let \( G : A \rightarrow B \) be a functor with a left adjoint \( F : B \rightarrow A \). Assume that

\[ \text{every } B \in B \text{ has a conservative morphism } B \rightarrow G(A) \text{ into } G(A) \text{ for some } A \in A. \]

\((*)\)

Then,

\[ \text{for every } B \in B, \text{ the unit map } \eta_B : B \rightarrow G(F(B)) \text{ is conservative} \]

\((***)\)

as well. The reason is that we have the factorization

\[
\begin{array}{ccc}
B & \xrightarrow{m} & GF(B) \\
\downarrow \circ & & \downarrow G(f) \\
\downarrow m & & \downarrow G(A)
\end{array}
\]

with \( f \) the transpose of \( m \), and if a composite \( h \circ g \) is conservative, then so is \( g \).

To apply the last remark, we let \( A, B \) be \( \text{biH}_0 \) and \( \text{H}_0 \), resp. The forgetful functor \( G : \text{biH}_0 \rightarrow \text{H}_0 \) has a left adjoint \( F \); for a Heyting algebra, \( F(H) \) is the free biHeyting extension of \( H \) via the Heyting map \( \eta_H : H \rightarrow GF(H) \). The posets of the form \( 2^p \) are not only Heyting algebras, but they are biHeyting as well: \( G(2^p) = 2^p \); (2.3) says that \((*)\) holds with an appropriate \( A \) of the form \( 2^p \). With writing \( \mathcal{F}_{\text{biH}} \) for \( F \) in this particular case, we conclude that

\[ \text{the canonical map } \eta_H : H \rightarrow \mathcal{F}_{\text{biH}}(H) \text{ of a Heyting algebra into its free biHeyting extension is conservative.} \]

\((2.4')\)

Assertion (2.4) is a consequence; if \( H \) is the free Heyting algebra on a set \( L \) generators ("propositional letters"), then \( \mathcal{F}_{\text{biH}}(H) \) is the free biHeyting algebra on \( L \); the Lindenbaum–Tarski (L · T) algebra of the \( L \)-formulas of IPL is \( H \), and the L · T algebra of \( L \)-formulas of biIPL is \( \mathcal{F}_{\text{biH}}(H) \); the conservativeness in (2.4) is equivalent to saying that this particular \( \eta_H : H \rightarrow \mathcal{F}_{\text{biH}}(H) \) is a conservative morphism.

Since the just-described phenomenon is a recurring theme in this paper, we introduce some terminology. We say that \( A \) is a conservative enrichment of \( B \) (along \( G : A \rightarrow B \); we usually omit mentioning \( G \) since it will be an "obvious" forgetful functor) if \((***)\) holds. Let \( \mathcal{S} \) be a set of objects in \( B \). We say \( \mathcal{S} \) is (or, the objects in \( \mathcal{S} \) are) representative in \( B \) if for any \( B \in B \) there are \( S_i \in \mathcal{S} \) (\( i \in I \)) and a conservative map \( B \rightarrow \prod_{i \in I} S_i \). (The Stone representation theorem is easily seen to be equivalent to saying that \( \{2\} \) is representative in \( \text{Dl} \).) The implication \((*) \Rightarrow (***) \) proved above can now be put in the following form:

\[ \text{if the objects in the image of } G \text{ are representative in } B, \text{ then } A \text{ is a conservative enrichment of } B. \]

\((2.5)\)

A conditionally biHeyting map is a d.l. map preserving all existing implications and coimplications in the domain lattice. We have

\[ e_A : A \rightarrow A^{**} \text{ is conditionally biHeyting.} \]

\((2.6)\)
In fact, as Michael Barr has pointed out to us, this is a consequence of the previous result (2.2). The point is that $2^o \cong 2$ (for any category, in particular preorder $C$, $C^o$ is the opposite of $C$). Because of this, in (2.1), $2^o$ may be used in place of 2, and we get that the evaluation

$$e' : A^o \rightarrow (2^o)^{\text{hom}(A^o, 2^o)}$$

is conditionally Heyting. But in general, for posets $I$ and $P$, $(I^P)^o = (I^o)^{P^o}$, and a function between the underlying sets of the d.l.'s $C$ and $D$ is a d.l. map $C \rightarrow D$ if it is a d.l. map $C^o \rightarrow D^o$. Thus, $\text{hom}(A^o, 2^o) = (\text{hom}(A, 2))^o$ and $(2^o)^{\text{hom}(A^o, 2^o)} = (2^{\text{hom}(A, 2)})^o$, thus, $e'$ is a conditionally Heyting map

$$e' : A^o \rightarrow (2^{\text{hom}(A, 2)})^o.$$ 

Also, $e'$ is the same function on underlying sets as $e_A$. Thus, $e_A$ as a map $A^o \rightarrow (A^{**})^o$ is conditionally Heyting; as a consequence, $e_A : A \rightarrow A^{**}$ is conditionally coHeyting.

In particular,

the lattices of the form $2^P$ are representative in $\text{biHo}.$

(2.6')

Let us investigate the posets of the form $2^P$ ($P$ any poset) more closely; they will occur frequently in the sequel. $2^P$ is the same thing as $\mathcal{P}_1(P)$, the set of upward-closed subsets of $P$, ordered by set-theoretic inclusion. $\mathcal{P}_1(P)$ is a complete lattice, in fact, a frame (the law $x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} x \wedge y_i$ holds, this is equivalent to saying that we have a complete Heyting algebra). Since $(2^P)^o = (2^o)^{P^o} \cong 2^{(P^o)}$, $2^P$ is also a coframe.

A prime element, or simply a prime, in a complete lattice $L$ is any $p \in L$ such that for any $\{x_i : i \in I\} \subseteq L$, $p \leq \bigvee_{i \in I} x_i$ implies $p \leq x_i$ for some $i \in I$. $\text{Pr}(L)$ denotes the set of primes of $L$. One observes that in $\mathcal{P}_1(P)$, the elements of the form $\uparrow(x) = \{y \in P : y \geq x\}$, $x \in P$, are primes; and in fact, these are all the primes.

We say that $L$ is a prime-generated lattice if $L$ is a complete lattice and every element $x$ of $P$ is a sup of primes: $x = \bigvee_{i \in I} p_i$, $p_i \in \text{Pr}(L)$ (equivalently, $x = \bigvee \{p \in \text{Pr}(L) : p \leq x\}$). Clearly, $\mathcal{P}_1(P)$ is prime-generated: any $X \in \mathcal{P}_1(P)$ is $X = \bigvee \{\uparrow(x) : x \in X\}$. In fact, we can see that any prime-generated lattice $L$ is isomorphic to one of the form $\mathcal{P}_1(P)$: take $P = (\text{Pr}(L))^o$, and note that the map

$$\phi : \mathcal{P}_1(P) \rightarrow L, \quad X \mapsto \bigvee X$$

is an isomorphism. Indeed, $\phi$ is order-preserving; for $X, Y \in \text{Pr}(L)$, $\bigvee X \subseteq \bigvee Y \Rightarrow X \subseteq Y$ follows from the definition of “prime”; thus, $\phi$ is order-reflecting and consequently 1–1; finally, $\phi$ is surjective because $L$ is prime-generated. As a consequence, prime-generated lattices are frames, coframes, and the opposite of a prime-generated lattice is again one.

$\text{Prg}$, the category of prime-generated lattices, has objects those named, and morphisms that preserve all (not necessarily finite) sup's and all inf's.
For any d.l. $A$, $A^{**} \overset{\text{def}}{=} 2^A$ is prime-generated, and $e_A : A \rightarrow A^{**}$ is a lattice embedding. Does $e_A$ have a universal property among lattice homomorphisms of $A$ into prime-generated lattices? The answer is “yes”;

*given any $h : A \rightarrow L$, a lattice map into a prime-generated lattice, there is a unique map $\ell : A^{**} \rightarrow L$ of complete lattices (preserving all $\lor$ and $\land$) such that*

\[
\begin{array}{c}
A \\
\downarrow \circ \\
L
\end{array}
\xrightarrow{e_A}
\begin{array}{c}
A^{**}
\end{array}
\xrightarrow{\ell}
\begin{array}{c}
h
\end{array}
\]

*commutes.* \hfill (2.7)

Because of (2.7), we call $A^{**}$ the prime-generated hull of $A$. Before proving (2.7), let us list some properties, either obvious or already seen above, of the map $e_A : A \rightarrow A^{**}$, which we now abbreviate as $\cdot : A \rightarrow \hat{A}$:

(i) $\cdot : A \rightarrow \hat{A}$ is a lattice homomorphism, $\hat{A}$ is prime-generated;

(ii) for every prime element $p$ of $\hat{A}$, $p = \bigwedge \{ \hat{x} : x \in A, p \leq \hat{x} \}$;

(as a consequence,

(ii') every element $x$ of $\hat{A}$ can be written in the form $x = \bigvee_{i \in I} \bigwedge_{j \in J} \hat{x}_{ij}$ with suitable elements $\hat{x}_{ij} \in \hat{A}$);

(iii) if $\bigwedge_{i \in I} \hat{x}_i \leq \hat{x}$ ($x_i, x \in A$), then $\bigwedge_{i \in I'} x_i \leq x$ for some finite subset $I'$ of $I$;

(as a consequence,

(iii') $\cdot$ is conservative);

(iv) if $p \in \hat{A}$ has the property that $p \leq \bigvee_{i \in I} \hat{x}_i$, $I$ finite imply $p \leq \hat{x}_i$ for some $i \in I$ (“$p$ is prime with respect to $A^{**}$”), then $p$ is prime in $\hat{A}$. \hfill (2.8)

We will show that, assuming (2.8)(i)-(iv), $\cdot : A \rightarrow \hat{A}$ has the universal property claimed of $e_A$ in (2.7); it will follow that, for given $A$, (2.8) characterizes $e_A$ up to isomorphism.

Because of (2.8)(ii), for $p$ a prime element of $\hat{A}$, $\ell(p)$ has to be defined as

\[
\ell(p) = \bigwedge \{ \hat{x} : x \in A, p \leq \hat{x} \}
\]

and in general, for $X \in \hat{A}$,

\[
\ell(X) = \bigvee_{p \leq X} \bigwedge \{ \hat{x} : p \leq \hat{x} \};
\]

here and below, $p$ ranges over the primes of $\hat{A}$. The general formula clearly specializes to the special one for $p = X$; as a consequence,

\[
\ell(X) = \bigvee_{p \leq X} \ell(p).
\]

Since $\{ p : p \leq \bigvee_{i \in I} X_i \} = \bigcup_{i \in I} \{ p : p \leq X_i \}$, $\ell$ preserves $\lor$. Define, for $q \in \text{Pr}(L)$,

\[
t(q) \overset{\text{def}}{=} \bigwedge_{x \leq \hat{x}} \hat{x};
\]

$x$ ranges over $A$. Using 2.8(iii) and the fact that $h$ preserves $\land$,
we see that
\[ t(q) \leq x \iff q \leq hx. \tag{2.9} \]
t(q) is prime with respect to \( \mathbb{A} \): \( t(q) \leq \bigvee_{i \in I} \mathbf{x}_i \Rightarrow q \leq h(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} hx_i \Rightarrow q \leq hx_i \) for some \( i \in I \Rightarrow t(q) \leq \mathbf{x}_i \) for some \( i \in I \). By (2.8) (iv), \( t(q) \) is a prime in \( \bar{\mathbb{A}} \).

Since
\[ q \leq \ell(p) \iff \text{for all } x \in \mathbb{A}, \ p \leq \mathbf{x} \Rightarrow q \leq hx \]
and
\[ t(q) \leq p \iff \text{for all } x \in \mathbb{A}, \ p \leq \mathbf{x} \Rightarrow t(q) \leq \mathbf{x}, \]
by (2.9), we have \( q \leq \ell(p) \iff t(q) \leq p \), from which, using the formula for \( \ell(X) \) and the fact that both \( q \) and \( t(q) \) are primes, we get \( q \leq \ell(X) \iff t(q) \leq X \). Extending \( t \) to an arbitrary element \( u \in \mathbb{L} \) by \( t(u) = \bigvee \{ t(q) : q \in \text{Pr}(\mathbb{L}), u \leq q \} \), we get that \( t \) is the left adjoint of \( \ell \). It follows that \( \ell \) preserves \( \bigwedge \).

Now, consider
\[ \ell(\mathbf{x}) = \bigvee_{p \leq \mathbf{x}} \ell(p) = \bigvee_{p \leq \mathbf{x}} \bigwedge_{i \leq y} h(y), \tag{2.10} \]
to show that \( \ell(\mathbf{x}) = h(x) \). \( \ell(\mathbf{x}) \leq h(x) \) is clear. To prove \( h(x) \leq \ell(\mathbf{x}) \), it suffices to show \( q \leq h(x) \Rightarrow q \leq \ell(\mathbf{x}) \) for all \( q \in \text{Pr}(\mathbb{L}) \), since \( \mathbb{L} \) is prime-generated. Assume \( q \in \text{Pr}(\mathbb{L}), q \leq h(x) \). Let \( p = t(q) \in \text{Pr}(\bar{\mathbb{A}}) \). By the definition of \( t(q), p \leq \mathbf{x} \). Eq. (2.9) says that \( q \leq \bigwedge_{i \leq y} h(y) \), and (2.10) says that \( q \leq \ell(\mathbf{x}) \). This completes the proof of (2.7).

Ghilardi showed us an unpublished proof, due to him and Meloni, of a stronger form of (2.7), which asserts the universal property of \( \mathbb{A}^{**} \) not just among prime-generated lattices, but more generally, among completely distributive ones. This version also follows from (2.7) when combined with Raney’s theorem [33] according to which every completely distributive lattice is a complete homomorphic image of a prime-generated one. Proposition (2.7) is presented here as motivation to the generalization to predicate logic, the topos of types, due to the first author [26]. The topos of types will be quoted and used in Section 8.

The operation \( (\_)^{**} \) on d.l.’s is a functor \( (\_)^{**} : \mathcal{D} \to \text{Prg} \); \( e_\_^{**} : \text{Id}_\mathcal{D} \to (\_)^{**} \) is a natural transformation; \( e_\_^{**} : \text{Id}_\mathcal{D} \to (\_)^{**} \); in fact, these facts are seen directly, without (2.7). Complementing fact (2.6) is the following:

For a Heyting morphism \( (h : H \to D) \in H_0, h^{**} : H^{**} \to D^{**} \) is Heyting as well \( \tag{2.11} \)

there is no version of this with “conditionally Heyting”). To see this, first we observe that

for a map \( \varphi : Q \to P \) of posets, a sufficient condition for \( \varphi^* : 2^P \to 2^Q \) to be a Heyting morphism is for \( \varphi \) to be upward surjective: for any \( q \in Q \) and \( p' \in p \) with \( p' \geq \varphi q \), there is \( q' \in Q \) with \( q' \geq q \) and \( \varphi q' = p' \). \( \tag{2.12} \)
The proof is easy on the basis of the formula given above for implications in $2^p$; it was also given in [11]. Fact (2.11) follows from (2.12) and

For a Heyting morphism $(h: H \to D) \in \mathcal{H}_0$, $h^*: D^* \to H^*$ is upward surjective.

Assume $p' \geq h^* q$. We want $q' \in \Pr(D)$ such that

\[ q \subset q' \]

and $h^* q' = p'$, which latter means

\[ x \in p' \Rightarrow hx \in q' \]

and

\[ z \notin p' \Rightarrow hz \notin q'. \]

Let $F = \{ y \in D. \exists v \in q. \exists x \in p'. v \land hx \leq y \}$, the filter generated by $q \cup h[p']$; note that (2.14) and (2.15) together mean $F \subset q'$. For any particular $z \in H$ with $z \notin p'$, we have $hz \notin F$; otherwise, $v \land hx \leq hz, v \in q, x \in p'$; it follows that $v \leq hx \to hz = h(x \to z)$ since $h$ is Heyting; hence, $h(x \to z) \in q$, and since $p' \geq h^* q, x \to z \in p'$, which implies by $x \in p'$ that $z \in p'$, contradiction. Thus, the set $I = \{ u: \exists z \notin p'. u \leq hz \}$ is disjoint from $F$. By using that $p'$ is a prime filter, $I$ is seen to be an ideal; we have the filter $F$ and the ideal $I$ on $D$ which are disjoint. The prime filter existence theorem thus gives $q' \in \Pr(H)$ with $F \subset q'$ and $q' \cap I = \emptyset$, as desired.

Any morphism $\psi: K \to L$ of complete lattices (preserving all sups and infs) has both a left and a right adjoint $\circ \psi \square$:

\[ K \xleftarrow{\circ} K \xrightarrow{\square} L. \]

In particular, with $\phi: Q \to P \in \text{Poset}$, the right and left adjoints of $\phi^*: 2^p \to 2^Q$ are computed according to the following formulas:

\[ p \in \square Y \iff \forall q \in Q \phi q \geq p \Rightarrow q \in Y, \]
\[ p \in \circ Y \iff \exists q \in Q \phi q \leq p \land q \in Y \quad (Y \in 2^Q, p \in P). \]

Now, with a morphism $\phi: H \to D$, on an element $y \in D$, the left adjoint $\circ$ of $\phi$ or may not be defined; to say that it is defined is to say that there is a (necessarily unique) element $\circ y \in H$ with

for all $x \in H \quad \circ y \leq x \iff y \leq x$.

Similarly, for the right adjoint $\square$ of $\phi$:

for all $x \in H \quad x \leq \square y \iff x \leq y$. 
Entities like \((\cdot : H \to D)\) form the category \(\text{Di}^-\); an arrow in this category is a pair \((\eta, \delta)\) forming a commutative square

\[
\begin{array}{ccc}
D & \xrightarrow{\delta} & D' \\
\downarrow & \searrow & \downarrow \\
H & \xrightarrow{\eta} & H'.
\end{array}
\]

Saying that \((\eta, \delta)\) is conditionally \(\Box\)-preserving has the obvious meaning: whenever \(y \in D, \Box y \in H\) exists, \(\Box (\delta y)\) also exists and it is equal to \(\eta(\Box y)\). Writing \(\Box : D \to H\) for the partial right adjoint of \(\cdot \), which is the partial function on \(D\) defined as explained above, this is expressed pictorially as the commutativity of

\[
\begin{array}{ccc}
D & \xrightarrow{\delta} & D' \\
\downarrow & \searrow & \downarrow \\
H & \xrightarrow{\eta} & H'.
\end{array}
\]

and of course, when those adjoints are totally defined, we get an ordinary commutative diagram. A further property of the functor \((\cdot)^* : \text{Di} \to \text{Pr}g\) is given in the next fact.

*For any \((\cdot : H \to D) \in \text{Di}^-\), the map \((e_H, e_D) : (\cdot : H \to D) \to (\star : H^* \to D^*)\) is conditionally \(\Box\)- and \(\circ\)-preserving.*

(2.18)

Suppose \(y \in D\) and \(\Box y \in H\) exists. Looking at the definitions of \(e_H, e_D,\) and \(\Box : D^{**} \to H^{**}\), we see that the required equality \(e_H(\Box y) = \Box e_D(y)\) is equivalent to the relation

\[
\Box y \in p \iff \forall q \in \text{Pr}(D) \quad \star(q) \geq p \Rightarrow y \in q.
\]

Here, the left-to-right implication is automatic. To show the other direction, assume \(\Box y \notin p\); we show the existence of \(q \in \text{Pr}(D)\) with \(\star(q) \geq p\) and \(y \notin q\). The first of the two requirements is the same as \(\cdot \ [p] \subseteq q\). Let \(F \subseteq D\) be the filter generated by \([p]\); \(F = \uparrow([p]) = \{z \in D : \exists x \in p, z \geq x\}\). We claim that \(y \notin F\). Otherwise, we have \(x \in p, y \geq x\); hence, \(\Box y \geq x, \Box y \in p\), contradiction. By the prime filter existence theorem, there is \(q\) as required.

By using an argument similar to the one above for (2.6) (involving \(2^a \cong 2\)), the assertion concerning the preservation of \(\circ\) is a consequence of the one for \(\Box\).

Let us call a structure of the form \(\mathfrak{H} = (h : H \to D)\), with \(h\) being an arrow in \(\text{Di}\), a *pre-S4 algebra* if \(h\) has a right adjoint \(\Box\). An arrow of pre-S4 algebras is an arrow in \(\text{Di}^-\) that preserves \(\Box\) in the sense explained above. A *pre-biS4 algebra* has, in addition, a left adjoint to the structure map \(h\); a morphism of pre-biS4 algebras preserves both \(\Box, \circ\). We have the categories \(\text{pre-S4}_0\) and \(\text{pre-biS4}_0\); they are (non-full) subcategories of \(\text{Di}^-\). A map in either of these categories is *conservative* if both of its components are conservative. With any \(\varphi : Q \to P \in \text{Poset}\), \((\varphi^* : P^* \to Q^*)\) is a pre-biS4 algebra. Facts (2.18) and (2.1) imply that the latter type of pre-(bi)S4 algebras are representative
in both pre-S₄₀ and pre-biS₄₀. It follows that pre-biS₄₀ is a conservative enrichment of pre-S₄₀.

Assume now of the pre-S₄ algebra \( \mathcal{A} = (h: H \to D) \) that \( D \) is a Heyting algebra, and that \( h \) is conservative. Then, we claim, \( H \) is also a Heyting algebra. In fact, for \( x, y \in H \), \( x \to y = \Box(\Box x \to \Box y) \), as the following sequence of equivalences show:

\[
\begin{align*}
& u \leq \Box(hx \to hy) \\
& hu \leq hx \to hy \\
& hx \land hu \leq hy \\
& x \land u \leq y,
\end{align*}
\]

where the last equivalence uses that \( h \) preserves \( \land \) and that it is conservative. It easily follows that if \((\eta, \delta): \mathcal{A} \to \mathcal{A}'\) is a morphism of pre-S₄ algebras, both \( \mathcal{A}, \mathcal{A}' \) satisfy the additional hypotheses, then \( \eta \) is a morphism of Heyting algebras provided \( \delta \) is. Note that in this situation both \( H \) and \( D \) are Heyting algebras but \( h \) is not necessarily a Heyting morphism.

There is a similar conclusion about coimplication being induced from \( D \) to \( H \) via a left adjoint \( \Box \). Thus, if \( h \) is conservative, \( D \) is a biHeyting algebra, and \( \mathcal{A} \) is a pre-biS₄ algebra, then \( H \) is a biHeyting algebra as well.

We are mainly interested in the case when, in addition, \( D \) is a Boolean algebra. A (bi)S₄ algebra is a pre-(bi)S₄ algebra \((h: H \to D)\) in which \( h \) is conservative and \( D \) is a Boolean algebra. Morphisms of (bi)S₄ algebras are those of pre-(bi)S₄ ones. We have the corresponding categories S₄₀ and biS₄₀. Thus, we have that if \((h: H \to D)\) is a (bi)S₄ algebra, \( H \) is a (bi)Heyting algebra; in fact, we have the forgetful functors S₄₀ \to H₀, biS₄₀ \to biH₀.

The "standard" (for our purposes) (bi)S₄ algebras are the ones of the form \( 2^P \to 2^{|P|} \), that is, \( \mathcal{P}_1(P) \to \mathcal{P}(|P|) \), the inclusion of the set of upward-closed subsets of \( P \) into the set of all subsets of \(|P|\); here, \( P \) is any preorder. More generally, for any \( \varphi: Q \to P \in \text{Preord} \), if \( Q \) is discrete and \( \varphi \) is surjective, then \( \varphi^*: 2^P \to 2^Q \) is a biS₄ algebra.

For a topological space \( X, \mathcal{C}(X) \to \mathcal{P}(|X|) \) is an S₄ algebra, but not necessarily a biS₄ one.

The notion of S₄ algebra corresponds to S₄ modal propositional logic (see [12, 23]). First of all, with \((h: H \to D) \in S₄\) we may assume, without loss of generality, that \( h \) is an inclusion. In that case, \( \Box: D \to H \) may be considered as a map \( \Box: D \to D \). Furthermore, we see that \( \Box: D \to D \) satisfies the following:

\[
\Box 1 = 1, \quad \Box (x \land y) = \Box x \land \Box y, \quad \Box (x \to y) \leq \Box x \to \Box y, \quad \Box x \leq x, \quad \Box \Box x = \Box x.
\]

(2.19)

Conversely, if \( D \) is a Boolean algebra with \( \Box: D \to D \) satisfying the listed identities, then \( H = \Box D = \{ \Box x: x \in D \} \), with the ordering inherited from \( D \), is a distributive lattice, \( \Box \) is a map of \( D \) into \( H \), and as such it is right adjoint to the inclusion \( h: H \to D \). In other words, S₄ algebras correspond essentially in a one-to-one manner to algebras of the form \((D, \Box)\) with \( D \) a Boolean algebra, and the unary operation \( \Box \) satisfying
On the other hand, S4 modal propositional logic is the extension of classical propositional logic with the unary connective $\Box$ obeying rules corresponding to the identities (2.19); see [12]. In other words, S4 algebras are an algebraic formulation of S4 modal propositional logic.

The coS4 operation $\circ$ corresponds to "closure", just as $\Box$ corresponds to "interior". In modal, or tense, logic, it means "past possibility", see e.g. [32]. The four propositional doctrines $H_0$, $bH_0$, $S4_0$, $bS4_0$ form the following square where all functors are forgetful:

$$
\begin{array}{c}
bS4_0 \longrightarrow S4_0 \\
\downarrow \\
bH_0 \longrightarrow H_0.
\end{array}
$$

(2.20)

We have mentioned that the lower horizontal is a conservative enrichment. Using the same argument (formalized in (2.5)), and using (2.3), (2.6'), we obtain that the two verticals and the diagonal are also conservative enrichments. Just notice that the representation theorem in (2.3) for Heyting algebras gives as representatives algebras the $2^P$, which are each the "H-" part of a $bS4$ algebra, namely $2^P \to 2^{1P}$.

The fact that $S4_0$ is a conservative enrichment of $H_0$ is the well-known fact that intuitionistic propositional logic has a faithful translation, the so-called Gödel interpretation, in classical propositional logic enriched to S4 modal logic. An algebraic treatment of this fact can be found in [23, 24]. The usual account (see [38, Section 12, p. 35], also for predicate logic to be discussed in the next section) runs as follows. Let $L$ be a fixed language for propositional logic (set of propositional letters). Define the mapping $\theta \mapsto \hat{\theta}$ of $L$-formulas of IPL to $L$-formulas of S4PL, S4 propositional logic (add $\Box$ as a new unary connective to the usual ones of CPL, classical propositional logic) as follows:

$$
\begin{align*}
\hat{P} &= \Box P \\
\hat{t} &= t \\
(\phi \land \psi) &= \hat{\phi} \land \hat{\psi} \\
(\phi \lor \psi) &= \hat{\phi} \lor \hat{\psi} \\
(\phi \to \psi) &= \Box(\hat{\phi} \to \hat{\psi}).
\end{align*}
$$

Then, for any $\theta \in IPL$,

$$
\vdash_{IPL} \theta \iff \vdash_{S4PL} \hat{\theta}.
$$

(2.21)

To deduce this conclusion, let $H$ be the Lindenbaum–Tarski algebra of $L$-formulas of IPL, that is, the free Heyting algebra on the set $L$ of generators; $(D, \Box)$ the Lindenbaum–Tarski algebra of $L$-formulas of S4PL, that is, the free S4 algebra on $L$. Then $C = \Box D$ is a Heyting algebra; let $\eta: H \to C$ be the morphism for which $\eta(P) = \Box P$. 

Clearly, \( \eta([\theta]) = [\hat{\theta}] \); here the two brackets are equivalence classes in the respective Lindenbaum–Tarski algebras. The left-to-right direction in (2.21) expresses that \( C \) is a Heyting algebra. The other direction says that \( \eta \) is conservative.

Let \( \hat{H} \rightharpoonup \hat{D} \) be \( \mathcal{F}_{S4}(H) \), \( \eta_H : H \to \hat{H} \) the unit map of adjunction \( \mathcal{F}_{S4}^*: G \). We know that \( \eta_H \) is conservative. Now, \( \eta \) is not quite the same as \( \eta_H \), but the difference is slight. In particular, \( \hat{H} \rightharpoonup \hat{D} \) can be described as follows. \( (D, \Box) \) is the free \( S4 \) algebra on the generators \( \hat{P} \), one for each \( P \in L \), subject to the condition \( \Box \hat{P} = \hat{P} \); \( \hat{H} = \sqcup \hat{D} \); \( \eta_H \) takes \( P \) to \( \hat{P} \) (\( P \in L \)). However, we have the \( S4 \) map \( (\eta', \delta) : D \to \hat{D} \) taking \( P \) to \( \hat{P} \) (\( P \in L \)) because \( (D, \Box) \) is free:

\[
\begin{array}{ccc}
D & \xrightarrow{\delta} & \hat{D} \\
\downarrow \uparrow \eta & & \uparrow \eta' \downarrow \\
H & \rightarrow & \hat{H}.
\end{array}
\]

The composite \( \eta' \eta \) maps \( P \) to \( \eta'(\eta(P)) = \eta'(\Box P) = \Box \delta(P) = \Box \hat{P} = \hat{P} \); it follows that \( \eta' \eta = \eta_H \) (since \( H \) is free on \( L \)). Since \( \eta_H \) is conservative, so is \( \eta \).

The remaining functor in (2.20), the upper horizontal, is seen to be a conservative enrichment by the following proposition, which is a form of Kripke's completeness theorem for \( S4 \) modal logic.

\[ \text{The algebras of the form } 2^P \to 2^{[P]}, \text{ } P \text{ a preorder, are representative in } S4_0 \text{ as well as bi} S4_0. \] (2.22)

**Proof.** First, a general construction. Given any functor \( F : A \to B \) (in the immediate application, poset-map), we have the bijective-on-objects/full-and-faithful factorization

\[
\begin{array}{ccc}
A & \xrightarrow{F} & X \\
\downarrow \uparrow F_{(1)} & & \uparrow F_{(2)} \\
A]X & \xrightarrow{F_{(1)}} & B \\
\end{array}
\]

where \( A]X \) is the category whose objects are those of \( A \), and an arrow \( A \to B \) in \( A]X \) is an arrow \( F(A) \to F(B) \) in \( X \); the functors \( F_{(1)}, F_{(2)} \) are the obvious ones, and they are bijective on objects, and full and faithful, respectively. Now, let \( (h : F \to D) \in S4_0 \), and consider the following diagram:

\[
\begin{array}{ccc}
D & \xrightarrow{\Box} & 2^D^* \\
\downarrow \Box & & \downarrow \Box \\
H & \rightarrow & \hat{H}^* \\
\end{array}
\]

\[
\begin{array}{ccc}
D & \xrightarrow{\Box} & 2^D^* \\
\downarrow \Box & & \downarrow \Box \\
H & \rightarrow & j^* \hat{H}^* \uparrow \Box \\
\end{array}
\]

\[
\begin{array}{ccc}
D & \xrightarrow{\Box} & 2^D^* \\
\downarrow \Box & & \downarrow \Box \\
H & \rightarrow & j^* \hat{H}^* \uparrow \Box \\
\end{array}
\]

Without the right adjoints, all denoted \( \Box \), the diagram commutes; also, the quadrilateral containing the left and the middle \( \Box \) commutes by (2.18). Since \( D \) is a Boolean
algebra, $D^*$ is discrete; for $L = D^* [H^*$, and $t_L : |L| \rightarrow L$ the obvious map,

$$
\begin{array}{c}
2^{[D^*]} \\
\downarrow k^* \\
2^{[H^*]} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qa
The notion of bicategory [3, 39] slightly generalizes that of 2-category; the basic concepts concerning 2-categories are treated in [25]. In a bicategory, the hom-set hom(A, B) of all arrows from A to B is in fact a category; there are arrows, 2-cells \( f \to g \) between arrows \( f, g \in \text{hom}(A, B) \). The typical bicategory is \( \text{CAT} \), the bicategory of all categories; here the arrows are functors, and the 2-cells are natural transformations. As a matter of fact, all bicategories considered in this paper are 2-categories. The reason that we call them bicategories is one that has no visible effect here, and it is that the operations in the doctrines are bicategorical and not 2-categorical: in the 2-category of coherent categories 2-categorical limits do not exist; the bicategorical ones do, and are vital, although not in this paper.

The process of lifting the contents of Section 2 into the context of this section is not a mechanical one. The reason for this is that it is not obvious what structure on categories one should consider. For example, the notion of coherent category (c.c.) generalizes that of distributive lattice (every d.l. is a c.c.); still, of course, this does not specify what c.c.'s ought to be. On the one hand, some straightforward generalizations of poset structures are unmanageable. For example, the right adjoint of \( A \times (\_): C \to C \), which is the straight generalization of implication, is exponentiation (of sets in \( \text{Set} \)), and belongs to higher-order logic, not (or at least not obviously) exhibiting "completeness" phenomena, which we are concerned with in this paper. On the other hand, there are new operations, especially the quantifiers, that have no counterpart in posets (that is, they trivialize when specialized to posets).

In [29] it is explained in detail how certain categorical operations correspond to logical operations. The basic ideas of categorical logic are due to Lawvere. Joyal played an important role in the development of the concepts of first-order categorical logic; specific results of Joyal will be mentioned below. The overall contribution of [29], beyond specific results (some of which will be used in this paper), is the setting up of a precise two-way correspondence between the categorical and the symbolic formulations of logic. This correspondence enables one to use in the categorical context results obtained in the symbolic context and vice versa. This ability will be exploited in this paper. On the one hand, we will use methods of model theory, such as compactness, and saturated and special models. By going the opposite way, our results, in the first instance formulated categorically, give new results of symbolic logic. We will emphasize those results obtained in this way that are statements about pure logic, that is, about the syntax of logic, and which, accordingly, have, via Gödel numbering, formulations (in fact, \( \Pi^0_2 \)-forms) in arithmetic.

A category \( A \) is coherent ("logical" in [29]), if

(i) \( A \) has finite limits;

(ii) for all \( X \in A \), \( S(X) \), the poset of all subobjects of \( X \) is (not only a meet semilattice, by (i), but also) a lattice: it has finite sups;

(iii) for any \( f: X \to Y \) in \( A \), \( f^*: S(Y) \to S(X) \) (defined by pulling back along \( f \)) is a lattice homomorphism;
(iv) for any \( f: X \to Y \) in \( A \), \( f^*: S(Y) \to S(X) \) has a left adjoint

\[
\exists_f: S(X) \to S(Y): A \leq_X f^*B \iff \exists_f A \leq_Y B \quad (A \in S(X), \ B \in S(Y)).
\]

(v) for any \( f \) as in (iv), and any pullback diagram as on the left, the right commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow a & & \downarrow b \\
X' & \xrightarrow{f'} & Y'
\end{array}
\quad \begin{array}{cc}
S(X) & \xrightarrow{3_f} & S(Y) \\
\downarrow a^* & & \downarrow b^* \\
S(X') & \xrightarrow{3_{f'}} & S(Y')
\end{array}
\]

Each \( S(Y) \) is in fact a distributive lattice, as a consequence of (iii) applied to monomorphisms \( X \to Y \). (v) is the so-called Beck–Chevalley condition for \( \exists \); it expresses that the operation \( \exists \) commutes with substitution (see also below). The standard coherent category is \( \text{Set} \).

A morphism of coherent categories, that is, a coherent functor, is a functor between coherent categories that preserves the operations defining coherent categories: finite limits, finite sups of subobjects, and the operations \( \exists_f \), for all \( f \) in the domain category. It is easy to guess what “preserves” should mean in each case here; see also [29].

\( \text{Coh} \) denotes the bicategory of coherent categories; in defining \( \text{Coh} \) we take a Grothendieck universe \( \mathcal{U} \) of which \( \text{Set} \) is an element; the objects of \( \text{Coh} \) are those coherent categories that belong to \( \mathcal{U} \). Arrows of \( \text{Coh} \) are the coherent functors, and the 2-cells all the natural transformations between the latter.

A morphism in \( \text{Coh} \) is conservative if it reflects isomorphisms: if an arrow is taken to an isomorphism, then it is an isomorphism itself. This is the same as to say that the induced morphisms on subobject lattices are all conservative. The notion of “conservative” will be defined in the same way in all other doctrines below. It is important to know that a conservative morphism, in any doctrine, reflects all the operations of the doctrine. For example, if a diagram is taken to a product diagram by a conservative morphism, then the diagram is a product itself.

Let \( A \in \text{Coh} \). A morphism \( A \to \text{Set} \) is called a model of \( A \). Note that any functor \( M: A \to \text{Set} \) is in particular an \( L_A \)-structure, where \( L_A \) is the many-sorted similarity type whose sorts are the objects of \( A \), and whose operation symbols are the arrows of \( A \); each operation symbol is unary, and it is sorted by its domain (sort of argument) and codomain (sort of value). It turns out that there is a canonically defined set \( \Sigma_A \) of \( L_A \)-sentences whose \( L_A \)-models are exactly the coherent functors \( A \to \text{Set} \); the natural transformations between coherent functors \( A \to \text{Set} \) are the homomorphism (in the algebraic sense) of models. Thus, the hom-category \( \text{hom}_{\text{Coh}}(A, \text{Set}) \) is the same as the category of models of the theory \( T_A = (L_A, \Sigma_A) \), the so-called internal (coherent) theory of \( A \); we write \( \text{Mod}(A) \) for \( \text{hom}_{\text{Coh}}(A, \text{Set}) \). Furthermore, each axiom in \( \Sigma_A \) is of the form \( \forall \vec{x}(\varphi \to \psi) \), where \( \varphi \) and \( \psi \) are coherent, that is, positive existential, formulas: built up from atomic formulas, also using equality (a separate equality symbol for each sort in \( L_A \), with both places sorted with the given sort), using finite conjunction, finite disjunction and existential quantification. We write \( \varphi \models^e \psi \) instead of \( \forall \vec{x}(\varphi \to \psi) \); we call it a coherent axiom. A coherent theory is one whose axioms are all coherent.
For the above, see especially Section 3.2 of [29].

Every coherent formula over $\mathcal{L}_A$ has an interpretation in $\mathcal{A}$. Let $\varphi$ be a coherent $\mathcal{L}_A$-formula whose free variables are among the ones in $\bar{x}$, a tuple of distinct (sorted) variables. With $\bar{x} = \langle x_i \rangle_{i < n}$, let $[\bar{x}] = \prod_{i < n} X_i$ where $x_i : X_i$ ($x_i$ is of sort $X_i$). Then $[\bar{x} : \varphi]$, the interpretation of $\varphi$ in the context $\bar{x}$, is a subobject of $[\bar{x}]$. For the definition of $[\bar{x} : \varphi]$, see especially Section 2.4 in [29]; $[\bar{x} : \varphi]$ is written there as $[\varphi]_\bar{x}$. We say the coherent axiom $\varphi \vDash \psi$ is true in $\mathcal{A}$, $\mathcal{A} \vDash \varphi \Rightarrow \psi$, if $[\bar{x} : \varphi] \leq_{\bar{x}} [\bar{x} : \psi]$ (we write $\leq_{\bar{x}}$ for the ordering on the subobjects of $X$). Returning to $\mathcal{T}_A$, $\Sigma_A$ may be defined as the set of all coherent $\mathcal{L}_A$-axioms that are true in $\mathcal{A}$.

The internal meaning $[\bar{x} : \varphi]$ of $\varphi$ is related to ordinary semantics by the formula

$$M([\bar{x} : \varphi]) = \{ d \in M(X) : M \vdash \varphi[\bar{d}/\bar{x}] \}. \quad (3.1)$$

Here, $M$ is any model of $\mathcal{A}$; on the left, we have the subset of $M([\bar{x}])$ that is the value of the coherent functor $M$ at the subobject $[\bar{x} : \varphi]$; on the right, we have the set (subset of $M([\bar{x}])$) of tuples that satisfy $\varphi$ in $M$ as an $\mathcal{L}_A$-structure in the Tarskian sense. (For (3.1), in fact in a more general form, with “models” in arbitrary categories rather than just $\text{Set}$, is 3.5.1 in [29].)

For any subobject $A \in S(X)$, $X \in \mathcal{A}$, there is a (simply defined) formula $A(x)$ (with $x : X$) such that $[x : A(x)] = A$. More generally, if $X$ is a product $\prod_{i < n} X_i$, and $A \in S(X)$, we have a formula $A(\bar{x})$, with $\bar{x} = \langle x_i \rangle_{i < n}$, $x_i : X_i$, such that $[\bar{x} : A(\bar{x})] = A$. Let us note, for the record, that for $\mathcal{A}$ a d.l., and hence, a c.c., a model $M$ of $\mathcal{A}$ is essentially the same as a homomorphism $\mathcal{A} \rightarrow 2$ ($M(1) = 1$, and all objects in $\mathcal{A}$ are subobjects of 1); the notion of “model” properly generalizes that of “prime filter”; it is appropriate to write $A^\ast$ for $\text{Mod}(\mathcal{A})$ ($\mathcal{A} \in \text{Coh}$).

If $B \in \text{Coh}$, and $C$ is any category, then $B^C$, the category of functors $C \rightarrow B$ and natural transformations between them, is again coherent, and in fact, the coherent operations in $B^C$ are computed “pointwise”: the evaluations $e_C : B^C \rightarrow B$ ($C \in C$) are coherent functors. In particular, we have, for any $\mathcal{A} \in \text{Coh}$, that $\mathcal{A}^{**} \overset{\text{def}}{=} \text{Set}^\ast$ is a coherent category. We will consider (in this paper) $\mathcal{A}^{**}$ only when $\mathcal{A}$ is small. Further, we have the canonical evaluation functor

$$e_A : \mathcal{A} \rightarrow \mathcal{A}^{**}$$
which is, because of the pointwise character of the coherent operations in $\text{Set}^*$, a morphism of coherent categories. The Gödel/Deligne/Joyal completeness/representation theorem is equivalent to the following statement:

For any small $A \in \text{Coh}$, $e_A : A \to A^{**}$ is a conservative coherent functor.  \hfill (3.2)

An equivalent form replaces $\text{Mod}(A)$ by the set $|\text{Mod}(A)|$ in the exponent; also, (3.2) is the same as to say that there are "enough models" in the sense that whenever $A, B \in S(X)$ in $A$, and $A \not\preceq_X B$, then there is $M \in \text{Mod}(A)$ such that $M(A) \not\subseteq M(B)$. For these matters, see especially 3.5.5. in [29], and Section 2.1 in [28].

Let $\vdash$ be a deducibility notion for first-order logic that is sound (with respect to the categorical interpretation): for any coherent category $A$ and coherent $\Sigma$ and $\sigma$, set of axioms and axiom, resp., over $\mathcal{L}_A$, we have $(\Sigma \vdash \sigma$ and $A \vDash \Sigma) \Rightarrow A \vDash \sigma$. The usual proof-systems, formulated for many sorted logic with (sorted) equality, with allowing possibly empty sorts, are all sound. With such $\vdash$, (3.2) is equivalent to the completeness for the coherent fragment of $\vdash : \Sigma \vdash \sigma \iff \Sigma \vdash \sigma$ for all coherent $\Sigma$ and $\sigma$. In fact, this equivalence can easily be seen on the basis of what has been stated here so far. Let us note that there are (instructive) purely categorical proofs for (3.2) as well; cf. Section 2.1 in [28] and, in a topos theoretical disguise, in [1], P. Deligne's Appendice of Exposé 6.

It should be noted (although this fact will not have an "active" role in this paper) that, conversely, for every coherent theory $T$, there is a coherent category $\mathcal{F}(T)$, or $\mathcal{F}_{\text{Coh}}(T)$ that "represents" $T$ in every reasonable sense; e.g., $\text{Mod}(T) \simeq \text{Mod}(\mathcal{F}(T))$; see Chapter 8 in [29].

A Boolean category is a coherent category in which every subobject lattice is a Boolean algebra. $\text{Set}$ is Boolean. A coherent functor between Boolean categories is automatically Boolean, that is (in addition to being coherent) it preserves Boolean complements of subobjects.

A Heyting category is a coherent category in which

\begin{enumerate}
  \item[(vi)] for any $f$ as in (iv), $f^*$ has a right adjoint $\forall_f : f^* B \leq_X A \iff B \leq_Y \forall_f A$ ($A \in S(X)$, $B \in S(Y)$).
\end{enumerate}

In a Heyting category, each subobject lattice is a Heyting algebra: for $A = [U \xrightarrow{m} X] \in S(X)$, $B \in S(X)$ we have $A \to B = \forall_m (m^* B)$. A Boolean category is Heyting.

For future use, we note a formula in a Heyting category for expressing a general $\forall_f$ in terms of implication and a $\forall$, with $\pi$ a product projection. Let $f : X \to Y$, $X \xrightarrow{m} A$, $Y \xrightarrow{n} B$, $Z \xrightarrow{p} X$, $\Phi = [p] \in S(X)$; we want to express $\forall_f \Phi \in S(Y)$ in terms of $\forall$, with $\pi : A \times B \to A$ the projection. Let $\pi^* : A \to A$ be the other projection. Let $\Phi = [mp] \in S(A)$, $\Phi$ as a subobject of $A$. Let $g : X \to A \times B$ be the composite $X \xrightarrow{(1, f)} X \times Y \xrightarrow{m \times n} A \times B$, and $\Gamma = [g] \in S(A \times B)$, the graph of $f$ as a subobject of $A \times B$. Then we have

$$\forall_f \Phi = \forall_{\pi}(\Gamma \to \pi^* \Phi) \wedge [n]. \hfill (3.2')$$
A **Heyting functor** is a functor between Heyting categories which is coherent and which preserves each operation \( \forall_f \) (in a sense similar to preserving \( \exists_f \)). For example, the formula (3.2') implies that if a coherent functor between Heyting categories preserves implications and \( \forall_f \)'s for product projections, then it is Heyting. One may talk about conditionally Heyting functors between coherent functors: this means preserving each \( \forall_f(A) \) that happens to exist in the domain category. \( H \) denotes the bi(2)-category of all Heyting categories, with all Heyting functors as arrows, and all *isomorphism* natural transformations between the latter. (Although in this paper this will not have any consequence it is important to restrict 2-cells to isomorphisms in order to obtain a bicategory with good algebraic properties.)

For any category \( K \), \( \text{Set}^k \) is Heyting. With \( f: X \rightarrow Y \) in \( \text{Set}^k \), \( A \in S(X) \) (that is, \( A \) a subfunctor of \( X \)), we have that \( \forall_f(A) \in S(Y) \) is given in the following way: for any \( M \in K \) and \( y \in Y(M) \),

\[
y \in \forall_f(A)(M) \iff \forall(h: M \rightarrow N) \in K. \forall x \in X(N). f_M(x) = Y(h)(y) \Rightarrow x \in A(N).
\]

In the special case when \( f \) is a product projection \( \pi: X \times Y \rightarrow Y \), (3.3) becomes

\[
y \in \forall_f(A)(M) \iff \forall(h: M \rightarrow N) \in K. \forall x \in X(N). (x, y) \in A(N).
\]

There is a correspondence between theories in many-sorted intuitionistic logic with equality and Heyting categories, entirely analogously to the coherent case. We note, however, that we will use the internal theory \( T_A = (L_A, \Sigma_A) \) always in the sense of coherent logic as fixed above.

In [29], we "conditionally" define the interpretation \( [\tilde{x}: \varphi] \) of any formula of \( L_{\omega_1^{\omega}} \), for \( L = L_{\omega_1} \), in any category \( C \). That is, \( [\tilde{x}: \varphi] \) will be defined and will be uniquely determined in case a certain set of instances of certain (possibly infinitary) categorical operations, all specified by universal properties, are defined. It goes without saying that this categorical interpretation coincides with the standard semantics when \( C = \text{Set} \). Also, it coincides with Kripke semantics in case \( C = \text{Set}^k \), etc. In particular, if \( C \) is Heyting, and \( \varphi \) is any formula of first-order logic, \( [\tilde{x}: \varphi] \), the internal meaning of \( \varphi \), will be defined. Let us make some remarks in this connection.

Suppose \( H \) is Heyting, \( M \in \text{Mod}(H) \); \( \varphi := \forall x A(x, y) \); here, \( A \in S(X \times Y), x: X, y: Y \). Let \( B \coloneqq [x: \forall x A(x, y)] \in S(Y) \). Let \( y \in M(Y) \). Then to say that \( y \in M(B) \) is not the same as to say that \( M \vdash \forall x A(x, y) \); the first interprets the universal quantifier internally in \( H \); the second in \( \text{Set} \). Note also that \( y \in M(B) \) does imply \( M \vdash \forall x A(x, y) \). Occasionally, we will use the notation \( [\forall x] A(x, y) \) for \( B(y) \), and similar other pieces of notation; \( [\forall x] A(x, y) \) stands for a formula whose internal meaning takes into account the internal meaning of the universal quantifier.

A fundamental result of categorical logic is the following theorem due to A. Joyal:

*For any small \( A \in \text{Coh}, c_A: A \rightarrow A^{**} \) is conditionally Heyting, in particular, every small Heyting category \( H \) has a conservative Heyting embedding into one of the form \( \text{Set}^k \).*
The proof of Joyal's theorem was given in [29]; see 6.3.5. in [29] (for an equivalent form) [disregard in (iii), [29], the clause referring to $X \to Y$, which is mistaken]. For future reference, we mention some points of the proof, in fact, in a more general form. Let $K$ be a subcategory of $\text{Mod}(A)$; we have a (restricted) evaluation $e: A \to \text{Set}^K$. Let $f: X \to Y \in A$, $A \in \text{S}(X)$; assume $\forall_f(A) \in \text{S}(Y)$ exists. Translating (3.3), we get that $e$ preserves $\forall_f(A)$ iff for any $M \in K$ and $y \in M(Y)$, 

$$y \in M(\forall_f(A)) \iff \forall(h: M \to N) \in K \forall x \in N(X). \ N(f)(x) = h(y) \Rightarrow x \in N(A).$$

As immediately seen, the left-to-right implication is automatic. Thus, the condition, in the contrapositive form, is that

$$\text{for any } M \in K \text{ and } y \in M(Y), \text{ if } y \notin M(\forall_f(A)), \text{ then there are } h: M \to N \text{ in } K \text{ and } x \in N(X) \text{ such that } N(f)(x) = h(y) \text{ and } x \notin N(A).$$

(3.5)

In the proof of Joyal's theorem, when $K = \text{Mod}(A)$, (3.5) is proved by constructing $N$, $h$ and $x$ by the method of diagrams in model theory, with the compactness theorem.

Joyal's theorem is a "canonical" version of Kripke's completeness theorem [17] for intuitionistic predicate logic with equality. Kripke's theorem refers to a poset, and in fact, to a special poset such as a tree, rather than a category in the position of $K$. (Indeed, a Heyting functor $H \to \text{Set}^c$, with a poset $P$ is the same as a Kripke model of (the theory corresponding to) the Heyting category $H$, with a system of possible worlds indexed by the elements of $P$. For example, the clause for $\forall$ in Kripke's semantics ("forcing") corresponds to (3.3').) However, we can obtain Kripke's original version from Joyal's by general arguments. We will describe the arguments because of their use later.

Let us call a poset $P = (P, \leq)$ a tree if for every $x \in P$, \( \downarrow x = \{ y \in P: y \leq x \} \) is linearly ordered by $\leq$ (thus, a tree may not be connected; it may be a "forest"), and a bush if for every $x, y \in P$ with $x \leq y$, the interval $[x, y] = \{ z \in P: x \leq z \leq y \}$ is linearly ordered. A bush (hence, a tree) is finitary if all intervals $[x, y]$ in it are finite. A functor $\varphi: P \to K$ is upward surjective (compare (2.12)) if for any $P \in P$, $K \in K$ and $f: \varphi P \to K$ in $K$, there is $g: P \to Q \in P$ with $\varphi g = f$. $\varphi$ is two-way surjective if both $\varphi$ and $\varphi^o: P^o \to K^o$ are upward surjective. A simple remark, generalizing (2.12), is that

if $\varphi: P \to K$ is upward (two-way) surjective, then $\varphi^*: \text{Set}^K \to \text{Set}^P$ is (bi) Heyting [for the "bi" version, see below].

(3.6)

Now, we have

for any (small) category $K$, there is $P$ a (small) finitary tree, respectively, bush, with a functor $\varphi: P \to K$ which is upward surjective, resp. two-way surjective; also, $\varphi$ is simply surjective on objects.

(3.7)

The construction of $\varphi: P \to K$ is straightforward; here is an indication for the case of trees. In what follows, $Q, R$ denote finitary trees. We write $Q \preceq R$ ($R$ is an end-extension of $Q$) if $Q$ is a subposet of $R$ and $R \leq Q \in Q$ implies $R \in Q$. Note that the
union of a chain of finitary trees ordered by the $\prec$-relation is again a finitary tree. Given $\psi: Q \to K$, $Q \in Q$, $K \in K$ and $f: \psi Q \to K$ in $K$, there is $R$ with $Q \prec R$, and there is $\theta: R \to K$ extending $\psi$ such that $f$ is in the image of $\theta$; just add a new element above $Q$, which is not below any element of $Q$. Attending to the simple surjectivity condition is even easier, although it is here that we are obliged to give up connectedness. An application of Zorn’s lemma finishes the proof.

Returning to Kripke completeness, first of all, it is quite obvious that from (3.4) we can get a small subcategory $K$ of $\text{Mod}(A)$ such that the evaluation $e: A \to \text{Set}^k$ is a conservative Heyting functor. Using $\varphi$ from (3.7), the resulting composite $\varphi^* \circ e: A \to \text{Set}^k$ is the desired Kripke model: $\varphi^*$, hence $\varphi^* \circ e$ too, is Heyting; since $\varphi$ surjective, $\varphi^*$, hence $\varphi^* \circ e$ too, is (clearly) conservative.

A biHeyting category is a Heyting category in which every subobject lattice is biHeyting (warning: not whose opposite is a Heyting category as well). The “trivial” examples of biHeyting categories are the Boolean (coherent) categories. We have the doctrine $\text{biH}$ of biHeyting categories and biHeyting morphisms.

A Grothendieck topos is prime-generated if all its subobject lattices are prime-generated (see [26]). Another way of defining the same concept is to say that an object in a topos is prime if any (canonical) cover of the object contains a single arrow that covers, and to say that a topos is prime-generated if it has a generating set of prime objects. Note that any presheaf topos $\text{Set}^k (K$ small) is prime-generated: the prime subobjects of the functor $X$ are the subfunctors $A$ of $X$ that are generated by a single element: for some $K \in K$ and some $x \in X(K)$, $A$ is the least subfunctor for which $x \in A(K)$. As [26, 27, 2] show, and as we will see in Section 8, there are many more prime-generated toposes than just the presheaf ones. Any prime-generated (Grothendieck) topos is biHeyting; in particular, so are the presheaf toposes. Lawvere was the first to consider biHeyting toposes; see [21, 22].

Our discussion of the concept of a “conservative enrichment” given in Section 2 applies without change in bicategorical doctrines; in particular, (2.5) applies. Now, note that the target categories of the Kripke/Joyal completeness theorem (see (3.4)), namely the presheaf categories, are (not just Heyting but also) biHeyting. Therefore, in the same way as in Section 2, (3.4) implies that

$$\text{biH is a conservative enrichment of } \mathcal{H}; \text{ the free biHeyting category over a Heyting category extends the latter in a conservative manner.}$$

(3.8)

The idea of biHeyting predicate logic seems to be clear, because its categorical formulation is clear. However, its “symbolic” formulation is non-traditional. The reason is that symbolic logic implicitly assumes that all operations are invariant under substitution. The coimplication is not, in general, invariant under substitution (pullbacks). One way of setting up the symbolic version is to make the symbol $\backslash$ for coimplication have an index $\bar{x}$, a tuple of distinct variables; thus, $\varphi \backslash_x \psi$ makes sense iff the free variables of $\varphi$, $\psi$ are all among the $\bar{x}$. In any case, we have, as a consequence of
biHeyting predicate logic is a conservative enrichment of Heyting (intuitionistic) predicate logic. \hfill (3.8')

We have a Kripke-type completeness theorem for biHeyting predicate logic.

The presheaf categories \(\text{Set}^k\) (\(K\) small) are small-representative in \(\text{biH}\); for any small \(H \in \text{biH}\), there is a small full subcategory \(K\) of \(\text{Mod}(H)\) (= the category of coherent functors \(H \to \text{Set}\)) such that the evaluation \(H \to \text{Set}^k\) is a conservative biHeyting morphism. \hfill (3.9)

Let us note that, similarly as for the case of \(H\), in (9) \(K\) can be taken to be a poset (genuine "Kripke model").

In this case, in contrast to the propositional case, the canonical version, with \(K = \text{Mod}(H)\), is almost certainly \textit{not} true in general, although we have not constructed a counterexample. We postpone the complete proof of (3.9) to Section 6 where a stronger result will be proved; here, also in preparation of the later work, we spell out the "Kripke semantics" of the coimplication. Let \(K\) be an arbitrary category, \(X \in \text{Set}^k\), \(A, B \in S(X)\). Then \(B \setminus A \in S(X)\) is given as follows: for any \(M \in K\) and \(x \in X(M)\),

\[
x \in (B \setminus A)(M) \iff \exists (h : N \to M) \in K. \exists y \in X(N). \ X(h)(y) = x \land y \in B(N) \land y \notin A(N).
\]

Thus, if \(K\) is a (not necessarily full) subcategory of \(\text{Mod}(H)\), \(X \in H\), \(A, B, B \setminus A \in S(X)\), then to say that the evaluation \(H \to \text{Set}^k\) preserves \(B \setminus A\) is the same as to say that for any \(M \in K\) and \(x \in X(M)\),

\[
x \in M(B \setminus A) \iff \exists (h : N \to M) \in K. \exists y \in N(X). \ h_X(y) = x \land y \in N(B) \land y \notin N(A). \hfill (3.10)
\]

In the last equivalence, only the left-to-right direction is non-automatic. When we want to prove (3.10), for a given choice of \(K\), we are called on to construct, for a given \(M\) and \(x\), some \((h : N \to M) \in K\) with \(y \in N(X)\) satisfying further conditions. It is then natural to impose a saturation condition on \(M\) for this to be possible. Indeed, \(K\) can be taken to be the subcategory of special models in a given cardinality; for further details, see Section 6.

Next, we turn to modal logic. To begin with, let \(\varphi : J \to K\) be an arbitrary functor between small categories, and consider the derived structure

\[
\varphi^* : \text{Set}^k \to \text{Set}' \quad (\varphi^*(X) = X \circ \varphi). \hfill (3.11)
\]
It is well-known that \( \varphi^* \) has both left and right adjoints:

\[
\begin{array}{c}
\text{Set}' \\
\downarrow \varphi^* \downarrow \\
\text{Set}^*
\end{array}
\]

for \( Y : J \to \text{Set} \), \( \pi(Y) : K \to \text{Set} \) is the left Kan extension of \( Y \) along \( \varphi \); \( \gamma(Y) \) is the right Kan extension (see [25]). In any situation of the form

\[
\begin{array}{c}
D \\
\downarrow \gamma \\
H
\end{array}
\]

with \( H, D \in \text{Coh} \), \( \gamma \) and \( \pi \) give rise to adjoints on subobject lattices as follows: for any \( X \in H \), we have adjoints \( \Box_X \), \( \Box_X \) to the map \( \cdot : S(X) \to S(\hat{X}) \) induced by the functor \( \cdot : S(X) \to S(\hat{X}) \) induced by the functor \( \cdot : S(X) \to S(\hat{X}) \) induced by the functor \( \cdot : S(X) \to S(\hat{X}) \) induced by the functor \( \cdot : S(X) \to S(\hat{X}) \) induced by the functor

\[
\begin{array}{c}
S(\hat{X}) \\
\downarrow \Box_X \\
S(X)
\end{array}
\]

(3.12)

In fact, for \( A \in S(\hat{X}) \), \( \Box_X(A) \) and \( \Box_X(A) \) are obtained as follows. With \( A = [U \xrightarrow{m} \hat{X}] \), \( \Box_X(A) = [V \xrightarrow{n} X] \), \( \Box_X(A) = [W \xrightarrow{\gamma} X] \), \( n \) is obtained from the first diagram, a pull-back, and \( p \) from the second, a regular epi/mono factorization:

\[
\begin{array}{c}
V \xrightarrow{n} X \\
\downarrow \Box \\
\downarrow \gamma U \\
\gamma X \xrightarrow{\gamma m} \gamma \hat{X} \xrightarrow{\gamma p} X.
\end{array}
\]

A pre-S4 category consists, by definition, of two coherent categories \( H \) and \( D \) and a coherent functor \( \cdot : H \to D \) such that \( \cdot \) has a local adjoint, that is, a system \( \langle \Box_X \rangle_{X \in H} \) of maps as in (3.12). This means that each \( X \in H \) has associated with it a pre-S4 algebra as part of the pre-S4 category structure. A pre-biS4 category also has the \( \Box_X \) as in (3.12), for each \( X \in H \). A morphism \( (\psi : H \to D) \to (\psi' : H' \to D') \) of pre-(bi)S4 categories is a pair \( (\eta : H \to H', \delta : D \to D') \) of coherent functors, with a specified isomorphism \( \iota : \delta \psi \cong \psi' \delta \):

\[
\begin{array}{c}
D \xrightarrow{\delta} D' \\
\psi \iota \psi' \iota \\
H \xrightarrow{\eta} H'
\end{array}
\]

(most of the time, \( \iota \) is the identity) such that the induced map

\[
\begin{array}{c}
S(\hat{X}) \xrightarrow{\delta} S(\delta \hat{X}) \\
\downarrow \psi \\
S(X) \xrightarrow{\eta} S(\eta X)
\end{array}
\]
(dots refer to the effect of both $\psi$ and $\psi'$) is a pre-(bi)$\text{S}4$ algebra map. $(\eta, \delta)$ is conservative if both components are, in the usual sense. Using isomorphism 2-cells, we have the doctrines pre-$\text{S}4$ and pre-bi$\text{S}4$.

The question why we consider the “pre-$\text{S}4$” structure instead of the more comprehensive and more natural structure (3.11’) consisting of global adjoints naturally arises. The answer is that we rely on certain constructions such as the one in (3.13’) below that give us functors that preserve the pre-$\text{S}4$ structure; we do not have similar constructions behaving similarly for the full adjoint structure. It is not impossible that there are interesting results, even of the character of a completeness theorem, for the full adjoint structure; but these will have to await further investigations. On the other hand, the pre-$\text{S}4$ structure encompasses $\text{S}4$ modal predicate logic as we point out in detail below; thus, the “partial” adjoint structure we isolate and investigate is relevant enough to merit consideration on its own right. Notice also the formal similarity of the modal operators as local adjoints to the quantifiers in coherent and Heyting categories (see Section 3); in each case, the operator is an adjoint of a map between subobject lattices. This indicates that modal operators are in a “good company” from a categorical point of view. To add just one more point to this discussion, the quantifiers in an elementary topos can also be regarded as traces of more global adjoints, notably ones between comma-categories derived from the topos. The consideration of these more global adjoints is no longer “first-order logic” however; they do not, in the usual ways at least, lend themselves to completeness theorems.

The standard pre-(bi)$\text{S}4$ categories are the ones of the form (3.11). With reference to (3.11) $\mathcal{X} \in \text{Set}^K$, $\hat{\mathcal{X}} = \varphi^*\mathcal{X}$, the direct expressions for the operations $\Box_x$, $\Diamond_x$ are as follows. For $A \in \mathcal{S}(\hat{\mathcal{X}})$, $M \in \mathcal{K}$, $x \in X(M)$,

\[
x \in (\Box_x A)(M) \iff \forall N \in J. \forall (h : M \rightarrow \varphi N) \in K. \ X(h)(x) \in A(N),
\]

\[
x \in (\Diamond_x A)(M) \iff \exists N \in J. \exists (h : \varphi N \rightarrow M) \in K. \ \exists y \in \hat{\mathcal{X}}(N). \ y \in A(N)
\]

& $X(h)(y) = x$.

We have

*The standard objects (3.11) are small representative both in pre-$\text{S}4$ and pre-bi$\text{S}4$.\)

(3.13)

Indeed, for the first of these doctrines, we have a canonical map that demonstrates the assertion:

*For $\mathcal{U} = (\psi : H \rightarrow D) \in \text{pre-}\text{S}4$, the evaluation-map $e_{\eta} : \mathcal{U} \rightarrow \mathcal{U}^{**}$:\)

\[
\begin{array}{c}
D \xrightarrow{e_D} D^{**} \\
\uparrow \psi' \downarrow \psi'' \\
H \xrightarrow{e_H} H^{**}
\end{array}
\]

is a conservative (see (3.2)) pre-$\text{S}4$ map.\)

(3.13')
Since in Section 6, we will prove more general results, here we only make some remarks concerning the proofs of (3.13), (3.13'); these remarks will be useful later. Let us note that, in contrast to (3.13'), for the pre-biS4 case, suitable subcategories of $H^*$, $D^*$ have to be taken for the exponents.

Let $\psi : H \to D$ be an arbitrary arrow in Coh. Suppose $K, J$ are subcategories of $H^* = \text{Mod}(H)$, $D^* = \text{Mod}(D)$, resp., such that $\psi^* : D^* \to H^*$ restricted to $J$ factors through $K$, thereby inducing $\varphi : J \to K$. Then clearly

$$
\begin{array}{ccc}
D & \xrightarrow{e_D} & \text{Set}^J \\
\downarrow{\psi^*} & & \downarrow{\varphi^*} \\
H & \xrightarrow{e_H} & \text{Set}^K
\end{array}
$$

(3.13'')

commutes; here, $e_D$, $e_H$ are evaluations again. Let $X \in H$, $\Phi \in S(\hat{X})$; assume $\Box \Phi$, $\Diamond \Phi$ exist. Using the above expressions for $\Box$, $\Diamond$, saying that $(e_H, e_D)$ preserves $\Diamond \Phi$, respectively $\Box \Phi$, is equivalent to saying that

for any $M \in K$ and $x \in M(X)$ such that $x \not\in M(\Box \Phi)$, there are $N \in J$ and $h : M \to \varphi N$ such that $h_x(x) \not\in N(\Phi)$;

(3.14)

resp.,

for any $M \in K$ and $x \in M(X)$ such that $x \in M(\Diamond \Phi)$, there are $N \in J$, $h : \varphi N \to M$ and $y \in N(\Phi) \subseteq N(\hat{X})$ such that $h_x(y) = x$.

(3.15)

Using (3.14), we can get the desired result (3.13') (with $K = H^*$, $J = D^*$) by the method of diagrams similarly to Joyal's theorem (3.4). More generally, we have

For any $\psi : H \to D \in \text{Coh}$, by a suitable choice of $\varphi : J \to K$ for (3.13''), (3.13'') is conservative and conditionally biS4.

(3.15')

For details, see Section 6.

A pre-(bi)S4 category $\psi : H \to D$ in which each $\psi : S(X) \to S(\hat{X})$ is (not just a pre-(bi)S4 but also) a (bi)S4 algebra is called a (bi)S4 category. If so, then $\psi$ is conservative, and $D$ is Boolean. If $\psi : H \to D$ is an S4 category, then $H$ is Heyting; for $f : X \to Y \in H$, $\Phi \in S(X)$, we have that $\forall_f(\Phi) = \Box \forall_f(\Phi)$ (we write, again, dots to indicate the effect of $\psi$), as the following sequence of equivalences shows:

$$
\frac{\Psi \leq_Y \Box \forall_f(\Phi)}{\Psi \leq_Y \forall_f(\Phi)} \\
\frac{\forall_f(\Phi)}{\hat{f}^* \Psi \leq_Y \Phi} \\
\frac{\hat{f}^* \Psi \leq_Y \Phi}{\hat{f}^* \Psi \leq_Y \Phi}
$$

the last equivalence uses that $\psi$ is conservative. The fact that if $\psi : H \to D$ is biS4 then $H$ is biHeyting follows from the corresponding fact for propositional logic.
S4 and \( \text{biS4} \) are the doctrines of S4, respectively, biS4 categories; they are full and 2-full sub-bicategories of pre-S4 and pre-biS4, resp. It is immediately seen that a morphism \((\eta, \delta):(H \to D) \to (H' \to D')\) of (bi)S4 categories makes \(\eta:H \to H'\) a (bi)Heyting morphism. Thus, we have the following square of doctrines and forgetful functors:

\[
\begin{array}{ccc}
\text{biS4} & \longrightarrow & \text{S4} \\
\downarrow & & \downarrow \\
\text{biH} & \longrightarrow & \text{H}
\end{array}
\]  

(3.16)

We have

Each forgetful functor in \(3.16\) gives its domain as a conservative enrichment of its codomain.  

(3.17)

To see this, consider the following. For a category \(K\), let \(|K|\) denote the discrete category on the same objects as those of \(K\); \(|K|\) is just the set of objects of \(K\) in fact. We see that \(\text{Set}^K\) is part of the biS4 structure \(\iota^*: \text{Set} \to \text{Set}^{|K|}\), induced by the obvious functor \(\iota: |K| \to K\). Together with (3.4) and (3.9), this gives (via (2.5)) that the two verticals and the diagonal in (3.16) are conservative enrichments. The treatment of the last remaining functor in (3.16), the upper horizontal, needs a completeness theorem for S4. This, as well as another one for biS4, will now be obtained by suitably applying (3.13).

First, a general construction that will be used later too. Let \(\varphi: J \to K\) be any functor. Recall the construction introduced at the start of the proof of (2.22). \(\lfloor J\rfloor K\) is the category whose "objects are those of \(J\), arrows those of \(K\)", Let \(|J|\) be the discrete category on the objects of \(J\), \(\iota = \iota_J: |J| \to J\) the obvious functor. Of course, \(|J|\lfloor J\rfloor K = J\lfloor J\rfloor K\). We have the commutative diagram

\[
\begin{array}{ccc}
J & \leftarrow & |J| \\
\varphi \uparrow & \downarrow^{\theta} & \downarrow^{\xi} \\
K & \leftarrow & \lfloor J\rfloor K
\end{array}
\]  

\(\theta = (\varphi \iota)_{(1)}\) \(\xi = (\varphi \iota)_{(2)}\)  

(3.18)

inducing

\[
\begin{array}{ccc}
\text{Set}^J & \leftarrow & \text{Set}^{(|J|)} \\
\varphi^* \downarrow & \downarrow^{\varphi^*\iota^*} & \downarrow^{\varphi^*} \\
\text{Set}^K & \leftarrow & \text{Set}^{(|K|)}
\end{array}
\]  

(3.19)

where we have indicated three right adjoints, all denoted \(\square\). Although we cannot say that the triangle having the first and second \(\square\)'s as sides commutes up to isomorphism, the traces on the subobject lattices left by the functors that are the three sides of that triangle will form commutative triangles as a consequence of the conservativeness of \(\iota^*\). This, in turn, follows from \(\iota\) being surjective on objects. On the other hand, the
triangle with the two rightmost $\Box$'s commutes up to isomorphism, by the same principle that was used for the corresponding part of (2.23). The same can be said with left adjoints $\circ$ in place of the $\Box$'s. Writing $L$ for $J \downarrow K$ ("whose objects are those of $J$, and whose morphisms are those of $K$"), we have $|J| = |L|$ and $\theta = \iota_L$, and we obtain

For any functor $\varphi: J \to K$, the construction in (3.18), (3.19) gives a biS$4$ morphism

$$
\begin{array}{ccc}
\text{Set}^t & \xrightarrow{r^*} & \text{Set}^{[L]} \\
\varphi^* & \downarrow & \iota_L^* \\
\text{Set}^k & \xrightarrow{\iota^*} & \text{Set}^t
\end{array}
$$

(3.20)

Note that $r^*$ here is always conservative. If, in particular, $\varphi: J \to K$ is chosen as for (3.13'), then composing (3.13') with (3.20), by (3.15') we get

For any $\psi: H \to D \in \text{Coh}$, the construction in (3.20), with a suitable $\varphi: J \to K$, gives the conditionally bi$S4$ morphism

$$
\begin{array}{ccc}
D & \xrightarrow{e^*_D} & \text{Set}^{[L]} \\
\varphi | & \downarrow & \iota_L^* \\
H & \xrightarrow{e^*_H} & \text{Set}^L
\end{array}
$$

(3.21)

in which $e^*_D$ is conservative.

Note that, if in addition, $\psi$ is conservative, then it follows, from the commutativity of the square, that $e^*_H$ is conservative as well. This suffices to conclude that

the (bi)S$4$ categories of the form $\iota^*_L: \text{Set}^t \to \text{Set}^{[L]}$ are small-representative in (bi)S$4$.

(3.22)

Note that (3.22) gives biS$4$ is a conservative enrichment of S$4$; the categories in (3.22) stated to be representative for S$4$ are in fact in biS$4$. Result (3.22) for S$4$ is a form (generalization of) Kripke's completeness theorem for S$4$ modal logic [15].

Let us discuss the symbolic logical meaning of the doctrines S$4$ and biS$4$. S$4$ predicate logic with equality (S$4$PredL) (compare [38]) may be described as follows. With any (possibly many-sorted) language $L$ (in the usual sense as for first-order logic), we define the formulas of S$4(L)$ as having atomic formulas, including ones formed by the use equalities (one for each sort), and being closed under the usual first-order operations, and also under the operation of passing from $\varphi$ to $\Box \varphi$; note, in particular, that the free variables of $\Box \varphi$ are the same as those of $\varphi$. The rules of inference (including the axiom schemes) of S$4$PredL are those of first-order logic with equality (with care for validity in possibly empty sorts), and the instances of the schemes corresponding to (2.19). (As the referee pointed out, the axioms $\forall x \forall y. x =_x y \to \Box (x =_X y)$, one for each sort $X$, are consequences of the foregoing.)

Given a theory $T = (L, \Sigma)$ in S$4$PredL, we may form the S$4$ category $\mathcal{S}_4(T) = (\vdash: H \to D)$ as follows. $D$ is defined as the "Lindenbaum–Tarski category" of $T$ as in [29] for coherent logic (see Chapter 8, Section 2), except that we use formulas of
S4(L) instead of coherent formulas. For example, an object of D is an entity [x:φ] ("formula in context"), with x a tuple of distinct variables, containing all the free variables of φ; in [29] [x:φ] is written as [φ(x)]. Arrows are defined as definable functional relations as in [29], once again using all formulas of S4(L). H is defined similarly, except that we use formulas, both for objects and morphisms, those of the form □φ only. Note how essential the role of equality is in the definition of the "Lindenbaum–Tarski category", through the specification of what the morphisms and composition are. Because of the rule on equality mentioned parenthetically above, equality will be available in H, and we can show that H is a coherent category. The coherent functor ::H→D is defined essentially as an inclusion.

It is easy to see that FS4(T) will faithfully serve in the categorical context in place of T. For example, Kripke models of T (see e.g. [38]) correspond essentially one-to-one to morphisms from FS4(T) to (ιP:SetP→SetP) with P a preorder. Thus, (the case for S4 of) (3.22) contains (via the consideration of trees and such as in the (bi)Heyting case above) Kripke's completeness theorem.

Consider T and FS4(T) = (::H→D). Note that every sort X will have a corresponding object [x ∈ X:□x] in H, and thus, FS4(T) has the following specific property, formulated for an arbitrary (::H→D) ∈ S4:

\[ \text{For every } S ∈ D \text{ there are } X ∈ H \text{ and a monomorphism } S → X; \]  

(3.23)
in other words, every object is a subobject of an H-object. An arbitrary S4 category may, of course, fail to have this property; thus, the categorical doctrine S4 is more general than S4PredL. We now spell out what a general object of S4 means as a theory.

Consider a language L as before, and a distinguished subset L0 of sorts of L. Define the set S4(L/L0) of formulas as follows. This is a subset of S4(L) as given above; φ from S4(L) is in S4(L/L0) iff all free variables in the scope of every occurrence of the symbol □ are of sorts belonging to L0. With this restriction, the rules of the logic are defined as before; e.g., the axioms for equality mentioned above apply only for sorts X ∈ L0. When we form the Lindenbaum–Tarski S4 categories FS4(T) for this kind of theory, we get, up to categorical equivalence, all the (small) S4 categories.

Two more comments on S4 logic. One is that if condition (3.23) is directly imposed on S4 categories, the resulting bicategory is not "good" (locally finitely presentable, in technical terms), and thus is not a proper "doctrine". On the other hand, there is a way (using Lawvere's formalism of "hyperdoctrines") of grasping the usual logical notion of an S4 theory precisely; this was given by [8]. The latter approach also has the advantage that it accommodates logic without equality as well. Our doctrine here, except for the fact that it misses logic without equality, is wider, and therefore, positive results for it (such as completeness, interpolation, etc) are mathematically stronger than the ones for the usual logical, or "hyperdoctrinal", formulation of S4. The last remarks apply to biS4 logic as well; in the "hyperdoctrinal" form, and restricted to countable theories, the result in (3.22) is due to [8]. For further comparisons, see Section 9.
The fact of the faithfulness of the Gödel interpretation of intuitionistic predicate logic in S4 logic (restricted to the case of logic with equality), given a proof-theoretic treatment in [38], is essentially the same as the fact that S4 is a conservative enrichment of \( H \), similarly to the corresponding situation pointed out in Section 2 for propositional logic. The point is that \( \mathcal{F}_{S4}(H) \), the free S4 category extending \( H \in H \) satisfies (3.23) with a “canonical choice of the X’s” (as easily seen), and thus, it is an S4 category of the “traditional kind”.

4. Invariance under substitution, distributivity, the axiom of constant domains, and Barcan’s formula

First of all, let us point out why universal quantification commutes with substitution. Let \( H \) be a coherent category. In \( H \), let \( f: A \to B \) and \( \Phi \in S(A) \); assume \( \forall_f(\Phi) \in S(B) \) exists. Let

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\alpha \downarrow \quad \Box \quad \uparrow b \\
A' \xrightarrow{f'} B'
\end{array}
\]

be a pullback diagram. Then, we claim, \( \forall_f(\alpha^*\Phi) \) exists, and is equal to \( b^*(\forall_f(\Phi)) \). What we just said is a careful “local” statement; in particular, if \( H \) is Heyting, the statement reduces to the “Beck–Chevalley condition for \( \forall_f \)”, that is, the commutativity of

\[
\begin{array}{c}
S(A) \xrightarrow{\forall_f} S(B) \\
\alpha^* \downarrow \quad \downarrow b^* \\
S(A') \xrightarrow{\forall_f'} S(B')
\end{array}
\]

under (4.1). For the proof, and for similar proofs below, we use “adjoint squares”. Suppose we have the categories and pairs of adjoint functors as in the diagram

\[
\begin{array}{c}
A \xrightarrow{\perp} B \\
\downarrow I \quad \downarrow J \\
C \xrightarrow{\perp} D \\
\downarrow H
\end{array}
\]

actually, the two left-to-right horizontals are assumed only to be partial functors; they are defined on the full subcategories of their “domains” with precisely the objects for which the value (as an object of the codomain category satisfying a certain universal property) of the respective right adjoint is defined. The claim is that if the square of the left-adjoint functors commutes up to isomorphism, then so does the square of the right adjoints; and more specifically, if \( A \in A \) such that \( G(A) \) is defined, then \( H(I(A)) \)
is defined as well, and is isomorphic to \( J(G(A)) \) (a "strong" notion of commutativity of a square with two parallel functors being only partial is used here as well as below). The verification of this assertion poses no difficulty. Naturally, the dual statement with "left" and "right" interchanged is also true. Of course, if the categories involved are posets, we have strict commutativities.

To return to the Beck–Chevalley condition for \( \forall \), one notes that the square

\[
\begin{array}{ccc}
S(A) & \xleftarrow{f^*} & S(B) \\
\downarrow \circ & & \downarrow \circ \\
S(A') & \xleftarrow{f'^*} & S(B')
\end{array}
\]

is left adjoint to (4.2), and it is commutative because of the B–C condition for \( \exists \) in a coherent category. The "local" version also follows, with (4.2) understood with partial horizontals.

There is a similar conclusion for modal logic. Let \((H \twoheadrightarrow D) \in \text{pre-S4}\). Then, for any \(f: A \rightarrow B \in H\), we have the adjoint squares

\[
\begin{array}{ccc}
S(\hat{B}) & \xleftarrow{3_f} & S(\hat{A}) \\
\downarrow \circ & & \downarrow \circ \\
S(B) & \xleftarrow{3_f} & S(A)
\end{array}
\]

The left-hand square is commutative since \( \cdot \) is a coherent functor. Hence, so is the right-hand square, that is, \( \Box_A(f^*(\Phi)) := f^*(\Box_B \Phi) \) for any \( \Phi \in S(\hat{B}) \) (where we used the non-symmetric symbol \( := \) to mean "if the right side is defined, so is the left, and they are equal").

In contrast to the above, the invariance of the coHeyting and the coS4 operations under substitution is far from being automatic. Of course, the invariance holds when the categories involved are Boolean, since then these operations are definable in terms of the classical ones. We will see below that, conversely, requiring the invariance under all substitutions of the coHeyting operation in a biHeyting category makes the category Boolean.

Let \( H \in H, f: A \rightarrow B \in H, \Phi \in S(A) \). Then we have the adjoint squares

\[
\begin{array}{ccc}
S(B) & \xrightarrow{(f \cdot \Phi)} & S(B) \\
\downarrow f^* & & \downarrow \forall_f \\
S(A) & \xrightarrow{(f \cdot \Phi)} & S(A)
\end{array}
\]

The commutativity of the latter square says that

\[
\forall_f(f^*\Phi \lor \Psi) = \Phi \lor \forall_f \Psi \tag{4.4}
\]

for all \( \Psi \in S(A) \); also note that the "left \( \leq \) right" part of the equality is the requirement; the other inequality is automatic. Let us call \( f: A \rightarrow B \) distributive if (4.4) holds for all \( \Phi \in S(B) \) and \( \Psi \in S(A) \); this make sense whenever \( \forall_f \) is totally defined, in
particular in any Heyting category. We conclude that, for a given arrow \( f: A \to B \), 
\( f^* \) preserves all existing \( \mathcal{P} \setminus \Phi \), \( f^*(\mathcal{P} \setminus \Phi) = (f^*\mathcal{P}) \setminus (f^*\Phi) \), if \( f \) is distributive; in case the underlying category is biHeyting, "if" can be replaced by "if and only if".

We will be particularly interested in the condition on \( f: A \to B \) being \textit{stably distributive} which is to say that for any pullback diagram (4.1), \( f' \) is distributive. If the arrow \( !_A: A \to 1 \) is stably distributive, we say \( A \) is a \textit{distributive object}. This means that for any object \( X \), the product projection \( \pi: A \times X \to X \) is distributive. We will use this terminology also when the category \( H \) is assumed merely to be coherent; then by saying that \( A \) is distributive we will mean that \( \forall : S(A \times X) \to S(X) \) is (completely) defined \textit{and} distributive in the above sense for all \( \pi \) as above.

For an intuitionistic theory \( T \) and a specific sort \( A \) in its language, requiring the axiom scheme

\[
\forall a (\varphi(\bar{x}) \lor \psi(a, \bar{x})) \Rightarrow \varphi(\bar{x}) \lor \forall a \psi(a, \bar{x})
\]

with \( \varphi \) and \( \psi \) arbitrary formulas with free variables as indicated (specifically, \( a \) not free in \( \varphi \)) is equivalent to saying that the object (corresponding to) \( A \) in \( \mathcal{F}_{\text{int}}(T) \), the Heyting category presented by \( T \), is distributive. The condition in its last form is the well-known \textit{axiom of constant domains}, so-called because of its role in Kripke models (see Section 6). The many-sortedness is not the essentially new element in our treatment here; rather, it is the fact that the axiom scheme is imposed on a relativized quantification. Imagine an ordinary (one-sorted) intuitionistic theory, with a specific predicate \( P(a) \). We may require (4.5) with \( \forall a(...) \) replaced by \( \forall a(P(a) \to ...) \), and consider theories with having the resulting "constant domains axiom relative to \( P \)". Our semantic theory in Section 6 of the axiom of constant domains will apply to this relativized variant, and yield a completeness theorem with Kripke models in which the interpretation of \( P \) is constant. This will come from the many-sorted version because the theory may be translated into a many-sorted variant in which \( P \) itself becomes a sort (the category theory automatically does this). Let us add that our extension of the original theory (due to G"orncemann [10]) seems to be a mathematically non-routine generalization.

We note that, with (4.1) a pullback diagram and

\[
b \text{ (hence also } a \text{) a monomorphism, if } f \text{ is distributive, then so is } f';
\]

thus, the stability of distributivity under the pullback along a monomorphism is automatic. Indeed, in this case any \( \Phi' \in S(B') \) and \( \Psi' \in S(A') \) are of the respective forms \( \Phi' = b^*\Phi, \quad \Psi' = a^*\Psi \) with \( \Phi \in S(B), \quad \Psi \in S(A) \); then

\[
\forall f. (f'^*\Phi' \lor \Psi') = \forall f. (f'^*b^*\Phi \lor a^*\Psi) = \forall f. (a^*f'^*\Phi \lor a^*\Psi)
\]

\[
= \forall f. a^*(f'^*\Phi \lor \Psi) = b^*\forall f (f'^*\Phi \lor \Psi)
\]

and

\[
\Phi' \lor \forall f. \Psi' = b^*\Phi \lor \forall f. a^*\Psi = b^*\Phi \lor b^*\forall f. \Psi = b^*(\Phi \lor \forall f. \Psi);
\]

thus, the "primed" version of (4.4) follows from the "unprimed" one.
Recall that for $\Phi \in S(A)$, with $\Phi = [X \to A]$, and for any $\Psi \in S(A)$,

$$\Phi \to \Psi = \forall_m(m^*\Psi).$$

Thus, the distributivity of $m$ means that

$$\Phi \to (\Gamma \lor \Psi) = \Gamma \lor (\Phi \to \Psi) \quad (\Gamma, \Psi \in S(A)); \tag{4.7}$$

indeed, $\Phi \to (\Gamma \lor \Psi) = \forall_m(m^*(\Gamma \lor \Psi)) = \forall_m(m^*\Gamma \lor m^*\Psi) = \Gamma \lor \forall_m(m^*\Psi) = \Gamma \lor (\Phi \to \Psi)$. Apply this to $\Gamma = \Phi, \Psi = 0_A$ (minimal subobject); the left-side of (4.7) is $1_A$, the right is $\Phi \lor \neg \Phi$. We conclude that

- if $\Phi$ is a subobject defined by a monomorphism which is distributive, then $\Phi$ is a complemented subobject; \tag{4.8}

the converse is also easy to see. In particular, a Heyting category is Boolean iff all arrows (or, all monomorphisms) in it are distributive.

After the last paragraph, it comes as a surprise that the condition of all objects being distributive is satisfied by many non-Boolean (bi)Heyting categories. For brevity, let us call this condition (L); also, we talk about an L-Heyting category, or L-biHeyting category.

Lawvere has shown (see [22]) that for the presheaf category $\text{Set}^{\text{co}}$ to satisfy (L), the following condition on $C$ is sufficient (and as Zolfaghari [40] has shown, it is also necessary):

for any arrow $f: A \to B$, there is a diagram of the form

$$A \xrightarrow{f} B \xleftarrow{i} C \xleftarrow{h} A.$$ \hspace{1cm} : f = hg, hj = 1_B, ig = 1_A.

Further, this condition is satisfied once $C$ has binary products, and for any objects $B$ and $A$, there is at least one arrow $B \to A$; the latter is true e.g. if $C$ is pointed, i.e., $0 = 1$ (e.g., if $C$ is Abelian). (To see this, consider, with any $a: B \to A$,

$$A \xleftarrow{(1_A, f)} A \times B \xrightarrow{a \times 1_B} B.$$ \hspace{1cm} )

In contrast, the presheaf category $\text{Set}^{\text{co}}$ is Boolean (if and) only if $C$ is a groupoid (all arrows are isomorphisms).

Next we look at what the condition (L) means in "symbolic" terms.

Let us start with the observation (easily verified) that, in general,

if $\Phi = [U \to A], \Psi \in S(A), f: A \to B$, then $\forall_m(m^*\Psi) = \forall_f(\Phi \to \Psi). \tag{4.9}$
Holding onto the notation in (4.9), let us see what the distributivity of $\forall f m$ means in terms of $\forall f$. With $\Gamma \in S(B)$, by (4.9) we have

$$\forall f m((f m)^{*} \Gamma \lor m^{*} \Psi) = \forall f m((m^{*} f^{*} \Gamma) \lor m^{*} \Psi) = \forall f m(m^{*}((f^{*} \Gamma) \lor \Psi))$$

$$= \forall f(\Phi \rightarrow ((f^{*} \Gamma) \lor \Psi),$$

and

$$\Gamma \lor \forall f m(m^{*} \Psi) = \Gamma \lor \forall f(\Phi \rightarrow \Psi).$$

Since all subobjects of $U$ are of the form $m^{*} \Psi$,

the distributivity of $f m$ (see (4.9)) is equivalent to the identity

$$\forall f(\Phi \rightarrow (f^{*} \Gamma \lor \Psi)) = \Gamma \lor \forall f(\Phi \rightarrow \Psi) \quad (\Psi \in S(A), \Gamma \in S(B)). \quad (4.10)$$

Now, let $n : V \rightarrow X$; we are interested in expressing the distributivity of the object $V$ in terms of quantifiers $\forall x$ for projections $\pi : X \times Y \rightarrow Y$. Given any projection $\hat{\pi} : V \times Y \rightarrow Y$, we can write it in the form $\hat{\pi} = \pi \circ (V \times n)$. Applying (4.10), with $A = X \times Y$, $B = Y$, $f = \pi : X \times Y \rightarrow Y$, $m = V \times n$, and as a consequence, $\Phi = \Xi \times Y$, we get that the distributivity of $\hat{\pi}$ is equivalent to the identity

$$\forall x(\Xi \times Y \rightarrow (\pi^{*} \Gamma \lor \Psi)) = \Gamma \lor \forall x(\Xi \times Y \rightarrow \Psi) \quad (\Psi \in S(X \times Y), \Gamma \in S(Y)).$$

With the logical notation the equality takes the form

$$\forall x(\Xi y \rightarrow (\Gamma y \lor \Psi x y)) \equiv \Gamma y \lor \forall x(\Xi y \rightarrow \Psi x y). \quad (4.11)$$

Let $T$ be a theory in intuitionistic logic. Recall that in the Heyting category $\mathcal{F}(T)$ presented by $T$ all objects are of the form $[\bar{x}, \xi(\bar{x})]$, in particular, they are domains of subobjects of products of sorts. Using what we saw above, including the automatic stability of distributivity under pullback along a monomorphism applied to pullbacks of the form

$$X \times Y \rightarrow Y$$

$$\downarrow \quad \downarrow$$

$$X \times V \rightarrow V$$

we can now conclude that

saying that $\mathcal{F}(T)$ satisfies (L) is equivalent to saying that the axiom scheme

$$\forall \bar{x}(\xi(\bar{x})) \rightarrow (\gamma(\bar{y}) \lor \psi(\bar{x}, \bar{y})) \quad \Rightarrow \quad \gamma(\bar{y}) \lor \forall \bar{x}(\xi(\bar{x}) \rightarrow \psi(\bar{x}, \bar{y}))$$

holds in $T$; \quad (4.12)

here, the formulas have only the free variables shown; $\bar{x}$ and $\bar{y}$ are disjoint.
Although condition (L) is satisfied by many natural non-Boolean examples, weak-looking conditions additional to (L) will force Booleanness. For example,

\[ \text{if an } L\text{-Heyting category has all its equalities decidable, that is, each } \]
\[ =_A = [A \rightarrow A \times A] \in S(A) \text{ is complemented, the category must be Boolean.} \]

(4.13)

Assume that \( \neq \) has the Boolean complement \( \neq \), let \( A \in S(X) \); with \( Y = X \) among others, we get the following special case of (4.12):

\[ \forall x(y \rightarrow (Ax \land x \neq y)) \equiv Ly \lor \forall x(Ax \rightarrow x \neq y). \]

(4.14)

Now, using that \( x = y \lor x \neq y \equiv T \) and \( Ax \rightarrow (x = y \land Ay) \equiv T \), we obtain that the left-hand side in (4.14) is identically true. Also, since \( x \neq y \Rightarrow \neg(y(x = y)) \), we have \( \forall x(Ax \rightarrow x \neq y) \Rightarrow \neg y(Ay) \). We conclude that \( Ay \lor \neg Ay \equiv T \), that is, \( A \) has a Boolean complement.

Here is another formula connecting two universal quantifiers. Let \( f : A \rightarrow B \), \( g : Y \rightarrow B \), and

\[ \begin{array}{c}
A \\
\downarrow f
\end{array} \begin{array}{c}
\square
\end{array} \begin{array}{c}
B
\end{array} \begin{array}{c}
g
\end{array} \begin{array}{c}
X
\end{array} \begin{array}{c}
h
\end{array} \begin{array}{c}
Y
\end{array} \]

be a pullback square. With \( m : X \rightarrow A \times Y \) the canonical monomorphism, the subobject \([m] \in S(A \times Y)\) is denoted by \( \hat{X} \). The usual construction of the pullback as a subobject of a product gives, with \( \pi : A \times Y \rightarrow Y \) the projection, and with \( G = [A \rightarrow A \times B] \in S(A \times B) \) the graph of \( f \), that \( \hat{X} = (A \times g)^*(G) = [ay : fa = gy] \in S(A \times Y) \). For any \( \Phi \in S(A \times Y) \), we claim

\[ \forall h(m \Phi) = \forall_x(\hat{X} \rightarrow \Phi) = \forall_x((A \times g)^*(G) \rightarrow \Phi) = [y : \forall a(fa = gy) \rightarrow \Phi ay]. \]

The verification is routine. We now derive a consequence of the formula.

**Assume** \( B \) is a Boolean category, \( B \rightarrow H \) is a coherent functor that takes every object of \( B \) to a distributive object. Then \( \cdot \) takes all arrows of \( B \) to stably distributive arrows.

(4.16)

Indeed, let (4.15) be a pullback in \( H \), with \( f = f_0 \) coming from \( B \). Using the above notation, \( G = G_0 \) with \( G_0 \) graph of \( f_0 \). Since \( G_0 \) has a complement (in \( B \)), so does \( G \) (in \( H \)). Any pullback of a subobject with complement has a complement; thus, \( \hat{X} \) has a complement, say \( \Psi \). It is clear that, as a consequence, \( \hat{X} \rightarrow \Phi = \Psi \land \Phi \) for any \( \Phi \in S(A \times Y) \). Let \( A \in S(Y) \); any subobject of \( X \) is of the form \( h^\Phi \) for a \( \Phi \in S(A \times Y) \).
Consider
\[ \forall_h (h^* \Lambda \lor m^* \Phi) = \forall_h (m^* (\pi^* \Lambda \lor \Phi)) = \forall_h (\pi^* \Lambda \lor (\pi^* \Lambda \lor \Phi)) \]
\[ = \forall_h (\pi^* \Lambda \lor \Psi \lor \Phi) = \bigvee \forall_h (\Psi \lor \Phi) = \bigvee \forall_h (\pi^* \Lambda \lor \Phi) \]
\[ = \Lambda \lor \bigvee \forall_h (m^* \Phi); \]
this proves our assertion.

Let us turn to invariance of the past-possibility operator under substitution. With data as for (4.3), assume that \( \forall_f \) and \( \forall_f' \) are totally defined, and consider the adjoint squares

\[
\begin{array}{cccc}
S(\mathcal{B}) & \xrightarrow{f^*} & S(\mathcal{A}) \\
\circ h & \circ & \circ \circ h & \circ \circ \\
S(\mathcal{B}) & \xrightarrow{f^*} & S(\mathcal{A}) \\
\end{array}
\]

The commutativity of the left-hand square expresses the invariance of \( \circ \) under substitution along \( f \). We conclude that

*for \( H \to D \in \text{pre-biS4}, \) if \( H, D \) are Heyting, \( \circ \) is invariant under substitution along \( f \) iff \( \cdot \) preserves \( \forall_f \).

(4.17)

In particular, if the structure map of a pre-biS4 category is Heyting, then its past-possibility operator is fully invariant under substitution.

In case \( \cdot : H \to D \) is also conservative, then, as we know, \( \setminus \) in \( H \) is expressible in terms of the same operation in \( D \) and \( \circ \);

if \( \circ \) and \( \setminus \) in \( D \) are invariant under substitution along \( f \), then so is \( \setminus \) in \( H 
\]
\[ f^* (\Psi \setminus \Phi) = f^* (\circ (\Psi \setminus \Phi)) = \circ f^* (\Psi \setminus \Phi) \]
\[ = \circ (f^* \Psi \setminus f^* \Phi) = \circ ((f^* \Psi) \setminus (f^* \Phi)) = (f^* \Psi) \setminus (f^* \Phi). \]

(4.18)

Assume \( \cdot : H \to D \in \text{pre-S4}, \) is conservative, and \( D \) is Heyting. We know that then \( H \) is Heyting as well. Let \( f : A \to B \in H, \Phi \in S(\mathcal{A}); \) we have \( \forall_f (\Phi) = \square \forall_f (\Phi). \) To say that \( \cdot \) preserves \( \forall_f \), i.e., \( (\forall_f (\Phi))' = \forall_f (\Phi) \), is to say that

\[ (\square \forall_f (\Phi))' = \forall_f (\Phi). \]

\( \Phi \) is a typical subobject of the form \( \Box \Gamma, \Gamma \in S(\mathcal{B}) \) (since \( \Phi = \square \Phi \)); we get

\[ (\Box \forall_f (\Box \Gamma))' = \forall_f (\Box \Gamma). \]

(4.19)

Let us assume (without loss of generality) that \( (\cdot) \) is an inclusion; in particular, each induced map \( S(Y) \to S(\mathcal{Y}) \) is an inclusion; let us drop the dots:

\[ \Box \forall_f \Box \Gamma = \forall_f \Box \Gamma \quad (\Gamma \in S(\mathcal{B}}). \]
We have to keep in mind that here everything takes place in \( D \); \( \Box \Gamma \) is meant as a subobject of \( B \) in \( D \), etc. Now, we claim that the last identity is the same as

\[
\Box \forall \gamma \Gamma = \forall \gamma \Box \Gamma \quad (\Gamma \in S(B)). \tag{4.20}
\]

Indeed, (4.20) applied to \( \Box \Gamma \) in place of \( \Gamma \) gives (4.19) since \( \Box \Box \Gamma = \Box \Gamma \). On the other hand, note that \( \Box \Gamma \leq \Gamma \) gives \( \Box \forall \gamma \Box \Gamma \leq \Box \forall \gamma \Gamma \), and \( f^* \forall \gamma \Gamma \leq \Gamma \Rightarrow \Box f^* \forall \gamma \Gamma \leq \Box \Gamma \Rightarrow f^* \Box \forall \gamma \Gamma \leq \Box \forall \gamma \Gamma \Rightarrow \Box \forall \gamma \Gamma \leq \forall \gamma \Box \Gamma \); we conclude (in general) that \( \Box \forall \gamma \Box \Gamma \leq \Box \forall \gamma \Gamma \leq \forall \gamma \Box \Gamma \). Now, it is clear that (4.19) implies (4.20).

Eq. (4.20) is the so-called Barcan's formula (see, e.g., [12]) of modal logic. What we have seen is that

\[
\text{In case } \cdot: H \to D \in \text{pre-S4} \text{ is conservative, } D \in H, f \in \text{Arr}(H), \text{ Barcan's formula for } \forall_f \text{ expresses that } \cdot \text{ preserves } \forall_f. \tag{4.21}
\]

For any \( H \to D \) in \( \text{Coh}^- \), we say that an object \( A \in H \) is a Barcan object if for all product projections \( \pi: A \times X \to X \) in \( H \), \( \forall_f \) is completely defined, and is preserved by \( \cdot \).

In case \( T \) is an ordinary S4-theory, with a specific sort \( A \), then to say that (the object corresponding to) \( A \) is Barcan in \( \mathcal{F}_s(T) \) is to say that the axiom scheme

\[
\forall a \Box \phi \to \neg \forall a \phi \quad (a: A) \tag{4.22}
\]

(with \( \phi \) an arbitrary formula, with possibly free variables) holds. The converse implication to (4.22) is automatic.

Assume \( \cdot: H \to D \in \text{S4} \). Then of course all \( \forall_f \) in \( D \) are distributive. Thus, if \( \cdot \) preserves \( \forall_f \), then the distributivity of \( \forall_f \) is reflected down along the conservative morphism \( \cdot \).

We conclude that

\[
\text{if } \cdot: H \to D \text{ is an S4 category, } f \in \text{Arr}(H), \text{ and Barcan's formula holds for } \forall_f, \text{ then } \forall_f \text{ satisfies the axiom of constant domains in } H. \tag{4.23}
\]

5. A preservation theorem

In this section, we are squarely in the context of model theory. We go through some basic definitions of model-theoretical concepts; in the literature, these are stated in the one-sorted context.

Let us fix a language \( L \); all structures and formulas are over \( L \) unless otherwise stated. Let \( \#_{\inf}(L) \) denote the cardinality which is the maximum of \( \aleph_0 \) and the cardinality of the set of all sorts, operation symbols and relation symbols in \( L \). \#_{\inf}(L) is the same as the cardinality of the set of all \( L \)-formulas.

Every structure is isomorphic to one, say \( M \), which is separated, that is, for which \( M(X) \cap M(X') = \emptyset \) for any two distinct sorts \( X, X' \). Because of this, we may usually assume that our models are separated. \( |M| = \bigcup_x M(X) \); if \( M \) is separated, \( |M| \) can be taken to be \( \bigcup X M(X) \); \( X \) ranges over all sorts. The cardinality of \( M \), \( |M| \), is the cardinality of the set \( |M| \).
Let $M$ be any separated structure. The diagram-language $L(M)$ is the extension of $L$ by individual constants, one for each $x \in |M|$; the constant corresponding to $x$ is denoted by $x$ as well; $x \in M(X)$ is of sort $X$. For any $A \subset |M|$, $L(A)$ is the sublanguage of $L(M)$ with only the constants in $A$ added to $L$. $M^* = (M, x)_{x \in M}$ is the obvious $L(M)$-structure expanding $M$. With $A \subset |M|$, we say that $M$ is $A$-saturated if for any set $\Phi(x)$ of formulas over $L(A)$ with a single free variable $x$, if every finite subset $\Phi'$ of $\Phi$ is satisfiable in $M^*$ ($M^* \models \exists x / \bigwedge_1 \Phi(x)$), then $\Phi$ itself is satisfiable in $M^*$ ($M^* \models \exists x / \bigwedge_1 \Phi(x)$, the last formula being possibly infinitary). With $\kappa$ an infinite cardinal, $M$ is $\kappa$-saturated if for all $A \subset |M|$ of cardinality $< \kappa$, $M$ is $A$-saturated. We have that

if $\kappa \geq \#_\inf(L)$ and $M$ is a structure of cardinality $\leq \kappa$, then there is a $\kappa^+$-saturated structure of cardinality at most $2^\kappa$, with an elementary embedding $h : M \to N$;

the proof is the same as for the one-sorted version in [4, 5.1.4, p. 294].

Let us fix an infinite cardinal $\lambda$ such that $\lambda \geq \#_\inf(I)$. Let $\lambda_0 = \lambda$, $\lambda_{n+1} = 2^{\lambda_n}$ for $n < \omega$. Let $k < \omega$ and let

$$M_0 \xrightarrow{h_0} M_1 \xrightarrow{h_1} M_2 \xrightarrow{} \cdots \xrightarrow{} M_n \xrightarrow{h_n} M_{n+1} \xrightarrow{} \cdots$$  \hspace{1cm} (5.1)

be a chain of structures with elementary embeddings as connecting morphisms such that $M_{n+1}$ is $\lambda_{n+k+1}$-saturated and of cardinality at most $\lambda_{n+k+1}$. Any structure $M$ of the form $\colim_{n < \omega} M_n$, the colimit of a chain as described, is called a $\lambda$-special structure; (5.1) is a specializing chain for $M$. In what follows in this section, $\lambda$ is sometimes dropped from "$\lambda$-special", since $\lambda$ will not be varied in this section; however, we need to be able to vary $\lambda$ in the applications.

Note the parameter $k$ is the above definition; let us say that the specializing chain, or the special structure, is $k$-based. It is clear that if a (special) structure has a $k$-based specializing chain, then it has an $\ell$-based specializing chain for any $\ell, k \leq \ell < \omega$. In fact, it turns out that the base-parameter is superfluous; every special structure has a $0$-based specializing chain, see below. Also note that every special structure has a specializing chain with each connecting morphism and each colimit coprojection being an inclusion; in this case, the special structure is the union of the specializing chain. In this case, we refer to the specializing chain in the form $\langle M_n \rangle_{n < \omega}$.

By the above existence theorem, we have that

any structure of cardinality at most $\lambda$ can be elementarily extended to a (0-based) $\lambda$-special structure.

We also have the uniqueness theorem

any two elementarily equivalent $\lambda$-special structures are isomorphic.

See 5.1.17, p. 300 in [4] for the proof; Proposition 5.1 below is in fact a generalization. The existence and uniqueness theorems together say that every complete theory over $L$ has a (0-based) $\lambda$-special model which is unique up to isomorphism. Our remark about the superfluousness of the base-parameter is a consequence.
If $M$ is special, $a \in M(A)$, then the $L(\{a\})$-structure $(M, a)$ is also special. What is clear directly from the definition is that if $M$ is $k$-based special, then $(M, a)$ is $\ell'$-based special, with $\ell' \geq k$ chosen so that $a \in |M|_\ell$; here, $\langle M_n \rangle_{n<\omega}$ is the specializing chain for $M$.

For later reference, let us note another well-known, and obvious, property of special structures:

for any sublanguage $L_0$ of $L$, if the $L$-structure $M$ is $\lambda$-special, then so is its $L_0$-reduct $M|_{L_0}$.

Let $A$ be a set of specific sorts in $L$. A morphism $h: M \to N$ is $A$-surjective if $h_A: M(A) \to N(A)$ is surjective for all $A \in A$. $|M|_A$ will abbreviate $\bigcup_{A \in A} M(A)$; and $h_A = \bigcup_{A \in A} h_A$; thus, the requirement is that $h_A: |M|_A \to |N|_A$ be surjective.

Let $\forall_A$ denote the least set of formulas containing all atomic $L$-formulas, and closed under the operations $\land, \lor, \forall x$ for all variables $x$, and $\forall a$ for variables $a$ of sorts $A$ in $A$. If $A = \emptyset$, $\forall_A$ is the set of coherent formulas; when $A$ is the set of all sorts, $\forall_A$ is the set of positive formulas. Note the easily seen fact that an $A$-surjective map preserves (the validity of) any formula in $\forall_A$. The following proposition generalizes 5.2.11, p. 314 in [4].

**Proposition 5.1.** Suppose $M$ and $N$ are $\lambda$-special structures such that for every sentence $\sigma \in \forall_A$, $M \models \sigma$ implies $N \models \sigma$. Then there is an $A$-surjective morphism $h: M \to N$.

Let $M, N$ be structures (not yet the ones in the assertion of the proposition), assumed separated. A relation $R \subseteq |M| \times |N|$ is an $\forall_A$-relation (for $M$ and $N$) if for any $\bar{x} = \langle x_i \rangle_{1 \leq i \leq m}$, $\bar{y} = \langle y_i \rangle_{1 \leq i \leq m}$, $x_i \in M(X_i)$, $y_i \in N(X_i)$ $(1 \leq i \leq m)$, if $x_i R y_i$ for $1 \leq i \leq m$ (which we abbreviate as $\bar{x} R \bar{y}$), then for any $\varphi \in \forall_A, M \models \varphi[\bar{x}]$ implies $N \models \varphi[\bar{y}]$; this includes the condition that $x R y$ and $x \in M(X)$ imply $y \in N(X)$. We have the following lemma.

**Lemma 5.2.** Suppose $M$ and $N$ are $\kappa$-saturated, $R \subseteq |M| \times |N|$ an $\forall_A$-relation with $\text{dom}(R) \subseteq |M|$ and $\text{range}(R) \subseteq |N|$ both of cardinality $< \kappa$. Then

(i) for any $x \in |M|$, there is $y \in |N|$ such that $R \cup \{(x, y)\}$ is an $\forall_A$-relation;

(ii) for any $b \in |N|_A$, there is $a \in |M|_A$ such that $R \cup \{(a, b)\}$ is an $\forall_A$-relation.

**Proof.** (i) Let $x \in |M|$, and let $\Phi(x)$ be the set of all formulas of the form $\varphi(\bar{y}, x)$ obtained from some $\varphi(\bar{x}, x) \in \forall_A$ and $\bar{y}$ such that for some $\bar{x}, \bar{x} R \bar{y}$ and $M \models \varphi[\bar{x}, x]$. A moment’s reflection shows that $\Phi$ is closed under finite conjunction. Given any $\varphi(\bar{y}, x)$ in $\Phi$ with $\bar{x}$ as in the definition, $\exists x \varphi(\bar{x}, x) \in \forall_A$ and $M^* \models \exists x \varphi(\bar{x}, x)$. Thus, since $R$ is an $\forall_A$-relation, $N^* \models \exists x \varphi(\bar{y}, x)$; thus, there is $y \in |N|$ with $N^* \models \varphi(\bar{y}, y)$; this shows that $\Phi$ is finitely satisfiable in $N^*$. Using the saturation of $N$, we have some $y \in |N|$ satisfying $\Phi(x)$ in $N^*$. It is clear that $R \cup \{(x, y)\}$ is then an $\forall_A$-relation.

(ii) is quite similar, interchanging the roles of $M$ and $N$, and using the availability of the universal quantifiers $\forall a$ with $a: A, A \in A$, in forming formulas in $\forall_A$. \[\square\]
Proof of 5.1. Let \( \langle M_n \rangle_{n<\omega} \) be specializing chains for \( M, N \), resp. (\( M \) and \( N \) are each the simple union of the specializing chain); we assume that both are \( k \)-based (thus, we are not using our remark above on the dispensability of the base-parameter; in fact, with an easy trick of changing languages, the uniqueness result, used in seeing that dispensability, can be deduced from Proposition 5.1). By induction on \( n < \omega \), we define \( R_n \), an \( \forall_\alpha \)-relation for \( M_{n+1} \) and \( N_{n+1} \), such that the following hold:

(iii) \( |M_n| \subseteq \text{dom}(R_n) \);
(iv) \( |N_{n+1}| \subseteq \text{range}(R_n) \);
(v) \( R_n \subseteq R_{n+1} \).

Suppose \( n = 0 \), or that for \( n - 1, R_{n-1} \) has been defined. In case \( n = 0 \), \( R \) is an \( \forall_\alpha \)-relation for \( M_0 \) and \( N_0 \); this is the assumption of Proposition 5.1 (with \( \bar{x} \) and \( \bar{y} \) being the empty tuples in the definition of "\( \forall_\alpha \)-relation"). Thus, in any case, we have an \( \forall_\alpha \)-relation \( R(=R_{n-1}) \) if \( n > 0 \) for \( M_n \) and \( N_n \); we want to extend it to another one, for \( M_{n+1} \) and \( N_{n+1} \), with additional properties (iii), (iv). We employ enumerations \( \langle x_\beta \rangle_{\beta<\alpha}, \langle b_\beta \rangle_{\beta<\alpha} \) of the sets \( |M_n|, |N_n| \), resp.; here, \( \nu = \lambda_k+n \) (unless either of those sets is empty, in which extreme case an obvious alternative is used). By induction on \( \beta < \nu \), we define the increasing chain \( \langle R^\beta \rangle_{\beta<\alpha} \) of \( \forall_\alpha \)-relations for \( M_{n+1} \) and \( N_{n+1} \). If \( \beta < \nu \), and the \( R^\gamma \) have been defined for \( \gamma < \nu \), then \( \hat{R} \overset{\text{def}}{=} R \cup \bigcup_{\gamma<\beta} R^\gamma \) is an \( \forall_\alpha \)-relation for \( M_{n+1} \) and \( N_{n+1} \). For \( \beta = \alpha \cdot 2 + 1 \), \( R^\beta \) is chosen as \( \hat{R} \cup \{(x, y)\} \) for a suitable \( y \), by the use of 5.2(i), with \( M_{n+1}, N_{n+1} \) and \( \hat{R} \) as \( M, N \), and \( R \), resp.; note that \( M_{n+1}, N_{n+1} \) are \( \nu^+ \)-saturated, and the domain and range of \( \hat{R} \) are of cardinality \( \leq \nu \). Similarly, if \( \beta = \alpha \cdot 2 + 1 \), \( R^\beta \) is chosen as \( \hat{R} \cup \{(a, b_\alpha)\} \) for a suitable \( a \), by applying 5.2(ii). This completes the definition of \( \langle R^\beta \rangle_{\beta<\alpha} \). We put \( R_n = \bigcup_{\beta<\alpha} R^\beta \). The construction ensures that \( R_n \) is as required.

Having defined the sequence \( \langle R_n \rangle_{n<\omega} \), we take the union \( R = \bigcup_{n<\omega} R_n \subseteq |M| \times |N| \). It is clear that \( R \) is an \( \forall_\alpha \)-relation for \( M \) and \( N \). Since the equality-formulas \( x = x' \) are in \( \forall_\alpha \), \( R \) is the graph of a function \( h \), and in fact, \( h = \bigcup_x h_x \), with \( h_x \) a function with domain and range included in \( M(X) \) and \( N(X) \), respectively, for each \( x \). Conditions (iii) and (iv) ensure that \( \text{dom}(h) = |M| \) and \( \text{range}(h_\alpha) = |N| \). Using that the atomic formulas are in \( \forall_\alpha \), we see that \( h \) defines a morphism \( h : M \to N \), which is in fact \( \alpha \)-surjective, by the previous sentence.

Proposition 5.1 just proved will be used in the next section in its present form. However, we note that it has the following consequence, of the form of a "preservation theorem", deduced by the standard methods (see [4, 5.2.13, p. 315]).

Corollary 5.3. If an \( L \)-sentence \( \sigma \) is preserved by all \( \alpha \)-surjective morphisms of \( L \)-structures, then \( \sigma \) is logically equivalent to an \( \forall_\alpha \)-sentence.

6. Kripke models with selected constant domains

Consider a category \( C \), and the functor category \( \text{Set}^C \). The latter is (among others) a coherent category, and its coherent structure is computed pointwise, that is, each
Observation 6.1. With $F$ c.s., $G$ any functor $\in \text{Set}^C$, and with the product projection $\pi : F \times G \to G$, $\forall_n(\Phi)$ is computed pointwise.

Proof. We claim that $\forall_n(\Phi)(C) = \forall_{nc}(\Phi(C))$ as subsets of $G(C)$. We have, for $x \in G(C)$,

$$x \in \forall_n(\Phi)(C) \iff \forall(C \to D). \forall y \in FD. (y, (Ff)x) \in \Phi D,$$

and $x \in \forall_{nc}(\Phi(C)) \iff \forall z \in FC. (z, x) \in \Phi C.$

Expression (6.2) is the instance of (6.1) with $f = 1_C$. But, assuming (6.2), to show (6.1), take any $C \to D$ and $y \in FD$; since $Ff$ is surjective, there is $z \in FC$ with $(Ff)z = y$; by (6.2), $(z, x) \in \Phi C$ and the functoriality of $\Phi$, $((Ff)z, (Ff)x) \in \Phi D$, that is, $(y, (Ff)x) \in \Phi D$, as desired.

As a consequence, every c.s. functor $F \in \text{Set}^C$ is a distributive object (see Section 4) in $\text{Set}^C$, that is, $\pi$ is a distributive arrow ($\forall_n$ is distributive), for any product projection $\pi : F \times G \to G$. Indeed, the set of evaluations $\epsilon_C : \text{Set}^C \to \text{Set} (C \in C)$ is a conservative family of coherent functors preserving each $\forall_n(\Phi)$, into $\text{Set}$ in which all $\forall_f$ are distributive; it follows that $\forall_n$ is distributive.

Suppose $H$ is a Heyting category, with a set $A$ of distinguished objects in $H$. We ask under what conditions we can have a conservative Heyting functor of the form $M : H \to \text{Set}^C$ so that, in addition, for each $A \in A$, $M(A)$ is a c.s. functor. An obvious necessary condition is that each $A \in A$ be a distributive object in $H$, since distributivity will be reflected through $M$ from $\text{Set}^C$. The essence of the next theorem is that the said condition is also sufficient.

Theorem 6.2. Let $H$ be a small coherent category, $A$ a set of distributive objects in $H$. Then there is a small (non-full) subcategory $K$ of $\text{Mod}(H)$ such that, with $e : H \to \text{Set}^K$ the evaluation, we have $e$ is conservative, conditionally Heyting, conditionally coHeyting, and for all $A \in A$, $e(A)$ is a componentwise surjective functor in $\text{Set}^K$.

Until the end of the proof of the theorem, we fix $H$ and $A$ as in the hypothesis of the theorem.

Given that $K$ is a subcategory of $\text{Mod}(H)$, the last requirement in the theorem is the same as to say that each $(h : M \to N) \in K$ is $A$-surjective, i.e., for each $A \in A$, the component $h_A : MA \to NA$ is a surjective function.

We obtain a necessary condition on objects $M \in K$, for $K$ to serve as in the theorem, as follows. Let $K \in K$, $A \in A$, $\pi : A \times X \to X$. Then $M = e_M \circ e$, and since, by the above
discussion, $e_M$ preserves $\forall_{(a)}$, $M : H \to \text{Set}$ preserves $\forall_x. M \in \text{Mod}(H)$ is called $A$-
standard{} if $M$ preserves $\forall_x$ whenever $\pi : A \times X \to X$ is a product projection, with $A \in A$; that is, for $\Phi \in S(A \times X)$,

$$M(\forall_x \Phi) = \{ x \in X : \forall a \in M(A). (a, x) \in \Phi \}. \quad (6.3)$$

Equivalently, $(6.3)$ can be written as

$$M \vDash \forall x \in X (\forall_x \Phi(x) \leftrightarrow \forall a \in A. \Phi(a, x)); \quad (6.3')$$

thus we see that being $A$-standard is a first-order condition, although expressing it requires stepping out of coherent logic.

We conclude that it is necessary to restrict the objects in $K$ to $A$-standard ones.

Using the diagram language of $M$, and transforming (the non-automatic part of) the condition, we see the $A$-standardness of $M$ means that

$$M^* \vDash \neg [\forall a] \varphi(a) \Rightarrow \exists a \in M(A). M^* \vDash \neg \varphi[a] \quad (6.3'')$$

holds for all $A \in A$, $a : A$, and all coherent (equivalently, all atomic) formulas $\varphi(a)$ over $L(M)$.

A morphism $h : M \to N$ in $\text{Mod}(C)$, $C$ any coherent category, is called pure if $h$ not
only preserves but also reflects all subobjects: for any $\Phi \in S(X)$, $x \in M(X)$, we have

$$h_x(x) \in N(\Phi) \iff x \in M(\Phi); \text{ that is, } x \notin M(\Phi) \Rightarrow h_x(x) \notin N(\Phi).$$

Let $\text{Diag}^+(M)$ denote the set of coherent sentences over $L(M)$ that are true in $M^*$. Let $\text{Diag}^-(M)$ denote the set of sentences $\neg \sigma$ over $L(M)$, with $\sigma$ coherent, that are true in $M^*$. It is clear that, with $N \in \text{Mod}(C)$, $(N, \hat{x})_{x \in |M|} \vDash \text{Diag}^+(M)$, respectively $(N, \hat{x})_{x \in |M|} \vDash \text{Diag}^-(M)$ iff $h : M \to N$, respectively $h : M \to N$ is pure, for $h$ defined by $h(x) = \hat{x}$ ($x \in |M|$). In the next three lemmas, $\lambda$ is any cardinal such that $\lambda \geq \#_{\text{int}}(H)$; $\#_{\text{int}}(H)$ is
the maximum of $\aleph_0$ and the cardinality of $\text{Arr}(H)$.

**Lemma 6.3.** Any $M \in \text{Mod}(H)$ of cardinality $\leq \lambda$ has a pure extension which is $A$-
standard{} and of cardinality $\leq \lambda$.

**Proof.** Let us show that any $M \in \text{Mod}(H)$ has a pure extension $M \to N$ that honors a single preassigned instance of $(6.3'')$ originating in $M$. In other words,

$$\text{if } a : A \in A, \varphi(a) \in L(M) \cup \{ a \} \text{ and } M^* \vDash \neg [\forall a] \varphi(a), \text{ then there are } M \to N \text{ and } a \in N(A) \text{ such that } (N, h(x))_{x \in |M|} \vDash \neg \varphi[a]. \quad (6.4)$$

Once $(6.4)$ is shown, a straightforward argument involving directed colimits will establish the lemma; note that a directed colimit of pure morphisms is pure.

Assume the hypotheses in $(6.4)$. It suffices to show that

$$\Sigma_H \cup \text{Diag}^+(M) \cup \text{Diag}^-(M) \cup \{ \neg \varphi(a) \}$$

is $L(M) \cup \{ a \}$-consistent. By compactness and the fact that both $\text{Diag}^+(M)$ and $\text{Diag}^-(M)$ are closed (up to logical equivalence) under finite conjunction, it is enough
to take $\rho \in \text{Diag}^+(M)$ and $-\sigma \in \text{Diag}^-(M)$ and see that

$$\Sigma_M \cup \{ \rho \land -\sigma \land -\varphi(a) \}$$

is $L_M(\overline{u})$-consistent where $\overline{u}$ is a repetition-free tuple of $M$-constants containing all those occurring in (6.5). Expression (6.5) inconsistent means that $\Sigma_M \vdash \rho \Rightarrow \sigma \lor \varphi(a)$; that is,

$$[\overline{a}; \rho] \leq [\overline{a}; \sigma \lor \varphi(a)].$$

It follows that

$$[\overline{u}; \rho] \leq [\overline{u}; \forall a(\sigma \lor \varphi(a))].$$

Since $\sigma$ does not contain $a$, the distributivity of $\land$ says that

$$[\overline{u}; \forall a(\sigma \lor \varphi(a))] = [\overline{u}; \sigma \lor \forall a \varphi(a)] = [\overline{u}; \sigma] \lor [\overline{u}; \forall a \varphi(a)].$$

and thus

$$[\overline{u}; \rho] \leq [\overline{u}; \sigma] \lor [\overline{u}; \forall a \varphi(a)].$$

However, the tuple $\overline{u}$ of elements of $M$ belongs to the first, but does not belong to either the second or the third subobject when interpreted in $M$; the first two of these facts hold because the choice of $\rho$ and $\sigma$; the third because of the hypothesis in (6.4). This is a contradiction. \qed

Let us make the connection between the concept of $\forall_A$-formula of Section 5 and that of $A$-standard model. By an $\forall_A$-formula, we mean one over the language $L_M$, with $A \subseteq L_M$ the set of objects fixed above. If $\varphi(\overline{x})$ is a $\forall_A$-formula, then we have its internal meaning $[\overline{x}; \varphi] \in S([\overline{x}])$; this uses the Heyting structure of $H$ to the extent that that structure is assumed in the definition of "$A \in A$ distributive"; $\varphi$ is not necessarily coherent. I claim that an $A$-standard model $M \in \text{Mod}(H)$ preserves the meaning of $\varphi$:

$$M([\overline{x}; \varphi]) = M(\forall_A \varphi);$$

(6.6)

here, the latter expression is the meaning of the formula $\varphi$ in full first-order logic over $L_M$ in the $L_M$-structure $M$. The proof is by induction on the complexity of $\varphi$. For atomic $\varphi$, as well as for those obtained by coherent operations, the inductive steps are obvious. The only remaining case is when $\varphi := \forall a \psi, a \in A$. But, with $X = [\overline{x}]$, $\Phi = [a \overline{x}; \psi], \pi : A \times X \to X$ the product projection, we have

$$M([\overline{x}; \varphi]) = M(\forall_A \Phi),$$

and

$$M(\overline{x}; \varphi) = \{ \overline{x} \in M(X) : \forall a \in M(A), M \vdash \psi[a, \overline{x}] \} = \{ \overline{x} \in M(X) : \forall a \in M(A), (a, \overline{x}) \in M(\Phi) \},$$

the last equality being the induction hypothesis; the desired equality (6.6) follows from (6.3). Let us note that the left-in-right inclusion in (6.6) is always true, without the $A$-standardness assumption on $M$. 

The last-stated fact enables us to restate Proposition 5.1. Let us emphasize that "\( M \in \text{Mod}(H) \) is special" means that the \( L_H \)-structure \( M \) is special in the sense of Section 5; thus, this concept refers to the full first-order logic over \( L_H \).

**Lemma 6.4.** Suppose \( M, N \) are \( \lambda \)-special models of \( H \), \( M \) \( A \)-standard, \( X \in H \), \( x \in M(X) \), \( y \in N(X) \), and for each \( \Phi \in S(X) \), \( x \in M(\Phi) \) implies \( y \in N(\Phi) \). Then there is an \( A \)-surjective map \( h: M \to N \) such that \( h(x) = y \).

**Proof.** As we noted in Section 5, it follows that the \( L_H \cup \{x\} \)-structures \((M,x),(N,y)\) are \( \lambda \)-special themselves. With any \( \forall_A \)-formula \( \varphi(x) \) over \( L_H \cup \{x\} \), we have \( \Phi = [x: \varphi] \in S(X); (M,x) \models \varphi(x) \Rightarrow x \in M(\Phi) \Rightarrow y \in N(\Phi) \Rightarrow (N,y) \models \varphi(x) \), where the first and third implications are by the remarks preceding the lemma. Thus, \((M,x)\) and \((N,y)\) satisfy the hypotheses of Proposition 5.1. The conclusion follows. \( \square \)

**Lemma 6.5.** Every \( M \in \text{Mod}(H) \) of cardinality \( \leq \lambda \) has a pure special \( A \)-standard extension in \( \text{Mod}(H) \).

**Proof.** By Lemma 6.3, \( M \) has a pure \( A \)-standard extension \( N \) of power \( \leq \lambda \). By the existence theorem for special structures (see Section 5), there is a \( \lambda \)-special \( N' \) which is an elementary extension of \( N \). Since \( N' \equiv N \), \( N' \) is \( A \)-standard as well (see the remark following (6.3') above). An elementary extension is clearly a pure extension; and the composite of two pure maps is pure. \( N' \) is the desired extension of \( M \). \( \square \)

**Proof of Theorem 6.2.** Let us choose and fix any \( \lambda \geq \#_\text{inf}(H) \). We take \( K \) to be the subcategory of \( \text{Mod}(H) \) whose objects are the \( \lambda \)-special \( A \)-standard models of \( H \), and whose arrows are the \( A \)-surjective maps between the latter. Thus, the last requirement on \( e \) holds true. The fact that \( e \) is conservative follows from the fact that \( H \) has enough models of power \( < \lambda \) (which, in turn, is a consequence of the Gödel completeness theorem, and the downward Löwenheim–Skolem theorem), and from Lemma 6.5. Let us see that \( e \) is (conditionally) Heyting. According to (3.5), we need to have the following: given \( M \in K \), \( X \overset{f}{\to} Y \in H \), \( \Phi \in S(X) \) and \( y \in M(Y) - M(\forall f(\Phi)) \) (assuming \( \forall f(\Phi) \) exists), there are \( h: M \to N \) in \( K \) and \( x \in N(X) - N(\Phi) \) such that \( h_Y(y) = (Nf)(x) \). With the given data, consider the following set of \( L_H \cup \{x\} \)-sentences:

\[
\Xi = \text{def} \quad \Sigma_H \cup \{ \Psi(f(x)) : \Psi \in S(Y), y \in M(\Psi) \} \cup \{ \neg \Phi(x) \}.
\]

If \( \Xi \) were not consistent, by taking a finite meet of \( \Psi \)'s, we would have \( \Psi \in S(Y) \) such that \( y \in M(\Psi) \) and \( [x: \Psi(f(x))] \leq [x: \Phi(x)] \), that is, \( f^*(\Psi) \leq \Phi \); hence also \( \Psi \leq \forall f(\Phi) \); however, this contradicts \( y \in M(\Psi) \) and \( y \notin M(\forall f(\Phi)) \). Thus, \( \Xi \) is consistent; it has a model, say \((N_0, x_0)\), of cardinality \( \leq \lambda \). By Lemma 6.5, let \((N, x)\) be a \( \lambda \)-special \( A \)-standard model of \( H \) purely extending \((N_0, x_0)\). Then, by inspecting \( \Xi \), we see that \((N, x)\) is a model of \( \Xi \). Also, \( N \in K \). By the definition of \( \Xi \), we have that \( M \) and \( N \) satisfy the hypotheses of Lemma 6.4, with \( y \) and \((Nf)x\) playing the roles of \( x \) and \( y \), respectively.
respectively. Hence, there is \( h: M \to N \in K \) such that \( h_1(y) = (Nf)(x) \). Also by the definition of \( \Xi \), \( x \in N(X) - N(\Phi) \). This completes the proof that \( e \) is Heyting.

The proof that \( e \) is coHeyting is similar. Now, we have \( M \in K, \Phi, \Psi \in S(X), \Psi \setminus \Phi \) exists, and \( x \in M(\Psi \setminus \Phi) \); and we want \( h: N \to M \in K \) with \( y \in N(\Psi) - N(\Phi) \) such that \( h_1(y) = x \). We let

\[
\Xi = \Sigma_\Phi \uplus \{ \neg \Gamma(x): \Gamma \in S(X), x \notin M(\Gamma) \} \cup \{ \Psi(x), \neg \Phi(x) \}.
\]

If \( \Xi \) were inconsistent, we would have \( \Gamma \in S(X) \) with \( x \notin M(\Gamma) \) such that \( \Sigma_\Phi \models \neg \Gamma(x) \Rightarrow \neg \Psi(x) \vee \Phi(x) \), i.e., \( \Sigma_\Phi \models \Psi(x) \Rightarrow \neg \Gamma(x) \vee \Phi(x) \), that is, \( \Psi \leq \Gamma \vee \Phi \); as a consequence, \( \Psi \setminus \Phi \leq \Gamma \), which contradicts \( x \in M(\Psi \setminus \Phi) \) and \( x \notin M(\Gamma) \). As before, any \( \lambda \)-special \( A \)-standard \( (N, y) \) satisfying \( \Xi \) will do what we want. This completes the proof of the theorem. \( \Box \)

Before we give a version of the theorem for "modal logic", we clarify what happens in the relevant standard structures. Let \( \varphi: J \to K \) be a functor, \( \varphi^*: \text{Set}^k \to \text{Set}^J \) the corresponding premodal structure; let \( A \in \text{Set}^J \) be a componentwise surjective functor; hence, in particular, \( A \) is a distributive object in \( \text{Set}^k \). It immediately follows that \( \varphi^*(A) = A \circ \varphi \) is also componentwise surjective (in \( \text{Set}^J \)), since all components of \( \varphi^*(A) \) are components of \( A \). Hence, as our first conclusion, we see that \( \varphi^*(A) \) is a distributive object in \( \text{Set}^J \). But also, considering the equalities

\[
\text{Set}^k \xrightarrow{\varphi^*} \text{Set}^J \xrightarrow{\langle J \rangle} \text{Set} = \text{Set}^k \xrightarrow{\langle qJ \rangle} \text{Set},
\]

one for each \( J \in J \), with \( \langle J \rangle \) the projection, we see each \( \varphi^* \circ \langle J \rangle \) preserves \( \forall_x \), for any product projection \( A \times X \to X \), hence, since the \( \langle J \rangle \) form a conservative family, \( \varphi^* \) itself preserves \( \forall_x \), this is our second conclusion. The two conclusions lead us to make the assumptions of the following theorem, which is, in fact, a generalization of Theorem 6.2, from one category to two (take \( \langle \varphi \rangle \) of the next theorem to be an identity functor to get the previous result).

**Theorem 6.6.** Assume that \( \varphi: H \to D \) is a coherent functor between small coherent categories, \( A \) is a set of distributive objects in \( H \) satisfying the following two conditions:

(a) for each \( A \in A \), \( A \) is distributive in \( D \);
(b) each \( A \in A \) is a Barcan object for \( \varphi \).

Then there are subcategories \( J, K \) of \( \text{Mod}(D), \text{Mod}(H) \), respectively, such that the induced functor \( \cdot^*: \text{Mod}(D) \to \text{Mod}(H) \) restricts to a functor \( \varphi: J \to K \); both components of the evaluation morphism

\[
D \xrightarrow{\varepsilon_D} \text{Set}' \xrightarrow{\varphi^*} \text{Set}^J \xrightarrow{\langle J \rangle} \text{Set} = \text{Set}^k \xrightarrow{\langle qJ \rangle} \text{Set},
\]

one for each \( J \), with \( \langle J \rangle \) the projection, we see each \( \varphi^* \circ \langle J \rangle \) preserves \( \forall_x \), for any product projection \( A \times X \to X \), hence, since the \( \langle J \rangle \) form a conservative family, \( \varphi^* \) itself preserves \( \forall_x \), this is our second conclusion. The two conclusions lead us to make the assumptions of the following theorem, which is, in fact, a generalization of Theorem 6.2, from one category to two (take \( \langle \varphi \rangle \) of the next theorem to be an identity functor to get the previous result).
satisfy Theorem 6.2, in particular, they are conservative; and
\[ e = (e_D, e_H) \] is conditionally S4 and coS4.

Proof. Now, let us take \( \lambda \geq \max(\#_{\inf}(D), \#_{\inf}(H)) \). We define \( K \) and \( J \) similarly to \( K \) as in the previous theorem: \( J \) is the subcategory of \( \text{Mod}(D) \) with objects the \( \lambda \)-special \( A \)-standard models of \( D \), with \( A = \{ A : A \in A \} \), and with morphisms the \( A \)-surjective arrows; \( K \) similarly for \( H \), but making sure that we use the same \( \lambda \). Because of assumption (b), we obviously have that for \( M \in \text{Mod}(D) \) \( A \)-standard, \( M \upharpoonright H = M \circ (\cdot) \) is \( A \)-standard; thus, \( \phi \) is indeed well-defined on objects. The fact that \( \ast \) takes an \( A \)-surjective arrow to an \( A \)-surjective one is clear; hence, \( \phi \) is well-defined on arrows.

Assumption (a) says that all objects in \( A \) are distributive (in \( D \)). Since \( K \) and \( J \) are chosen here for \( H \) and \( D \), resp., as \( K \) is in the proof of Theorem 6.2, it follows that \( e_H, e_D \) satisfy the conclusions of that theorem. It remains to verify the biS4 character of \( e \).

To deal with \( \Box \) according to (3.14), we let \( X \in H, \Phi \in S(\hat{X}), \Box \Phi \in S(X), M \in K, x \in M(X) \sim M(\Box \Phi) \); we want \( N \in J \) and \( h : M \to N \upharpoonright H \) such that \( h_X(x) \not\in N(\Phi) \) (note that \( N(\Phi) \subset N(\hat{X}) = (N \upharpoonright H)(X) \)). To this end, consider the set
\[ E = \Sigma_D \cup \{ \forall \Phi(x) : \Phi \in S(X), x \in M(\Psi) \} \cup \{ \neg \Phi(x) \}; \quad (x : X) \]
of sentences over \( L_D \cup \{ X \} \). Suppose \( E \) is inconsistent; then, \( \exists \Phi \leq \Phi \) for some \( \Psi \in S(X) \) such that \( x \in M(\Psi) \); thus, \( \exists \Phi \leq \Box \Phi \), which contradicts \( x \in M(\Psi) \) and \( x \not\in M(\Box \Phi) \); \( E \) is consistent. By also using Lemma 6.5, let \( (N, y) \) be an \( A \)-standard \( \lambda \)-special model of \( E \). Then, \( (N \upharpoonright H, y) \) is an \( A \)-standard \( \lambda \)-special model of \( H \). Notice that the hypotheses of Lemma 6.4 are satisfied, with \( (M, x) \) and \( (N \upharpoonright H, y) \) as \( M \) and \( N \), resp; this is ensured by the fact that \( (N, y) \) satisfies the middle term of \( E \). Thus, by Lemma 6.4 we get \( h \) as desired; \( h_X(x) \not\in N(\Phi) \) is also part of \( E \).

Next, we turn to \( \Diamond \); see (3.15). Let \( X \in H, \Phi \in S(\hat{X}), \Diamond \Phi \in S(X), x \in M(\Diamond \Phi) \); we want \( N \in J, h : N \upharpoonright H \to M \) and \( y \in N(\Phi) \subset N(\hat{X}) \) such that \( h_X(y) = x \). Consider the set
\[ E = \Sigma_D \cup \{ \exists \Psi(x) : \Psi \in S(X), x \notin M(\Psi) \} \cup \{ \exists \Phi(x) \}; \quad (x : X) \]
Suppose \( E \) is \( L_D \cup \{ X \} \)-inconsistent; then \( \exists \Phi \leq \exists \Psi \) for some \( \Psi \in S(X) \) such that \( x \notin M(\Psi) \); thus, \( \exists \Phi \leq \Psi \), which contradicts \( x \in M(\exists \Phi) \) and \( x \notin M(\Psi) \); \( E \) is consistent. The rest of the proof is as in the previous proof for \( \Box \). \( \Box \)

We draw conclusions from Theorems 6.2 and 6.6 for four (families of) doctrines that are introduced next; with the natural forgetful functors, they form the following diagram:

\[
\begin{array}{ccc}
B-\text{biS4} & \rightarrow & B-\text{S4} \\
\downarrow & & \uparrow \\
B-\text{biH} & \rightarrow & B-\text{H}
\end{array}
\] (6.7)
Here $\mathcal{B}$ is a fixed abstract set; we could once and for all fix it to be $\mathbb{N}$, and we would get all the cases of interest. If $\mathcal{B} = \emptyset$, we get the doctrines of Section 3; what we say below will complete the proofs of the assertions made in Section 3. We start by defining $\mathcal{B}$-H.

A $\mathcal{B}$-Heyting category $(\mathcal{B} \to \mathcal{H})$ is a Heyting category $\mathcal{H}$ together with a map $\mathcal{B} \to \mathcal{H}$ that maps every $B \in \mathcal{B}$ to a distributive object in $\mathcal{H}$. The $\mathcal{B}$-Heyting categories form the doctrine (bicategory) $\mathcal{B}$-$\mathcal{H}$. An arrow $(h, i):(\mathcal{B} \to \mathcal{H}) \to (\mathcal{B}' \to \mathcal{H}')$ in $\mathcal{B}$-$\mathcal{H}$ is a Heyting functor $\mathcal{H} \to \mathcal{H}'$ making the triangle

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{h} & \mathcal{H}' \\
\mathcal{B} & \xrightarrow{b} & \mathcal{B}' \\
\end{array}
\]

commute up to the specified isomorphism $i$. A 2-cell $v:(h, i) \to (h', i')$, both arrows $(h, i), (h', i')$ from $(\mathcal{B} \to \mathcal{H})$ to $(\mathcal{B}' \to \mathcal{H}')$, is any $v:h \cong h'$ compatible with the $i$'s; that is,

\[
\begin{array}{ccc}
hb & \xrightarrow{v} & h'b \\
& i & \xrightarrow{z} & i' \\
& b & \xrightarrow{b} & b' \\
\end{array}
\]

(Usually, all those isomorphisms are identities.)

$\mathcal{B}$-bi$\mathcal{H}$ is defined similarly; now, the $\mathcal{H}$ and the $h$ are to be biHeyting. The condition on the objects $bB$ in the image of $\mathcal{B}$ is now equivalent to saying that each $\pi^*, \pi$ a product projection $\pi:(bB) \times X \to X$, preserves $\setminus$. The standard objects in $\mathcal{B}$-$\mathcal{H}$ as well as $\mathcal{B}$-bi$\mathcal{H}$ are the ones of the form $\mathcal{B} \to \text{Set}^K$, $K$ any small category, with each $B \in \mathcal{B}$ mapped to a componentwise surjective functor.

A $\mathcal{B}$-$\mathcal{S}4$ category is an $\mathcal{S}4$ category $(\mathcal{H} \to \mathcal{D})$ with an indexing map $b: \mathcal{B} \to \mathcal{H}$ such that, for every $B \in \mathcal{B}$, we have that $\cdot$ preserves $\forall, \exists$ for all product projections $\pi:(bB) \times X \to X$, that is, such that every $bB (B \in \mathcal{B})$ is a Barcan object (see Section 4). As we know (4.23), it follows that $bB$ is a distributive object in $\mathcal{H}$. The doctrine $\mathcal{B}$-$\mathcal{S}4$ is defined in the (by now) obvious way; we have a forgetful functor $\mathcal{B}$-$\mathcal{S}4 \to \mathcal{S}4$. What we said in the penultimate sentence tells us that we have a forgetful functor $\mathcal{B}$-$\mathcal{S}4 \to \mathcal{B}$-$\mathcal{H}$ as well; this is the right vertical in (6.7).

The specification of $\mathcal{B}$-bi$\mathcal{S}4$ is clear. Now, the condition on the objects $bB$ in the image of $\mathcal{B}$ is equivalent to saying that each $\pi^*, \pi$ a product projection $\pi:(bB) \times X \to X$, commutes with $\cup$. The structures of the form $\mathcal{B} \to \text{Set}^K \xrightarrow{\rho} \text{Set}^{[K]}$, with an arbitrary (small) category $K$, $\rho$ the canonical "restriction", and all $B \in \mathcal{B}$ mapped to componentwise surjective functors, are the standard $\mathcal{B}$-$\mathcal{S}4$ as well as the standard $\mathcal{B}$-bi$\mathcal{S}4$ categories.

**Theorem 6.7.** In each doctrine in (6.7), the named standard objects are small-representative.

**Proof.** For $\mathcal{B}$-$\mathcal{H}$ and $\mathcal{B}$-bi$\mathcal{H}$, the result is immediate from Theorem 2. For $\mathcal{B}$-$\mathcal{S}4$ and $\mathcal{B}$-bi$\mathcal{S}4$, note that the assumption (a) of Theorem 6.6 are satisfied for any object in
in a Boolean category, every object is distributive. Condition 6.6(b) is built into the definition of B-(bi)S4. Now, the result follows from Theorem 6.6, just like (3.22) did from (3.13); note that, in (3.19), obviously, $\xi^*$ maps every componentwise surjective functor to another such.

**Corollary 6.8.** The enrichments along the forgetful functors in (6.7) are all conservative. In particular, there is a faithful interpretation of intuitionistic logic with the axiom of constant domains required for quantification over selected sorts, in S4 modal logic with Barcan's formula required for quantification over the same sorts.

7. Constant and variable sets

Let $K$ be a (small) category; the functor-(presheaf-)category $\text{Set}^K$, a Grothendieck topos, is an often cited example for a universe of variable sets. The domain of variations is $K$; an object of $\text{Set}^K$, a functor $K \to \text{Set}$ is a set variable, or parametrized, over $K$. The Kripke/Joyal theory of intuitionistic and modal logic uses this kind of universe of variable sets to model these logics. We distinguish the constant sets among the variable ones, and ask what logical properties the constant ones have in relation to the general variable ones. The constant sets, among the variable ones, are the constant functors: those of the form $\Delta(S): K \to \text{Set}$, with $S \in \text{Set}$, for which $\Delta(S)(K) = S$ and $\Delta(S)(k) = 1_S$ for all $K \in K$ and $k \in \text{Arr}(K)$. In fact, we have an obvious functor $\Delta: \text{Set} \to \text{Set}^K$ picking out the constant functors in $\text{Set}^K$. $\Delta$ is a coherent functor from a Boolean category into a Heyting category. Can we say more? The crucial observation is that

for every $f \in \text{Arr}(\text{Set})$, $\Delta(f)$ is stably distributive in $\text{Set}^K$.

Let us formulate the last assertion slightly more generally. Consider a functor $\phi: K \to G$, and assume that $G$ is a groupoid. Then we have the induced functor $\phi^*: \text{Set}^G \to \text{Set}^K$, and $\text{Set}^G$ is Boolean. Our first context had $G = 1$, the terminal category. Then, under the above conditions,

for every $f \in \text{Arr}(\text{Set}^G)$, $\phi^*(f)$ is stably distributive in $\text{Set}^K$.

Indeed, the assertion includes that for each object $A$ in $\text{Set}^G$, $\phi^*(A)$ is distributive in $\text{Set}^K$; this follows from the fact $\phi^*(A)$ is componentwise surjective (in fact, it is componentwise bijective) (see Observation 6.1). Given that $\text{Set}^G$ is Boolean, the assertion follows from (4.16).

We have a doctrine at our hands, to be called the doctrine of constant vs. variable sets, a particular locally finitely presentable 2-category, denoted $\text{CV-H}$. An object $(B \rightharpoonup H)$ of $\text{CV-H}$, a CV-Heyting category, is given by a Boolean category $B$, a Heyting category $H$, and a coherent (hence, also Heyting) functor ( ) such that, for every $g \in \text{Arr}(B)$, $\hat{g}$ is a stably distributive arrow in $H$. A morphism $(B \rightharpoonup H) \rightarrow (B' \rightharpoonup H')$ in the doctrine is, naturally, a pair $(F: B \to B', G: H \to H')$ of Heyting functors, together
with a specified isomorphism $\theta$, making the square

$$
\begin{array}{ccc}
H & \overset{\theta}{\rightarrow} & H' \\
\downarrow & \cong & \downarrow \\
B & \overset{\phi}{\rightarrow} & B'
\end{array}
$$

(7.1)

commute up to the isomorphism $\theta$ (most of the time, $\theta$ will be the identity). To complete the definition of the doctrine, 2-cells are taken to be appropriate pairs of natural isomorphisms.

The arrow (7.1) is conservative if both $F$ and $G$ are conservative.

Our $(A : \text{Set} \rightarrow \text{Set}^k)$, or more generally, the $\varphi^*: \text{Set}^G \rightarrow \text{Set}^k$ above, are objects of CV-H.

In the definition above of "CV-Heyting category", requiring additionally the category $H$ and the morphism $G$ to be biHeyting gives the notion "CV-biHeyting category" and the doctrine CV-biH. The structures singled out last are in CV-biH as well. Notice that in the definition of a CV-biH category, the condition of distributivity is equivalent to saying that for every $f \in \text{Arr}(B)$, $f^*$ preserves coimplication ($\setminus$).

The main result in this section is the following theorem.

**Theorem 7.1.** The class of objects of the form $(A : \text{Set} \rightarrow \text{Set}^k)$, with $K$ any small category, is (small-)representative in CV-H, as well as in CV-biH. That is, for any CV-(bi)Heyting-category $\mathcal{B} = (B \rightarrow H)$ there are a small set $I$, small categories (in fact, preorders) $P_i (i \in I)$, and a conservative CV-(bi)Heyting map

$$
\begin{array}{ccc}
H & \longrightarrow & \text{Set}^{\prod_{i \in I} P_i} \\
\downarrow & \cong & \downarrow \\
B & \longrightarrow & \text{Set}^{I}
\end{array}
$$

First, a slight variant.

**Theorem 7.1’.** For any small CV-(bi)Heyting object $\mathcal{B} = (B \rightarrow H)$, there are $\varphi : K \rightarrow G$, a functor into a groupoid, and a conservative CV-(bi)Heyting map

$$
\begin{array}{ccc}
H & \longrightarrow & \text{Set}^G \\
\downarrow & \cong & \downarrow \\
B & \longrightarrow & \text{Set}^{I}
\end{array}
$$

Proof of Theorem 7.1’. Let $A$ denote the set $\{ \hat{A} : B \in B \}$ of objects in $H$; each $A \in A$ is distributive. By Theorem 6.2, we have a subcategory $K$ of $\text{Mod}(H)$ such that the evaluation $e_H : H \rightarrow \text{Set}^k$ is conservative, (bi)Heyting, and for all $A \in A$ and $M \rightarrow N$, $h_A$ is a surjective function. We claim that each $h_A$ is in fact bijective. The reason is that with $A = \hat{B}$, $=_{\hat{B}} \in S(B \times B)$ has a Boolean complement $\neq_{\hat{B}}$ (in $B$), and thus $=_{\hat{A}} \in S(A \times A)$ has a Boolean complement $\neq_{\hat{A}} = (\neq_{\hat{B}})^*$; since $h_A$ preserves $\neq_{\hat{A}}$ (use the component $h_{\hat{A}}$), it follows that $h_A$ is injective. Let $G$ be the groupoid $\text{Mod}^*(B)$ of all models of $B$; composition with $\cdot$ defines a (reduct) functor $\cdot^*: K \rightarrow G$. We have the
Proof of Theorem 7.1. Let us start with the data given by Theorem 7.1'. By (3.7), let \( P \) be a tree (bush), and \( \rho: P \to K \) be upward (two-way) surjective. Observe the fact that \( \varphi \circ \rho: P \to G \) as any functor from a bush into a groupoid is isomorphic to a functor constant on connected components. In fact, with \( I \) the set of connected components of \( P \), with \( \psi: P \to I \) the quotient map, and with \( \theta: I \to G \) defined by putting \( \theta(i) = \varphi(p_i) \) for \( i \in I \) with \( p_i \) any selected element of the component \( i \), we have that \( \tau: \varphi \circ \rho \cong \theta \circ \psi \); \( \iota \) will exist because in any component \( i = [p_i] = [p] \), there is a unique reduced path from \( p_i \) to \( p \); in fact, \( \iota \) is uniquely determined once its components \( \iota_p \) are (arbitrarily) fixed, for all \( i \in I \). Modify \( \iota \) by adding elements to it to make \( \theta \) surjective an objects. We have the square

\[
\begin{array}{ccc}
K & \to & P \\
\rho \downarrow & & \downarrow \psi \\
G & \to & I
\end{array}
\]

which gives rise, by (3.6), to the conservative CV-(bi)H morphism

\[
\begin{array}{ccc}
\text{Set}^k & \to & \text{Set}^\rho \\
\leftarrow & \cong & \leftarrow \\
\text{Set}^G & \to & \text{Set}^I
\end{array}
\]

Note that the CV-biHeyting object \( \text{Set}^I \to \text{Set}^\rho \) is isomorphic to the product of the \( \Delta_i: \text{Set} \to \text{Set}^{P_i} \) (i.e., with \( P_i \) the fiber over \( i \), empty if \( i \) is an “additional” element). Composing the CV-(bi)H map given in Theorem 7.1’ with the last one gives the desired CV-(bi)H map. □

To complete the picture, let us mention CV-S4 and CV-biS4 categories. A CV-(bi)S4 category is a structure of the form \( B \to H \to \Delta \) consisting of objects and morphism of Coh in which \( B \) is Boolean, \( H \to \Delta \in (bi)S4 \), and in which for all arrows \( f \) in \( B \), \( \forall_j \) satisfies Barcan’s formula (see (4.20)). In the CV-biS4 case, the last condition is to say that \( \circ \) is substitutive with respect to substitutions along arrows \( \bar{f}, f \in \text{Arr}(B) \). The definitions of the doctrines CV-S4 and CV-biS4 are as expected. The standard objects in CV-(bi)S4 are the ones of the form \( \text{Set}^A \to \text{Set}^B \to \text{Set}^{[\lambda]} \). The four
CV-doctrines are connected by forgetful functors:

\[
\begin{array}{ccc}
\text{CV-biS}_4 & \longrightarrow & \text{CV-S}_4 \\
\downarrow & & \downarrow \\
\text{CV-biH} & \longrightarrow & \text{CV-H}
\end{array}
\]  

(7.2)

**Theorem 7.2.** In each doctrine in (7.2), the designated standard objects are small-representative. As a consequence, each functor (7.2) gives its domain as a conservative enrichment of the codomain.

The result is related to the completeness theorem for S4 predicate logic with constant domains; see [13, 9.8, p. 176]. In [19, 4.13, p. 353], is a related result, with more structure in the logic, and with explicit topos-theoretic semantics. In both sources, symbolic logic is interpreted via Kripke semantics, resp. in categories, and traditional proof-systems are used. Since in both of these contexts, "all sorts are constant", or, translated into our context, every object is a subobject of a constant object (compare (3.23)), the proofs of these earlier variants of our result are essentially easier. On this last point, see also Section 9.

8. Theories satisfying Lawvere's condition

We start by introducing a square of four doctrines, with connecting forgetful functors:

\[
\begin{array}{ccc}
\text{L-biS}_4 & \longrightarrow & \text{L-S}_4 \\
\downarrow & & \downarrow \\
\text{L-biH} & \longrightarrow & \text{L-H}
\end{array}
\]  

(8.1)

The objects of L-H, resp. L-biH, are the Heyting, resp. biHeyting, categories satisfying Lawvere's condition (L) (see Section 4): for each product projection \(\pi\), \(\forall_n\) is distributive. Arrows and 2-cells are as in H, resp biH: e.g., L-H is a full and 2-full subbicategory of H. The objects of L-S4 are the S4 categories \((\cdot:\text{H} \to D)\) in which \(\cdot\) preserves \(\forall_n\) for all product projections \(\pi\) in H, that is, all objects are Barcan; the arrows and 2-cells of L-S4 are those of S4: L-S4 is a full and 2-full subbicategory of S4. By (4.23), as a consequence, each \(\forall_n\) is distributive (in H; in other words, the axiom of constant domains holds for \(\forall_n\)); it follows that the mapping \((\cdot:\text{H} \to D) \mapsto \text{H}\) defines a forgetful functor L-S4 \(\to\) L-H. L-biS4 is defined in the expected manner; note that by (4.17), the (new) condition is the same as that \(\circ\) be invariant under substitution along product projections. The main result of this section is the following theorem.

**Theorem 8.1.** The enrichments along the forgetful functors in (8.1) are all conservative.
As before, we deduce this result from appropriate completeness theorems. The completeness results are no longer of “Kripke-type”; the target categories are not presheaf categories. However, they are still Grothendieck toposes, in fact prime-generated (see Section 3) ones. The main tool is the first author’s construction of the “topos of types”, a categorical generalization of the “prime-generated hull” explained in the first section. Another tool is an embedding theorem due to the authors of this paper (in their [29]), which is a sharpening of M. Barr’s Boolean embedding theorem for Grothendieck toposes. Considering the completeness theorems of this section merely as tools to show the result of Theorem 8.1, a statement of pure logic, one still sees how essential the use of toposes are. For instance, the adjoints necessary to produce the right structures in the target categories are given by basic properties of Grothendieck toposes stemming from their infinitary structure.

**Theorem 8.2.** Let $H$ be a small $L$-Heyting category. Then there is a prime-generated (hence, biHeyting) Grothendieck topos $\mathcal{E}$ satisfying condition $(L)$ and a conservative, Heyting and conditionally coHeyting functor $H \to \mathcal{E}$.

$\mathcal{E}$ will be constructed as the topos of types of $H$ introduced in [26]. We now recall the definition and the properties needed.

For a while, let $H$ denote an arbitrary small coherent category.

Let $X \in H$. A filter on $X$ is a set $F \subseteq S(X)$ of subobjects of $X$ closed under finite intersection (in particular, $1_X \in F$) and closed upward. The set of filters on $X$ is denoted $\mathcal{F}(X)$. For $F \in \mathcal{F}(X)$, and $f: X \to Y$, $\exists_f(F)$ is defined to be

$$\exists_f(F) = \{ B \in S(Y); f^* B \in F \} \subseteq S(Y);$$

it is immediate that $\exists_f(F)$ is a filter on $Y$. Suppose $\mathcal{E}$ is a coherent category in which the subobject lattices are complete; e.g., $\mathcal{E}$ is a Grothendieck topos. A $p$-model of $H$ in $\mathcal{E}$ is any model $M$ in $\mathcal{E}$ (a coherent functor $M: H \to \mathcal{E}$) satisfying

$$M \models \bigwedge_{x \in F} B(y) \Rightarrow \exists x \bigwedge_{A \in F} (f(x) = r Y * A(x))$$

for all $X \xrightarrow{f} Y \in H$ and $F \in \mathcal{F}(X)$. $p\text{-Mod}_\mathcal{E}(H)$ denotes the category of all $p$-models of $H$ in $\mathcal{E}$; $p\text{-Mod}_\mathcal{E}(H)$ is a full subcategory of $\text{Mod}_\mathcal{E}(H)$.

A $\bigwedge$-geometric functor between Grothendieck toposes is a geometric functor (the inverse image part of a geometric morphism) preserving, in addition, all intersections of subobjects; $\bigwedge$-geo($\mathcal{E}$, $\mathcal{F}$) denotes the category of all $\bigwedge$-geometric functors $\mathcal{E} \to \mathcal{F}$; $\bigwedge$-geo($\mathcal{E}$, $\mathcal{F}$) is a full subcategory of geo($\mathcal{E}$, $\mathcal{F}$). Any $p$-model $M: H \to \mathcal{E}$ induces, by composition, a functor

$$M^*: \bigwedge\text{-geo}(\mathcal{E}, \mathcal{F}) \to p\text{-Mod}_\mathcal{E}(H).$$

The topos of types, or prime-generated hull, $\mathcal{E}$ of $H$ is defined as a prime-generated Grothendieck topos $\mathcal{E}$ with a $p$-model $M: H \to \mathcal{E}$ satisfying the following universal
property: for any prime generated Grothendieck topos $\mathcal{F}$, $M^*$ in (8.3) is an equivalence of categories. We write $\mathcal{F}(H)$ for the topos of types of $H$, and

$$\cdot : H \to \mathcal{F}(H)$$

for the canonical $p$-model of $H$ in $\mathcal{F}(H)$. In [26], the existence and various properties of the topos of types are proved; the definition through a universal property ensures the uniqueness of the topos of types up to an equivalence. Note that the fact that $\cdot^*$ is an equivalence contains the statement that for any $M \in p\text{-Mod}_p(H)$ there is $\tilde{M} \in \land\text{-geo}(\mathcal{F}(H), \mathcal{F})$, unique up to isomorphism, such that

$$\tilde{M} \circ (\cdot) = M;$$

we will use this below, although it should be emphasized that this is done only for a simplification in notation.

We note that a $\aleph_0$-saturated (see Section 5) model of $H$ is a $p$-model in $\text{Set}$. Indeed, with the notation for (8.2), if $M \in \text{Mod}(H)$, $y \in M(Y)$ with $y \in M(B)$ for all $B \in \exists \gamma(F)$, then the set $\{ f(x) = y \} \cup \{ A(x) : A \in F \}$ is finitely satisfiable in $M$; the second term of the union is closed under finite conjunction, and if $A \in F$, then $M \vDash \exists x(f(x) = y \land A(x))$, since for $B = \{ y \in Y : \exists x(f(x) = y \land A(x)) \}$, we have $f^*B = A$. Together with the existence theorem for $\aleph_0$-saturated models, this observation shows, among others, that $H$ has "enough $p$-models", that is,

$$\cdot : H \to \mathcal{F}(H)$$

is conservative.

With $X \in H$, a filter-subobject of $\hat{X}$ is any subobject of $\hat{X}$ which is the intersection of subobjects of the form $\hat{A}$, $A \in S(X)$ in $H$; a filter-object $P$ of $\mathcal{F}(H)$ is one which is the domain of a monomorphism $m : P \to \hat{X}$ giving rise to a filter-subobject $[m]$. We have that

the set of filter-objects generate $\mathcal{F}(H)$

in the sense used in theory of Grothendieck toposes; this is used to reduce assertions about objects in $\mathcal{F}(H)$ to filter-objects.

For any $X \in H$, the induced mapping $\cdot : S(X) \to S(\hat{X})$ satisfies the universal property of the canonical map of $S(X)$ into its prime-generated hull.

This follows from (1.8) in [26] and (2.8). [(2.8)(ii) is loc. cit. (iii); (i) is contained in loc. cit. (i); to see (iii), assume $\bigwedge_{i \in I} x_i \leq x$; take the filter $F$ generated by the $x_i$; assuming that $x \notin F$, by the prime filter existence theorem, there is a prime $p$ with $F \subset p$, but $x \notin p$;
but also, with the notation of loc.cit., \( \bigwedge \{ a(\hat{A}); A \in p \} \leq \bigwedge_{i \in I} \hat{x}_i \leq \hat{x} \), and by loc.cit. (iii) and (ii), \( x \in p \), contradiction. (iv) is contained in loc.cit. (i).

In particular, we are going to freely use for \( \vdash S(X) \rightarrow S(\hat{X}) \) the properties of the prime-generated hull listed in (2.8).

Let us show that

\( \vdash \mathcal{H} \rightarrow \mathcal{F}(\mathcal{H}) \) is conditionally Heyting and coHeyting.

Suppose \( f: X \rightarrow Y \) in \( \mathcal{H}, A \in S(X) \) and \( \forall_f A \in S(Y) \) exists. Let \( \Phi \in S(\hat{X}) \). Since \( f^*(\forall_f A) \leq \hat{A} \), the left-to-right direction in

\[ \Phi \leq \hat{x}(\forall_f A) \iff f^*\Phi \leq \hat{A} \]

is clear. To show the converse, first note that the required property for \( \Phi = \bigvee_{i \in I} \Phi_i \) is inherited from the same for the \( \Phi_i \)'s. Thus, it suffices to deal with \( \Phi \) a filter-subobject, \( \Phi = \bigwedge_{i \in I} \hat{B}_i \). Then, \( f^*\Phi = \bigwedge_{i \in I} f^*\hat{B}_i, f^*\Phi \leq \hat{A} \) implies that, for some finite \( I' \subset I \),

\[ \bigwedge_{i \in I} f^*\hat{B}_i = f^*\left( \bigwedge_{i \in I'} \hat{B}_i \right) \leq \hat{A} \]

and \( f^*(\bigwedge_{i \in I'} \hat{B}_i) \leq A \). Thus, \( \bigwedge_{i \in I'} B_i \leq \forall_f A \), from which \( \Phi \leq (\forall_f A) \) follows.

Since \( S(X) \) is the prime-generated hull of \( S(X) \), the fact that \( \cdot \) is conditionally coHeyting, referring as it does to propositional logic only, is contained in the fact that the canonical embedding of a distributive lattice into its prime-generated hull is conditionally coHeyting; see (2.6).

**Lemma 8.3.** Let \( \mathcal{E} \) be a biHeyting Grothendieck topos, \( \mathcal{G} \) a generating set of objects in \( \mathcal{E} \). Assume that for every \( A, B \in \mathcal{G} \), the projection \( \pi: A \times B \rightarrow B \) is distributive. Then \( \mathcal{E} \) satisfies (L): all product projections \( X \times Y \rightarrow Y \) are distributive.

**Proof.** We place ourselves in a Grothendieck topos \( \mathcal{E} \). Let us quote two formulas from [29]; both concern the expression of \( \forall_f \), for \( f: A \rightarrow B \), by other universal quantifiers. The first uses a cover (epimorphic family) \( \langle A_i \twoheadrightarrow A \rangle_{i \in I} \) of \( A \), the other a cover \( \langle B_i \rightarrow B \rangle_{i \in I} \) of \( B \). We have, for any \( \Phi \in S(A) \),

\[ \forall_f(\Phi) = \bigwedge_{i \in I} \forall_{f_i} x_i^* \Phi, \]

and, with reference to the pullback squares

\[ \begin{array}{ccc} A \xrightarrow{f} B \\ \pi_i \uparrow \downarrow \ \\ A_i \rightarrow B_i \end{array} \]

\[ \forall_f(\Phi) = \bigvee_{i \in I} \exists_{\hat{\pi}_i} \forall_{f_i} x_i^* \Phi \]
The formulas are not too hard to verify directly.

Turning to the proof of the lemma, consider first \( \pi : X \times B \rightarrow B \) with \( B \in \mathcal{G} \) (and arbitrary \( X \)). By assumption, there is a cover \( \langle A_i \rightarrow A \rangle_{i\in I} \) of \( X \) with \( A_i \in \mathcal{G}(i \in I) \). Then the composite in the following is a projection itself, for a product of objects in \( \mathcal{G} \):

\[
A_i \times B \xrightarrow{\pi_i \times B} X \times B \xrightarrow{\pi} B \equiv A_i \times B \xrightarrow{\pi_i} B;
\]

and \( \langle A_i \times B \rightarrow X \times B \rangle_{i\in I} \) is a cover of \( X \times B \). We calculate, with \( \Phi \in S(B), \Psi \in S(X \times B) \),

\[
\forall \pi(\pi^* \Phi \vee \Psi) = \bigwedge_{i \in I} \forall \pi_i(\pi_i^* \Phi \vee (\pi_i \times B)^* \Psi) \quad \text{(by the first formula)}
\]

\[
= \bigwedge_{i \in I} (\Phi \vee \forall \pi_i(\pi_i \times B)^* \Psi) \quad \text{(by } \pi_i \text{ distributive)}
\]

\[
= \Phi \vee \bigwedge_{i \in I} \forall \pi_i(\pi_i \times B)^* \Psi \quad \text{(since } \mathcal{G} \text{ is biHeyting,}
\]

\[
\Phi \vee (\wedge) \text{ preserves } \bigwedge
\]

\[
= \Phi \vee \forall \pi \Psi \quad \text{(by a second application of the first formula), as desired.}
\]

Next, to the general case. Start with \( \pi : X \times Y \rightarrow Y \), and a cover \( \langle B_i \rightarrow Y \rangle_{i\in I} \) with \( B_i \in \mathcal{G} \); let \( \Phi \in S(Y), \Psi \in S(X \times Y) \). We have the pullbacks

\[
\begin{align*}
X \times Y & \xrightarrow{\pi} Y \\
\downarrow_{\pi_i} & \quad \uparrow_{\beta_i} \\
X \times B_i & \xrightarrow{\pi_i} B_i
\end{align*}
\]

and

\[
\forall \pi(\pi^* \Phi \vee \Psi) = \bigvee_{i \in I} \exists \beta_i \forall \pi_i(\pi_i^* \Phi \vee \Psi) = \bigvee_{i \in I} \exists \beta_i \forall \pi_i(\pi_i^* \beta_i^* \Phi \vee \pi_i^* \Psi)
\]

\[
= \bigvee_{i \in I} \exists \beta_i(\beta_i^* \Phi \vee \forall \pi_i \pi_i^* \Psi) = \bigvee_{i \in I} (\exists \beta_i \beta_i^* \Phi \vee \exists \beta_i \forall \pi_i \pi_i^* \Psi)
\]

\[
= \bigvee_{i \in I} \exists \beta_i \beta_i^* \Phi \vee \bigvee_{i \in I} \exists \beta_i \forall \pi_i \pi_i^* \Psi
\]

\[
= \Phi \vee \bigvee_{i \in I} \exists \beta_i \forall \pi_i \pi_i^* \Psi = \Phi \vee \forall \pi \Psi,
\]

we used the fact that the \( \beta_i \) cover \( Y \) in inferring \( \bigvee_{i \in I} \exists \beta_i \beta_i^* \Phi = \Phi \). \( \square \)

Consider \( : H \rightarrow \mathcal{F}(H) \).
Lemma 8.4. Suppose $H$ is a Heyting category that satisfies $(L)$. Then in $\mathcal{F}(H)$, for any filter-objects $P$ and $Q$, $\pi: P \times Q \to Q$ is distributive.

Proof. Considering the pullback

$$
P \times Y \rightarrowtail Y \quad \xleftarrow{\uparrow} \quad \xrightarrow{\uparrow} \quad P \times Q \rightarrow Q
$$

with an appropriate mono $Q \rightarrowtail Y$, and remembering that distributivity of the lower horizontal is inherited from that of the upper one, we see that it suffices to consider $\pi: P \times Y \rightarrowtail Y$ with $Y \in H$. Let $X \in H$, $\Pi \in S(\hat{X})$ a filter-subobject such that $\Pi = [P \rightarrowtail X]$. Let $\Phi \in S(\hat{Y})$, $\Gamma \in S(P \times Y)$; we let $\Psi = 3_{m \times Y} \in S(\hat{X} \times Y)$; among others, $\Gamma = (m \times \hat{Y}) \Psi$. We desire to show that $\forall_x (\pi^* \Phi \vee \Gamma) \leq_f \Phi \vee \forall_x \Gamma$. Since $\sigma$ is prime-generated, this is equivalent to saying that for all primes $q \in \Pr(S(\hat{Y}))$,

$$
q \leq_f \forall_x (\pi^* \Phi \vee \Gamma) \Rightarrow q \leq_f \Phi \vee \forall_x \Gamma;
$$

since $q$ is prime, $q \leq \Phi \vee \forall_x \Gamma \iff q \leq \Phi$ or $q \leq \forall_x \Gamma$. Also, since

$$
q \leq_f \forall_x \Gamma \iff \pi^* q \leq_{P \times Y} \Gamma \iff \Pi \times q \leq_{\hat{X} \times \hat{Y}} \Psi,
$$

and

$$
q \leq_f \forall_x (\pi^* \Phi \vee \Gamma) \iff \pi^* q \leq_{P \times \hat{Y}} \pi^* \Phi \vee \Gamma \iff \Pi \times q \leq_{\hat{X} \times \hat{Y}} \Pi \times \Phi \vee \Psi,
$$

we can rewrite the aimed-at implication as

$$
\Pi \times q \leq_{\hat{X} \times \hat{Y}} \Pi \times \Phi \vee \Psi \Rightarrow q \leq_f \Phi \text{ or } \Pi \times q \leq_{\hat{X} \times \hat{Y}} \Psi.
$$

Let us fix $q \in \Pr(\hat{Y})$ and assume

$$
\Pi \times q \leq_{\hat{X} \times \hat{Y}} \Pi \times \Phi \vee \Psi, \quad q \leq_f \Phi \text{ and } \Pi \times q \leq_{\hat{X} \times \hat{Y}} \Psi \quad (8.4)
$$

to derive a contradiction. It follows that there is $M \in \text{p-Mod}(H)$ such that $\bar{M}(\Pi \times q) \leq \bar{M}(\Psi)$; that is, there are $x \in M(X) (= \bar{M}(\hat{X}))$ and $y \in M(Y)$ such that $x \in \bar{M}(\Pi)$ and $y \in \bar{M}(q)$, but $\langle x, y \rangle \notin \bar{M}(\Psi)$. We claim that

there are $N \in \text{p-Mod}(H)$ and a pair $(\hat{x}, \hat{y})$ of elements $\hat{x} \in \bar{N}(\Pi)$ and $\hat{y} \in \bar{N}(q)$ such that $\langle \hat{x}, \hat{y} \rangle \notin \bar{N}(\Psi)$, and in addition, $\hat{y}$ is generic for $q$, that is, for any $B \in S(Y)$, $\hat{y} \in N(B)$ iff $q \leq B$.

To show the claim, we let $\Psi = \bigvee_{i \in I} \bigwedge_{j \in J_i} C_{ij}$, with appropriate $C_{ij} \in S(X \times Y)$. We have for every $i \in I$ some $j_i \in J_i$ such that $\langle x, y \rangle \notin M(C_{ij})$ (since $\bar{M}$ respects $\land$). Consider the following set of sentences over the language $L_H \cup \{x, y\} \{x: X, y: Y\}$:

$$
\Sigma_H \cup \{B(y): q \leq_f \hat{B}\} \cup \{-D(y): q \leq_f \hat{D}\} \cup \{-\neg C_{ij}(x, y): i \in I\} \cup \{A(x): II \leq_x A\}.
$$

(8.5)
We claim it is consistent. Suppose not. The second and fifth terms of the union are clearly closed under finite conjunction (up to equivalence in $T_H$). The same is true of the third term because $q$ is prime. We have $B, D \in S(Y), A \in S(X)$ with $q \leq \check{B}, q \leq \check{D}, \Pi \leq \check{A}$ and a finite subset $I'$ of $I$ such that, for $C = \bigvee_{i \in I'} C_{ij} \in S(X \times Y)$, we have

$$T_H \vdash B(y) \wedge \neg D(y) \wedge \neg C(x, y) \wedge A(x) \Rightarrow \bot.$$ 

that is,

$$T_H \vdash A(x) \wedge B(y) \Rightarrow D(y) \vee C(x, y).$$

(8.5')

Note that the choice of the $j_i$ ensures that $\langle x, y \rangle \notin M(C)$.

Let $A = [U \rightarrow X]$. With $\rho : U \times Y \rightarrow Y$ the projection, $E = (n \times Y)^*C \in S(U \times Y)$, (8.5') says that

$$\rho^*(B) \leq_{U \times Y} \rho^*(D) \vee E.$$

Hence,

$$B \leq_{Y} \forall \rho(\rho^*(D) \vee E) = D \vee \forall \rho(E),$$

the last equality because of (L) for $H$. Since $q \leq \check{B}$, we obtain

$$q \leq_{Y} \check{D} \vee (\forall \rho(E))' = \check{D} \vee \forall \rho(E),$$

the last equality because $\cdot$ is Heyting. However, $q$ is a prime and $q \leq_{Y} \check{D}$. It follows that

$q \leq_{Y} \forall \rho(E)$, which is to say that $\check{\rho}^*(q) \leq_{U \times Y} \check{E}$, and equivalently, $\check{A} \times q \leq_{x, y} \check{C}$. However, this last is contradicted by $M, x$ and $y$: $x \in \check{M}(A), y \in \check{M}(q)$ and $\langle x, y \rangle \notin \check{M}(C)$. This contradiction proves the consistency of (8.5).

Let $(N, \check{x}, \check{y})$ be an $N_0$-saturated model of (8.5). Then $N$ is a p-model. Since $q, \Pi$ are filter-subobjects, the second and fifth terms in (8.5) ensure that $\check{y} \in \check{N}(q), \check{x} \in \check{N}(\Pi)$. The third term makes $\check{y}$ generic for $q$. Finally, the fourth term ensures that $\langle x, y \rangle \notin \check{N}(\Psi)$. We have proved the claim.

Armed with the data as in the claim, we return to (8.4) and derive a contradiction. We have $\Phi = \bigvee_{i \in I} \bigwedge_{j \in J_i} \check{B}_{ij}$, for suitable $B_{ij} \in S(Y), q \notin \Phi$ means that for every $i \in I$ there is $j_i \in J_i$ such that $q \notin \check{B}_{ij}$. Since $\check{y}$ is generic for $q$, $\check{y} \notin N(B_{ij})$ for all $i \in I$. But then $\check{y} \notin \check{N}(\Phi)$ (\check{N} respects $\vee$). Looking at the first condition in (8.4), and the facts that $\langle \check{x}, \check{y} \rangle \in \check{N}(\Pi \times q), \langle \check{x}, \check{y} \rangle \notin \check{N}(\Pi \times \Phi)$, we conclude $\langle \check{x}, \check{y} \rangle \in \check{N}(\Psi)$, contradicting the last-mentioned property of $\langle \check{x}, \check{y} \rangle$ in the claim. \qed

**Proof of Theorem 8.2.** The theorem follows, with $H \rightarrow \delta$ taken to be $\vdash : H \rightarrow \mathcal{T}(H)$, by the properties, listed above, of the topos of types, and the two lemmas. \qed

**Proof of Theorem 8.1** (first part). First of all, note that the “conservative enrichment” character of the lower horizontal in (8.1) is an immediate consequence of Theorem 8.2: $\delta \in L$-biH.
Turning to other parts of (8.1), let us quote a result, Theorem 6.2.1, slightly restated (weakened) from [29]:

Any Grothendieck topos has a conservative geometric embedding into a Boolean Grothendieck topos preserving all stably distributive infs of subobjects and all stably distributive $\forall_f$'s. (8.6)

Here, a particular inf $\bigwedge_{i=1}^{n} A_i$ of subobjects $A_i \in S(X)$ is distributive if for any $A \in S(X)$, $A \vee \bigwedge_{i=1}^{n} A_i = \bigwedge_{i=1}^{n} (A \vee A_i)$; it is stably distributive if for any $f: Y \to X$, the inf $\bigwedge_{i=1}^{n} f^* A_i \in S(Y)$ is distributive. It turns out that, in a Grothendieck topos, one can write any inf as a suitable $\forall_f(\Phi)$, and then (stable) distributivity of the inf becomes the same as that of the $\forall_f(\Phi)$; thus, in fact, in the quote the reference to infs is not needed. However, in our use of the quoted result, the uses of the two mentioned distributivities are separate.

Let $H \in \mathbb{L}-(\mathfrak{bi})H$; we need some $(\phi: \mathfrak{E} \to \mathfrak{B}) \in \mathbb{L}\mathfrak{biS4}$ and a conservative $(\mathfrak{bi})$Heyting functor $\mathsf{c}: H \to \mathfrak{E}$; this will establish the conservative character of the enrichments along the two verticals and the diagonal in (8.1). We let $\mathsf{c}: H \to \mathfrak{E}$ be the embedding given by Theorem 8.2. Since $\mathfrak{E}$ is prime-generated, and as a consequence, all subobject lattices in $\mathfrak{E}$ are coframes, all infs are distributive in $\mathfrak{E}$. By Theorem 8.2, for every product projection $\pi$, $\forall_x$ is (stably) distributive in $\mathfrak{E}$. Therefore, (8.6) is applicable to obtain a geometric embedding $\phi: \mathfrak{E} \to \mathfrak{B}$ into a Boolean topos $\mathfrak{B}$ preserving all infs and all $\forall_x$'s ($\pi$ projection). For any $X \in \mathfrak{E}$, $\phi_X: S(X) \to S(\phi X)$ preserves all infs; hence, $\phi_X$ has a left adjoint $\circ_X: S(\phi X) \to S(X)$. As a geometric functor, $\phi$ itself has a right adjoint; thus, in particular, also a local right adjoint (see Section 3). We have that $(\phi: \mathfrak{E} \to \mathfrak{B})$ is a $\mathfrak{biS4}$ category. Since all $\forall_x$'s are distributive in $\mathfrak{E}$ as given by Theorem 8.2, $(\phi: \mathfrak{E} \to \mathfrak{B})$ is in fact in $\mathbb{L}\mathfrak{biS4}$. □

The proof of the remaining case of Theorem 8.1 requires a completeness theorem for $\mathbb{L}\mathfrak{S4}$. We prove a result of a greater generality.

**Theorem 8.5.** Let $(\phi: H \to D) \in \text{Coh}$ with $H \in \mathcal{H}$ such that $\phi$ preserves all $\forall_\rho$, $\rho$ a product projection in $H$. Then the induced functor $\mathcal{T}(\phi): \mathcal{T}(H) \to \mathcal{T}(D)$ between the prime-generated hulls preserves all $\forall_x$ for product projections $\pi$ in $\mathcal{T}(H)$.

**Proof.** We are going to use $^*$ to denote both $\phi$ and $\mathcal{T}(\phi)$, and $\cdot$ for the canonical $H \to \mathcal{T}(H)$, $D \to \mathcal{T}(D)$.

In analogy to Lemma 8.3, but more directly, using the two formulas quoted in the proof of Lemma 8.3, and using the fact that $\mathcal{T}(\phi)$ preserves infs, we obtain that it suffices to show that $\mathcal{T}(\phi)$ preserves $\forall_\rho$ for projections $\pi: P \times Q \to Q$ with $P, Q$ filter-objects of $\mathcal{T}(H)$. Further, by an easy argument similar to (4.6), similarly to the reduction achieved at the start of the proof of Theorem 8.2, we can replace $Q$ here by an object $\hat{Y}, Y \in \mathcal{H}$. Thus, we consider $P \in \mathcal{T}(H), X, Y \in \mathcal{H}, \Pi = [P \xrightarrow{\mu} \hat{X}], \Pi$ a filter-subobject, $\pi: P \times \hat{Y} \to \hat{Y}$ a product projection. Since $\mathcal{T}(\phi)$ preserves infs, $\mathcal{T}(\phi)$ has
a local left adjoint $\circ$. With a look at the familiar adjunction

\[
\begin{array}{rcccl}
S(\hat{P} \times \hat{Y}) & \xleftarrow{\circ} & S(\hat{Y}) & \overset{\forall}{\longrightarrow} & S(\hat{Y}) \\
\circ & \downarrow & \circ & & (8.7) \\
S(P \times Y) & \xrightarrow{\pi^*} & S(Y) & \overset{\forall}{\longrightarrow} & S(Y)
\end{array}
\]

we see that what we need is the commutativity of the left-hand square (8.7). Since all maps in (8.7) preserve $\forall$, it suffices to show the equality $\circ \hat{\pi}^* \Phi = \pi^* \circ \Phi$ for filter-subobjects $\Phi \in S(\hat{Y})$. This we do by showing that $\pi^* \circ \Phi$ has the universal property of $\circ \hat{\pi}^* \Phi$; that is, for any $\Gamma \in S(P \times \hat{Y})$,

\[
\pi^* \circ \Phi \leq \Gamma \iff \hat{\pi}^* \Phi \leq \hat{\Gamma}.
\]

Since $\Phi \leq (\circ \Phi)^\wedge$ and $\pi^* \circ \Phi \leq \Gamma \Rightarrow \hat{\pi}^* \Phi \leq \hat{\pi}^* (\circ \Phi)^\wedge \leq \hat{\Gamma}$, only the right-to-left implication requires proof. Assume

\[
\hat{\pi}^* \Phi \leq \hat{\Gamma}, \tag{8.8}
\]

to show $\pi^* \circ \Phi \leq \Gamma$.

Since the models $\tilde{M}$, for $M \in p\text{-Mod}(H)$, form a sufficient family of $\bigwedge$-geometric functors from $\mathcal{F}(H)$ to Set, the task is reduced to showing that for any such $M$, $\tilde{M}(\pi^* \circ \Phi) \leq \tilde{M}(\Gamma)$. Thus, in addition, we assume $M \in p\text{-Mod}(H)$, $\langle x, y \rangle \in \tilde{M}(\pi^* \circ \Phi)$, i.e.,

\[
x \in \tilde{M}(\Pi) \subset M(X), \quad y \in \tilde{M}(\circ \Phi) \subset M(Y) \tag{8.9}
\]

and we want

\[
?: \; \langle x, y \rangle \in \tilde{M}(\Gamma). \tag{8.10}
\]

We are going to construct $N \in p\text{-Mod}(D)$, $\hat{x} \in N(\hat{X})$, $\hat{y} \in N(\hat{Y})$ with the following properties:

\[
\hat{x} \in \tilde{N}(\hat{\Gamma}), \quad \hat{y} \in \tilde{N}(\Phi) \text{ and } \langle \hat{x}, \hat{y} \rangle \in N(\hat{C}) \Rightarrow \langle x, y \rangle \in M(C) \text{ for all } C \in S(X \times Y). \tag{8.11}
\]

Supposing we have (8.11), note that the first two relations in (8.11) say that $\langle \hat{x}, \hat{y} \rangle \in \tilde{N}(\circ \Phi)$; then, by (8.8), $\langle \hat{x}, \hat{y} \rangle \in \tilde{N}(\hat{\Gamma})$. Now, $\Gamma = (m \times \hat{X})^* \bigvee_{i \in I} \bigwedge_{j \in J} \hat{C}_{ij}$ for suitable $C_{ij} \in S(X \times Y)$; thus, $\hat{\Gamma} = (m \times \hat{X})^* \bigvee_{i \in I} \bigwedge_{j \in J} \hat{C}_{ij}$. It follows that there is $i \in I$ such that for all $j \in J_i$, $\langle \hat{x}, \hat{y} \rangle \in N(\hat{C}_{ij})$. By the last relation in (8.11), $\langle x, y \rangle \in M(C_{ij})$ for the same $i$ and for all $j \in J_i$, from which $\langle x, y \rangle \in M(\Gamma)$, as desired.

It suffices to satisfy (8.11). Consider the following set $\Sigma$ of sentences over the language $L_D \cup \{x, y\}$:

\[
\Sigma_D \cup \{\Delta(x); \; A \in S(X), \; A \ni x \} \cup \{B(y); \; B \in S(\hat{Y}), \; \hat{B} \ni \hat{y} \Phi\}
\]

\[
\cup \{\neg \hat{\Pi}(x, y); \; R \in S(X \times Y), \; \langle x, y \rangle \notin M(R)\}.
\]
Using that $\Pi$ and $\Phi$ are filter-subobjects, if $N \in p$-Mod$(D)$, $(N, \hat{x}, \hat{y}) \vdash \exists$, then (8.11) holds. It suffices to show that $\exists$ is consistent; any $\aleph_0$-saturated model of it will do.

Suppose $\exists$ fails to be consistent. Since the second, third and fourth terms in the union are closed (up to equivalence) under finite conjunction, we then must have $A \in \mathcal{S}(X)$, $B \in \mathcal{S}(Y)$, and $R \in \mathcal{S}(X \times Y)$ such that

$$\hat{A} \ni \pi_1 \Pi, \quad \hat{B} \ni \pi_2 \Phi \quad \text{and} \quad \langle x, y \rangle \not\ni \hat{M}(R)$$ (8.12)

and such that

$$\Sigma \vdash \hat{A}(x) \land \hat{B}(y) \rightarrow \hat{R}(x, y).$$ (8.13)

Let $A = \left[ U \rightarrow X \right]$, $\rho : U \times Y \rightarrow Y$ the product projection. Then (8.13) is equivalent to saying that

$$\hat{\rho}^* B \leq (\hat{n} \times \hat{Y})^* \hat{R}.$$ (8.14)

Now we use the assumption that $(\phantom{\text{lo}}) = \varphi : H \rightarrow D$ preserves all $\forall_\rho$, $\rho$ a product projection in $H$. Thus, for $C = \forall_\rho (\hat{n} \times Y)^* R \in \mathcal{S}(Y)$, we have that $\forall_\rho (\hat{n} \times \hat{Y})^* \hat{R}$ exists are equals $\hat{C}$. From (8.14), we infer $B \leq \forall_\rho (\hat{n} \times \hat{Y})^* \hat{R}$; thus $B \leq \hat{C}$. Since $\Phi \leq \hat{B} \leq \hat{C}$ (by (8.12)), we obtain $\circ \Phi \leq \hat{C}$. By (8.9), it follows that $y \in M(C)$; also $x \in M(A)$ by (8.9), (8.12); equivalently, $\langle x, y \rangle \in M(\pi_1^* A \land \pi_2^* C)$; here, $X \hat{\times} X \times Y \hat{\rightarrow} Y$ are the product projections. Now, $C = \left[ y : \forall x (\hat{A}x \rightarrow \hat{R}xy) \right]$, which makes it clear that $\pi_1^* A \land \pi_2^* C \leq R$. We conclude $\langle x, y \rangle \in M(R)$, contradicting (8.12). \[Q.E.D.\]

**Corollary 8.6.** For $(\varphi : H \rightarrow D) \in H$, we have that $\mathcal{F}(\varphi) : \mathcal{F}(H) \rightarrow \mathcal{F}(D)$ is also Heyting (in other words, $\mathcal{F}(\varphi)_*: \mathcal{F}(D) \rightarrow \mathcal{F}(H)$ is an open geometric morphism; see e.g. [30]).

**Proof.** Note that a functor $\varphi : H \rightarrow D \in \text{Coh}$, with Heyting categories $H$, $D$, is Heyting if it preserves all $\forall_\rho$, $\rho$ a projection, and preserves implications of subobjects. Note that $\mathcal{S}(\hat{X})$, $\mathcal{S}(\hat{X})$ are the prime-generated hulls of $\mathcal{S}(X)$, $\mathcal{S}(\hat{X})$, resp., and as a consequence, $\mathcal{F}(\varphi)$ induces the canonical inf-preserving map $\mathcal{S}(\hat{X}) \rightarrow \mathcal{S}(\hat{X})$, which, by (2.11), is Heyting, since $(\phantom{\text{lo}}) : \mathcal{S}(X) \rightarrow \mathcal{S}(\hat{X})$ is. The assertion follows. \[Q.E.D.\]

The Corollary 8.6 is analogous to a result of A.M. Pitts concerning a construction related to $\mathcal{F}(\phantom{\text{lo}})$; see Theorem 2.1 in [30].

**Theorem 8.7.** The structures of the form $\psi: \mathcal{E} \rightarrow \mathcal{A}$, with $\mathcal{E}$, $\mathcal{A}$ prime-generated Grothendieck toposes, $\psi$ a conservative $\land$-geometric functor preserving all $\forall_\rho$ for product projections $\pi$ in $\mathcal{E}$, are representative in $\text{L-S4}$ and in $\text{L-biS4}$.

**Proof.** Let $(\varphi : H \rightarrow D) \in \text{L-(bi)S4}$. Consider

$$D \longrightarrow \mathcal{F}(D)$$

$$\uparrow \circ \uparrow \mathcal{F}(\varphi)$$

$$H \longrightarrow \mathcal{F}(H)$$ (8.15)
It is easy to see, and it was pointed out in [26] too, that for \( \mathcal{D} \) Boolean, \( \mathcal{F}(\mathcal{D}) \) is also Boolean. As before, let us write \((\cdot)^{\cdot}\) for the effect of both \( \varphi \) and \( \mathcal{F}(\varphi) \). Since the filter-objects generate \( \mathcal{F}(\mathcal{H}) \), to show that \( \mathcal{F}(\varphi) \) is conservative it suffices to show that if \( \Phi, \Psi \) are filter-subobjects of \( \hat{X}, X \in \mathcal{H} \), and \( \Phi \leq \Psi \), then \( \hat{\Phi} \leq \hat{\Psi} \). We have some \( A \in S(X) \) such that \( \Psi \leq A \) and \( \Phi \leq \hat{A} \). Therefore, for all \( B \in S(X) \) such that \( \Phi \leq B \), we have \( B \leq \hat{B} \), that is, \( B \leq A \), and \( \hat{B} \leq \hat{A} \) since \( \varphi \) is conservative. Now, \( \hat{\Phi} = \bigwedge \{ \hat{B} : B \in S(X), \Phi \leq B \} \). Hence, by the contrapositive of (2.8)(iii), \( \hat{\Phi} \leq \hat{A} \) follows; \( \hat{\Phi} \leq \hat{\Psi} \) is a consequence.

Given any \( X \in \mathcal{H} \), on the corresponding subobject lattices, (8.15) reduces to the first of the two diagrams:

\[
\begin{array}{ccc}
S(\hat{X}) & \rightarrow & S(\hat{X}) \\
\uparrow^{\hat{\cdot}} & & \uparrow^{\hat{\cdot}} \\
S(X) & \rightarrow & S(X)
\end{array}
\]

which is isomorphic to the second. By (2.18), it follows that \((\cdot, \cdot)\) in (8.15) is an arrow in \((\text{bi})\mathcal{S}4\). □

**Proof of Theorem 8.1 (conclusion).** The fact that the upper horizontal in (8.1) is a conservative enrichment follows from Theorem 8.7 (for L-\(\mathcal{S}4\)). □

9. Alternative methods for the Kripke-type completeness results

It is easily seen that the conservative enrichment results depend only on completeness theorems for countable theories. More specifically, each of our doctrines is locally finitely presentable, and thus every object is a filtered colimit of finitely presentable objects. Moreover, all our forgetful functors are finitary, that is, preserve filtered colimits. Also, a filtered colimit of conservative morphisms is conservative. It is then immediate that the fact of the unit of the adjunction being conservative (see \((**\) in Section 2) is inherited from finitely presentable objects to all objects. It is clear that the internal theory \( T_H \) of a finitely presentable object \( H \), in any of the doctrines, is a countable theory; that is, one in a countable language.

The just stated fact is a justification for a narrowing of interest, common in logic, to countable theories; the theorems of "pure logic" will not need theories other than countable ones. In this section, we give alternative proofs, valid for countable theories only, for some Kripke-type completeness results. Note that the proofs in Section 6 use possibly uncountable models (that is, coherent functors to \( \text{Set} \) from the categories involved) even when the categories are countable. Here, we will work with countable models only.

There is a general way of substituting recursively saturated countable models for the special models, in case the theories (categories) involved are recursively presented.
This substitution does work in our cases without difficulty. Using relative recursive-
ness (relative to an arbitrary real), this method can be extended to all countable
theories. We will not elaborate on this; see e.g. [4]. We will point out those cases when
a method genuinely different from the preceding ones gives a better, that is, more
canonical, completeness theorem.

First, we state a version of the omitting types theorem, specifically suited for the
purposes of coherent logic. The theorem is essentially equivalent to the basic omitting
types theorem of [14]. To emphasize the naturalness of the present formulation, we
outline a direct proof of it.

Throughout this section, the symbol ⊩ is used for a notion of deducibility complete
for finitary coherent logic.

Let $\bar{x}, \bar{u}$ be finite tuples of distinct variables. A finite tuple $\bar{y}$ of not necessarily
distinct variables is a copy of $\bar{x}$ in $\bar{u}$ if, with $\bar{x} = \langle x_i \rangle_{i<n}$, $\bar{u} = \langle u_j \rangle_{j<m}$, we have that
$\bar{y} = \langle u_{j_i} \rangle_{i<n}$ for some $j_i < m$ ($i < n$) (not necessarily distinct $j_i$'s) such that $\bar{y}$ matches $\bar{x}$.
i.e., $x_i$ and $y_{j_i}$ are sorted in the same way for each $i < n$. Thus, in this case, $\varphi^{\bar{y}}_{\bar{x}}$, the result
of the (proper) substitution of $\bar{y}$ for $\bar{x}$ (possibly involving passing to an alphabetic
variant of $\varphi$ first) is well-formed.

Let $T = (L, \Sigma)$ be a countable coherent theory; we assume $T$ is consistent. Let
$\Xi(\bar{x}), \Gamma(\bar{x})$ be sets of coherent $L$-formulas with at most the free variables indicated; $\bar{x}$ is
a finite string of distinct variables. We are interested in a condition that it sufficient for $T$
 to be consistent with the following infinitary formula:

$$A := \forall \bar{x}\left( \bigwedge \Xi(\bar{x}) \Rightarrow \bigvee \Gamma(\bar{x}) \right);$$

we call such $A$ an infinitary entailment, or simply an $\infty$-condition. Also, to be faithful to
the spirit of the notation so far, we write

$$A := \bigwedge \Xi(\bar{x}) \Rightarrow \bigvee \Gamma(\bar{x}).$$

An $L$-structure $M$ meets (satisfies) the condition $A$ if $M \models A$ in the normal sense.

Consider an arbitrary $L$-entailment $\lambda := \theta \Rightarrow v$. The condition $\text{LC}(A, \lambda)$ on $A$ and
$\lambda$ is defined to be the following one:

$$\text{LC}(A, \lambda). \text{ For any copy } \bar{y} \text{ of } \bar{x} \text{ in } \bar{u}, \text{ if } T \vdash \theta \Rightarrow v \vee \zeta^{\bar{y}} \text{ for all } \zeta \in \Xi \text{ and }$$

$$T \vdash \theta \wedge \gamma^{\bar{y}} \Rightarrow \bar{u} \Rightarrow v \text{ for all } \gamma \in \Gamma, \text{ then } T \vdash \theta \Rightarrow v.$$ 

We say that $T$ is locally consistent with $A$ if $\text{LC}(A, \lambda)$ holds for all coherent $L$-
entailments $\lambda$.

**Proposition 9.1** (Coherent omitting types theorem). Let $T$ be a coherent theory over
the countable language $L$; assume $T$ is consistent. Let $A$ be a countable set of $\infty$-condi-
tions over $L$. If each $\Lambda \in A$ is locally consistent with $T$, then $T$ has a countable model
meeting each $\infty$-condition in $A$. 


Proof. First, let us formulate a model-existence principle. Suppose $\mathcal{X}$ is a countable set of variables, each sorted by sorts in $L$. Let $\theta$ and $\Omega$ be (countable) sets of coherent formulas, with free variables included in $\mathcal{X}$; we are going to treat free variables as individual constants, and accordingly, we call the formulas involved "sentences". Also, all formulas (sentences) are coherent unless otherwise specified. Let us make the following assumptions on $(\mathcal{X}, \theta, \Omega)$:

(i) $\theta \cup (\neg) \Omega$ is $L(\mathcal{X})$-consistent

$((\neg) \Omega = \{ \neg \phi : \phi \in \Omega \})$; $L(\mathcal{X})$-consistency means that there is an $L(\mathcal{X})$-model for the set of sentences in question; note that, because of possibly empty domains, it is not true that if a set of sentences is consistent with respect to a language, it remains consistent with respect to an extension of the language; the extension may contain an individual constant in a sort that was rendered empty by the given axioms; this is why the parameter $L(\mathcal{X})$ is mentioned);

(ii) $\theta \vDash \phi \lor \psi \Rightarrow \theta \vDash \phi$ of $\theta \vDash \psi$

($\theta \vDash \phi$ means logical consequence with respect to $L(\mathcal{X})$-models; remember that members of $\mathcal{X}$ are individual constants; $\phi$ and $\psi$, here and below, range over coherent $L(\mathcal{X})$-sentences);

(iii) $\theta \vDash \exists x \phi \Rightarrow$ there is $y \in \mathcal{X}$ matching $x$ such that $\theta \vDash \phi^y$;

(iv) $(\neg) \Omega \vDash (\phi \land \psi) \Rightarrow (\neg) \Omega \vDash \neg \phi$ or $(\neg) \Omega \vDash \neg \psi$.

We claim that, under the hypotheses (i)–(iv), $\theta \cup (\neg) \Omega$ has an $L(\mathcal{X})$-model in which every element is the denotation of some $x \in \mathcal{X}$. Indeed, let $N$ be any $L(\mathcal{X})$-model of $\theta \cup (\neg) \Omega$, given by (i); consider the subset $|M|$ of $N$ whose elements are the denotations of the constants in $\mathcal{X}$. The set $|M|$ is closed under the operations of $L$ as seen by an application of (iii). Thus, we may consider the submodel $M$ of $N$ on the set $|M|$. We show by induction on the complexity of the $L(\mathcal{X})$-sentence $\phi$ that $\theta \vDash \phi \Rightarrow M \vDash \phi$, and $(\neg) \Omega \vDash \neg \phi \Rightarrow M \vDash \neg \phi$. For $\phi$ atomic, this is clear by the construction of $M$. If $\phi := t$ or $\phi := f$, the assertions are a consequence of (i). For $\phi := \psi \land \theta$, the first assertion is automatic by induction, and the second follows by (iv) and induction. The remaining clauses are treated in a similarly direct manner. The claim is established.

Now, back to the data of the proposition. Let us fix a countable set $\mathcal{X}_0$ of variables such that for each sort $S$ of $L$, there are infinitely may variables in $\mathcal{X}_0$ of sort $S$; all variables of all formulas are to be from $\mathcal{X}_0$; $\mathcal{X}$, to be named later, will be a subset of $\mathcal{X}_0$. An approximation (of a description of a model) is a triple $(\vec{u}, \theta, v)$ with the free variables in $\theta$ and $v$ included in $\vec{u}$, and such that $T \vDash \theta \Rightarrow v$; $\theta$ is to be true (at the given free variables as constants) in the model to be constructed, $v$ to be false. We note the following properties (v)–(xi) of this notion; assume throughout that $(\vec{u}, \theta, v)$ is an approximation.

(v) $(\emptyset, t, f)$ is an approximation. By $T$ being consistent.

(vi) $\theta \land \neg v$ is $L(\vec{u})$-consistent. This is direct from the definition of approximation.
(vii) For any formula of the form $\varphi \lor \psi$ with free variables among the $\vec{u}$, if $\vdash \theta \Rightarrow \varphi \lor \psi$, then either $(\vec{u}, \theta \land \varphi, v)$, or $(\vec{u}, \theta \land \psi, v)$ is an approximation.

(viii) For any formula of the form $\exists x \varphi$ with free variables among the $\vec{u}$, if $\vdash \phi \Rightarrow \exists x \varphi$, then, for any variable $y$ matching $x$ such that $y$ is not in $\vec{u}$, $(\vec{u}, \theta \land \varphi^y, v)$ is an approximation.

(ix) For any formula of the form $\varphi \land \psi$ with free variables among the $\vec{u}$, if $\vdash \theta \Rightarrow \varphi \land \psi$, then either $(\vec{u}, \theta, v \lor \varphi)$, or $(\vec{u}, \theta, v \lor \psi)$ is an approximation.

(x) If $\varphi \not\vdash \psi \in \Sigma$ (the axioms of the given theory $T$), and $\vec{y}$ is a copy of $\vec{x}$ in $\vec{u}$, then either $(\vec{u}, \theta \land \varphi^\vec{y}, v)$, or $(\vec{u}, \theta, v \lor \varphi^\vec{y})$ is an approximation. This follows, by contraposition, from the fact (immediately seen, semantically) that

$$\varphi \not\vdash \psi, \quad \theta \land \varphi^\vec{y} \not\vdash \vec{u}, \quad \theta \not\vdash v \lor \varphi^\vec{y} \not\vdash \vec{u}.$$
structure!). Let \( H \) be countable coherent category, with \( A \) a set of distributive objects in \( H \). We seek a subcategory \( K \) of \( \text{Mod}_{\text{cSo}}(H) \), the category of countable models of \( H \) such that the evaluation \( e: H \to \text{Set}^K \) is conservative, conditionally Heyting, and maps every \( A \in A \) to a componentwise surjective functor. As it was pointed out after the statement of Theorem 6.2, necessarily, all objects of \( K \) have to be \( A \)-standard, and all arrows of \( K \) have to be \( A \)-surjective. The next theorem says that the maximal choice for \( K \) under the said restrictions works.

**Theorem 9.2.** Let \( H \) be countable coherent category, with \( A \) a set of distributive objects in \( H \). Let \( K = \text{Mod}_{\text{cSo}}^A(H) \) be the category of countable \( A \)-standard models of \( H \), with arrows the \( A \)-surjective natural transformations. Then the evaluation \( e: H \to \text{Set}^K \) is conservative and conditionally Heyting.

**Proof.** By Lemma 6.3 (with \( \lambda = \aleph_0 \)), \( e \) is conservative. According to (2.5), the Heyting character of \( e \) depends on the truth of the following statement:

\[
\text{Given any } f: X \to Y, \Phi \in \mathcal{S}(X) \text{ in } H, \ M \in K \text{ and } y \in M(Y) - \forall_f(\Phi), \text{ there is } h: M \to N \text{ in } K \text{ with } x \in M(X) - M(\Phi) \text{ and } h(y) = (Mf)(x). \quad (*)
\]

Assume the data and hypotheses in (\( * \)). As in the proof (in [29]) of Joyal's theorem (3.4), \( h: M \to N \) is sought as a model \( \hat{N} = (N, h(c))_{c \in M} \) of the theory \( T = (\mathcal{L}(M) \cup \{x\}, \Sigma), x:X \), where

\[
\Sigma = \text{Def}^+ (M) \cup \{ f(x) = y \} \cup \{ \neg \Phi(x) \}.
\]

(\( T \) is, essentially, a coherent theory; e.g., \( \neg \Phi(x) \) is the same as the coherent entailment \( \Phi(x) \Rightarrow f \); remember that \( x \) is a constant here!)

The additional conditions are that (i) for each \( A \in A \), \( h_A \) is a surjection, and (ii) \( N \) is \( A \)-standard. As to (i), to say that \( h_A \) is surjective is equivalent to saying that \( \hat{N} \) satisfies the infinitary entailment

\[
\Lambda_A \overset{\text{def}}{=} \bigwedge \emptyset \Rightarrow \bigvee_{a \in MA} a = a.
\]

Turning to (ii), consider an instance of the \( A \)-standardness condition given by \( A \in A \), \( \pi: A \times U \to U \) a product projection in \( H \), \( \Psi \in \mathcal{S}(A \times U) \). To say that \( N(\forall_x \Psi) = \forall_{x\in A}(N \Psi) \) as subsets of \( NU \) is to say that, for \( u \in NU \),

\[
u \in N(\forall_x \Psi) \iff \forall a \in NA. (a, u) \in N \Psi.
\]

Here, the left-to-right implication is automatic; thus, the implication from right to left is the required condition on \( \hat{N} \). Assume now that \( N \) is given with \( h: M \to N \) such that \( h_A : MA \to NA \) is a surjection; this will certainly hold if \( \hat{N} \) satisfies the \( \Lambda_A \) above. Then, the quantifier "\( \forall a \in NA " \) can be replaced by the quantifier "\( \forall a \in MA " \); more precisely,

\[
\forall a \in NA. (a, u) \in N \Psi \iff \forall a \in MA. (h_Aa, u) \in N \Psi.
\]
For any given \( u \in NU \), the right-hand side here is the same as to say that we have \( \mathcal{N} \models \wedge_{a \in MA} \Psi au \). Therefore, the condition of \( N \) satisfying the given instance of \( A \)-standardness is that it satisfy the infinitary entailment

\[
A_{(A, v, \Psi)} \overset{\text{def}}{=} \wedge_{a \in MA} \Psi au \Rightarrow [\forall a] \Psi au
\]

(for the \([ \ ]\) notation, see p. 48). We conclude that to prove \((*)\), we need a model of \( T \) satisfying all the conditions \( A_\Delta \) and \( A_{(A, v, \Psi)} \) given above.

We apply Proposition 9.1. \( T \) is consistent, by (the proof of) the Joyal theorem. Let us show that the infinitary entailments are locally consistent with \( T \). We use the \( A \)-standardness of \( M \) in the form stated in the next lemma; here, \( A \) is an object in \( A, a: A \); we abbreviate the theory \((L_\mathcal{H}(M), \Sigma_\mathcal{H} \cup \text{Diag}^+M)\) as \( T_M = (L_M, \Sigma_M) \).

**Lemma.** Let \( \varphi(a, \bar{w}), \psi(a, \bar{w}) \) be coherent formulas over \( L_M \), and assume that for all \( a \in MA \), we have

\[
T_M \vdash \varphi(a, \bar{w}) \Rightarrow_{\bar{w}} \psi(a, \bar{w})
\]

(equivalently,
\[
T_M \vdash \varphi(a, \bar{w}) \land a = a \Rightarrow_{\bar{w}} \psi(a, \bar{w})
\]

Then
\[
T_M \vdash [\forall a] [\forall \bar{w}] (\varphi(a, \bar{w}) \rightarrow \psi(a, \bar{w}))
\]

and, as a consequence,
\[
T_M \vdash \varphi(a, \bar{w}) \Rightarrow_{\bar{w}} \psi(a, \bar{w})
\]

**Proof of Lemma.** Assume the hypothesis. Consider the formula

\[
\tau(a) := [\forall \bar{w}] (\varphi(a, \bar{w}) \rightarrow \psi(a, \bar{w}))
\]

As a first step, we show that

\[
(M, c)_{c \in |M|} \vdash \tau(a) \quad \text{for all } a \in MA.
\]

Fix \( a \in MA \) for a moment. By compactness, there is \( \delta \in \text{Diag}^+M \) such that in (9.1), \( \text{Diag}^+M \) can be replaced by \( \delta \). Let \( \bar{z} \) be a tuple of elements of \( M \) including \( a \) and including those that appear in \( \delta, \varphi(a, \bar{w}) \) or \( \psi(a, \bar{w}) \). Choose a tuple \( \bar{z} \) of variables matching \( \bar{z} \) and disjoint from \( a \) and \( \bar{w} \); let \( z \) in \( \bar{z} \) correspond to \( a \) in \( \bar{z} \). Let \( \delta(a, \bar{z}), \varphi(a, \bar{w}, \bar{z}), \psi(a, \bar{w}, \bar{z}) \) be \( L_\mathcal{H} \)-formulas such that \( \delta, \varphi(a, \bar{w}) \) and \( \psi(a, \bar{w}) \) are identical to the substitution instances \( \delta(\bar{z}), \varphi(a, \bar{w}, \bar{z}), \psi(a, \bar{w}, \bar{z}) \), resp. It follows that

\[
T_\mathcal{H} \vdash (\delta(\bar{z}) \land z = a) \land \varphi(a, \bar{w}, \bar{z}) \Rightarrow_{\bar{w} \bar{z}} \psi(a, \bar{w}, \bar{z}),
\]
that is,
\[
[aaw\bar{z}:\delta(z) \land z = a] \land [aaw\bar{z}:\varphi(a, \bar{w}, z)] \leq [aaw\bar{z}:\psi(a, \bar{w}, z)].
\]

We infer that
\[
[aaw\bar{z}:\delta(z) \land z = a] \leq [aaw\bar{z}:\varphi(a, \bar{w}, z)] \rightarrow [aaw\bar{z}:\psi(a, \bar{w}, z)],
\]
and
\[
[aaw\bar{z}:\delta(z) \land z = a] \leq \forall_*([aaw\bar{z}:\varphi(a, \bar{w}, z)] \rightarrow [aaw\bar{z}:\psi(a, \bar{w}, z)]).
\]
for \( \pi: [aaw\bar{z}] \rightarrow [aaw\bar{z}] \) the projection; and this is the same as
\[
[aaw\bar{z}:\delta(z) \land z = a] \leq [\forall w](\varphi(a, w, z)[\rightarrow \psi(a, w, z))].
\]
Since \( M^\delta(z) \land z = a \) \( [a/a, \bar{z}/\bar{z}] \), the assertion follows.

Now, consider \( [\forall a] \tau \). From (9.4), and the fact that \( M \) is \( A \)-standard, we conclude that
\[
(M, C)_{C \in M} \vDash [\forall a] \tau.
\]
This says that \( [\forall a] \tau \in \text{Diag}'M' \); (9.2) follows. Note that
\[
T_{M} \vdash [\forall a] \tau \land \varphi(a, \bar{w}, z) \Rightarrow_{xw} \psi(a, \bar{w}, z).
\]
Eq. (9.3) follows.

Note that for any coherent entailment \( \theta \Rightarrow_{u} v \) over \( L_M \cup \{x\} \) \((x \notin \bar{u})\),
\[
T \vdash \theta \Rightarrow_{u} v \iff T_M \vdash f(x) = y \land \theta \Rightarrow_{xu} v \lor \Phi(x). \quad (9.5)
\]

Let us check that \( A_{A} \) \((A \in A)\) is locally consistent with \( T \). Let \( \bar{u} \) contain \( a \) but not \( x \) (the latter from \( T \)); let \( \theta \Rightarrow_{u} v \) be a coherent entailment over \( L_M \cup \{x\} \). Assume \( T \vdash \theta \land a = a \Rightarrow_{xu} v \) for all \( a \in M(A) \). By (9.5),
\[
T_M \vdash f(x) = y \land \theta \land a = a \Rightarrow_{xu} v \lor \Phi(x) \quad \text{for all } a \in M(A).
\]
By (9.3),
\[
T_M \vdash f(x) = y \land \theta \Rightarrow_{xu} v \lor \Phi(x);
\]
and by (9.5) again, \( T \vdash \theta \Rightarrow_{u} v \) as desired.

Next, we turn to the local consistency of \( A_{A,A,v} \). Now, we have \( \theta \Rightarrow_{u} v \), \( u \in \bar{u}, x \notin \bar{u}, a \notin \bar{u} \), and as assumptions
\[
T \vdash \theta \Rightarrow_{u} v \lor \varphi a u \quad \text{for all } a \in M(A),
\]
and
\[
T \vdash \theta \land [\forall a] \varphi a u \Rightarrow_{u} v.
\]
That is,
\[ T_M \vdash f(x) = y \land \theta \Rightarrow \vee \Psi au \lor \Phi(x) \quad \text{for all } a \in M(A) \]
(9.6)
\[ T_M \vdash f(x) = y \land \theta \land [\forall a] \Psi au \Rightarrow \vee \Phi(x). \]
(9.7)

By (9.6) and the lemma,
\[ T_M \vdash [\forall x u \forall a] (f(x) = y \land \theta \land [\rightarrow] v \lor \Psi au \lor \Phi(x)). \]
(9.8)

Let \( \sigma \) be the formula \([\forall a] (f(x) = y \land \theta \land [\rightarrow] v \lor \Psi au \lor \Phi(x)) \), and let \( \bar{y} \) be all the \( M \)-constants in \( \sigma \). Then, by "intuitionistic logic" \((\forall a (P \rightarrow Qa) \equiv P \rightarrow \forall a Qa) \) provided \( a \) is not free in \( P \),
\[ [\bar{y} x u : \sigma] = [\bar{y} : f(x) = y \land \theta \land [\rightarrow] (v \lor \Psi au \lor \Phi(x))]; \]
by the distributivity of \( A \),
\[ [\bar{y} x u : [\forall a] (v \lor \Psi au \lor \Phi(x))] = [\bar{y} x u : v \lor \Phi(x) \lor [\forall a] \Psi au]; \]
thus,
\[ [\bar{y} x u : \sigma] = [\bar{y} x u : f(x) = y \land \theta \land [\rightarrow] v \lor \Phi(x) \lor [\forall a] \Psi au]. \]

With (9.8), this implies
\[ T_M \vdash f(x) = y \land \theta \Rightarrow v \lor \Phi(x) \lor [\forall a] \Psi au, \]
which, together with (9.7), gives by "coherent logic",
\[ T_M \vdash f(x) = y \land \theta \Rightarrow v \lor \Phi(x), \]
that is, by (9.5)
\[ T \vdash \theta \Rightarrow v, \]
as desired. \( \square \)

Theorem 9.2 is a generalization of Görnemann's [10] completeness theorem on intuitionistic logic with the constant domains axiom. In fact, also the proof given here is a generalization of the original proof; see Chapter 3, Section 3 of [5]. Görnemann's theorem was formulated for ordinary one-sorted logic; but the essential hypothesis on \( H \) for the original proof to go through without essential change is that every object of \( H \) is a subobject of some \( A \in A \). Given this assumption, the proof of (\( \ast \)) reduces to the treatment of implication. We outline the argument, to provide a comparison with the proof of the general result, Theorem 9.2. The argument below is essentially the same as e.g. the proof of 3.3.2 in [5].

Assume the hypotheses and the notation of Theorem 9.2; assume, in addition, that \( H \) is Heyting, and for every object \( X \in H \), there is a monomorphism \( X \to A \) into some \( A \in A \). The fact that \( e \) is conservative is proved as before (6.3). To show that \( e \) is
Heyting, it suffices to show that \( e \) preserves \( \forall_x \) for every product projection \( \pi: A \times X \to X \) with \( A \in A \), and that \( e \) preserves the implication \( \Phi \to \Psi \), with \( \Phi, \Psi \in S(A), A \in A \); this follows from formula (3.2') and the second additional hypothesis. However, the preservation of the \( \forall_x \) by \( e \), with the said \( \pi \), is automatic. In fact, for \( M \in K \), and the projection \( e_M: \text{Set}^K \to \text{Set}, M \cong e \circ e_M \), and \( M \) preserves \( \forall_x \) (the definition of \( K \)); since the \( e_M \) are jointly conservative, the assertion follows.

It remains to show the preservation of the implications. Interestingly enough, in the proof we will have another use of the (second) additional assumption.

We need to show

\[
\text{Given } A \in A, \Phi, \Psi \in S(A), M \in K, a \in M(A) \text{ such that } a_0 \notin M(\Phi \to \Psi), \text{ there is } h: M \to N \in K \text{ (in particular, } h \text{ } A\text{-surjective) such that } h_A(a_0) \in N(\Phi) \setminus N(\Psi). \quad (**)
\]

It suffices to show that

\[
\Sigma \overset{\text{def}}{=} \Sigma_H \cup \text{Diag}^+(M) \cup \{\Phi(a_0), \neg \Psi(a_0)\}
\]

has an \( L(M) \)-model whose \( L_H \)-reduct is \( A \)-standard, and in which every element is the denotation of some \( L(M) \)-constant; the latter ensures that the resulting \( h: M \to N \) is \( A \)-surjective.

Define \( (\theta, \psi) \) to be an \textit{approximation} if \( \theta, \psi \) are coherent \( L(M) \)-sentences, and for \( \Sigma_0 \overset{\text{def}}{=} \Sigma_H \cup \text{Diag}^+(M) \), we have \( \Sigma_0 \vdash \theta \iff \psi \); there are no free variables in the entailment involved. We claim the following properties; \( (\theta, \psi) \) denotes an arbitrary approximation.

(xii) \( (\neg (\exists (\forall a_{\phi}), \neg \exists (\forall a_{\phi})) \text{ is an approximation. This is by } a_0 \notin M(\Phi \to \Psi) \text{ and (the proof of) Joyal's theorem (3.4).} \)

(xiii) \( \text{If } \Sigma_0 \vdash \theta \Rightarrow \phi \Rightarrow \psi, \text{ then either } (\theta \land \phi, \psi) \text{ or } (\theta \land \psi, \psi) \text{ is an approximation. Immediate.} \)

(xiv) \( \text{If } A \in A, a: A, \text{ and } \Sigma_0 \vdash \forall a_{\phi(a)} \Rightarrow \psi, \text{ then there is } a \in M(A) \text{ such that } (\theta, \psi \lor \phi(a)) \text{ is an approximation. Otherwise, } \Sigma_0 \vdash \theta \Rightarrow \psi \lor \phi(a) \text{ for all } a \in M(A). \text{ This means that } M^* \vdash \theta[\to (\psi \lor \phi(a))] \text{ for all } a \in M(A), \text{ hence, by } M \text{ being } A\text{-standard, } M^* \vdash [\forall a] (\theta[\to (\psi \lor \phi(a))] \text{ and } M^* \vdash \theta[\to (\forall a)(\psi \lor \phi(a))]; \text{ using that } M \text{ is distributive, we get } M^* \vdash \theta[\to (\forall a)(\psi \lor \phi(a))], \text{ i.e., } M^* \vdash \theta[\to \psi], \text{ hence, } \theta[\to \psi] \in \text{Diag}^+(M), \text{ and } \Sigma_0 \vdash \theta \Rightarrow \psi, \text{ contradiction.} \)

(xv) \( \text{If } A \in A, a: A, \text{ and } \Sigma_0 \vdash \theta \Rightarrow \exists a_{\phi(a)}, \text{ then there is } a \in M(A) \text{ such that } (\theta \land \phi(a), \psi) \text{ is an approximation. Otherwise, } \Sigma_0 \vdash \theta \land \phi(a) \Rightarrow \psi, \text{ that is, } M^* \vdash \phi(a)[\to (\theta[\to \psi]) \text{ for all } a \in M(A); \text{ using that } M \text{ is } A\text{-standard, we get as before that } [\forall a] (\phi(a)[\to (\theta[\to \psi])) \in \text{Diag}^+(M). \text{ But, intuitionistically, } \forall a(\phi(a) \to \psi) \vdash (\exists a_{\phi(a)} \to \psi) \text{ is valid provided } \psi \text{ does not have } a \text{ free; we conclude that } M \vdash (\exists a_{\phi(a)}[\to (\theta[\to \psi]))], \text{ hence } \Sigma_0 \vdash \theta \land \exists a_{\phi(a)} \Rightarrow \psi, \text{ and } \Sigma_0 \vdash \theta \Rightarrow \psi, \text{ contradiction.} \)
(xv) For arbitrary $X \in H, x:X$, if $\Sigma_0 \vdash \theta \Rightarrow \exists x\phi(x)$, then there is $x \in M(X)$ such that $(\theta \land \phi(x), x)$ is an approximation. This follows from (xv) since there is a monomorphism $m:X \rightarrow A$, with $A \in A$. Writing $\Phi = [m] \in S(A)$, and $\phi(a)$ for a formula for which $\Sigma_0 \vdash \phi(x) \Rightarrow \phi(m(x))$, we have that

$$\Sigma_0 \vdash \forall x \phi(x) \Rightarrow \exists a(\Phi(a) \land \phi(a));$$

applying (xv) to $\exists a(\Phi(a) \land \phi(a))$, we easily obtain the desired conclusion.

(xvi) Like (x) above, for $\Sigma_H$ in place of $\Sigma$ in (x); immediate.

Using (xi)–(xvi), by a construction such as the one in the proof of Proposition 9.1, we obtain a pair $(\Theta, \Omega)$ of sets of $L(M)$-sentences, satisfying (i)–(iv) above, with $\mathcal{X}$ the set of $L(M)$-constants, and such that $\text{Diag}^+(M) \subset \Theta, \Phi(a_0) \in \Theta, \Psi(a_0) \in \Omega$, and such that, the $L(M)$-model $N$ of $\Theta \cup \neg \Omega$ satisfies $\Sigma_H$ (use(xvi)), and finally such that $N$ is $A$-standard (use (xiv)). Then, $N$ is a model of the above $\Sigma$, with the required additional properties.

As mentioned above, we do not know whether Theorem 9.2 holds with “biHeyting” replacing “Heyting”.

We have a result similar to Theorem 9.2 for modal logic.

**Theorem 9.3.** With the assumptions of Theorem 6.6, and with $H, D$ countable, and with $K = \text{Mod}^4_{\leq N_0}(H), J = \text{Mod}^4_{\leq N_0}(D)$ (see Theorem 9.2), the evaluation $(e_K, e_J): (\cdot: H \rightarrow D) \rightarrow (\text{Set}^x \xrightarrow{\psi} \text{Set}^1)$ is conservative and conditionally $S4$.

The proof is similar to that of Theorem 9.2; we omit the details. Again, we do not have the co$S4$ part of the picture. However, thanks to a proof given by Ghilardi and Meloni in [8], we do have the bi$S4$ case of Theorem 9.3 for bi$S4$ categories rather than pre-bi$S4$ ones.

Let $(\psi:H \rightarrow D) \in \text{Coh}$, with $\psi$ conservative, $D$ Boolean, and both $H$ and $D$ countable; we may write $\cdot$ for $\psi$; let $A$ be a set of Barcan objects in $H$. Let $K$ and $J$ be as in Theorem 9.3; note that all models of $H$ are $A$-standard, since $D$ is Boolean. Thus, $|J|$ is the discrete category of all countable models of $D$. Let $L = J \downarrow K$; the objects of $L$ are the countable models of $D$; the arrows the $H$-homomorphisms between the $H$-reducts that are $A$-surjective. Consider

$$D \xrightarrow{e_D} \text{Set}^{|L|}$$
$$\psi \downarrow \quad \uparrow \iota^*_L$$
$$H \xrightarrow{e_H} \text{Set}^L$$

(9.9)

as constructed, from $K$ and $J$, in (3.21); $e_D, e_H$ are, essentially, evaluations. Clearly, for all $A \in A$, $e_H(A)$ is a componentwise surjective functor. Theorem 9.3 implies (just as (3.15') implies (3.21)) that (9.9) is a conservative, conditionally $S4$ morphism. The following argument due to Ghilardi and Meloni [8], applied here in a somewhat more
general context, shows that

\[ (9.9) \text{is conditionally co} \text{S}4. \quad (9.10) \]

Let \( U \in \mathcal{U}, \Phi \in S(U), \bigcirc \Phi \in S(U) \) exist, \( M \in \text{Mod}_{\bigcirc \Phi}(D), u \in M(\bigcup \Phi) \subset M(U) \); we want to construct \( N \in \text{Mod}_{\bigcirc \Phi}(D) \) with \( A \)-surjective \( h: N|H \to M|H \) and \( \dot{u} \in N(\Phi) \subset N(U) \) such that \( h_0(\dot{u}) = u \). We construct \( N \) and \( h \) approximating them by finite pieces. Let \( \mathcal{X}_0 \) be a countable set of individual constants as in the proof of OTT.

An approximation is, by definition, a triple \((V, f, \varphi)\) of a finite subset \( V \) of \( \mathcal{X}_0 \), a function \( f \) with domain a subset of \( V \) consisting of variables in \( V \) of sorts \( \mathcal{X} \) for some \( X \in H \) (such variables are called \( H \)-variables), and with range a subset of \([M]\), and of a formula \( \varphi \) over \( L_n \) with free variables included in \( V \) such that the condition below holds. Let \( x \) be a repetition free tuple of the variables in \( \text{dom}(f); f(x) \) is the corresponding tuple of elements of \( M \), \( \dot{x} \) one that lists the remaining variables in \( V \). Let \( X \in H \) with \( \dot{X} = [\dot{x}] \) and consider \([\dot{x}; \exists \dot{z}\varphi] \in S(\dot{X})\), \( \circ [\dot{x}; \exists \dot{z}\varphi] \in S(X) \). The condition on \((V, f, \varphi)\) is that \( \dot{f}(\dot{z}) \in M(\circ [\dot{x}; \exists \dot{z}\varphi]) \). One is tempted to say \( M_{\dot{f}}(\dot{x}) \) for the same; this is correct as long as we imagine the subscript \( \dot{x} \) attached to \( \circ; M_{\dot{f}}(\dot{x}) \). Implicit in the condition is that \( f \) maps a variable to an element of the same sort as the variable. Let us denote the set of all approximations by \( \mathcal{A} \).

With \( \dot{u} \in \mathcal{X}_0 \), \( \dot{u}: \dot{U} \) fixed, the assumptions tells us that

\[ (\text{xvii}) \] \([\dot{u}], [\dot{u} \mapsto u], \Phi \dot{u} \in \mathcal{A} \].

\[ (\text{xviii}) \text{If} (V, f, \varphi) \in \mathcal{A} \text{ and } T_D \vdash \varphi \Rightarrow \psi_1 \vee \psi_2, \text{then} (V, f, \varphi \land \psi_1) \in \mathcal{A} \text{ for } i = 1 \text{ or } i = 2. \]

This is because \( \varphi \equiv (\varphi \land \psi_1) \lor (\varphi \lor \psi_2) \), hence, \( \circ \exists \dot{z}\varphi \equiv \circ \exists \dot{z}(\varphi \land \psi_1) \lor \circ \exists \dot{z}(\varphi \lor \psi_2) \) (with a slightly imprecise but suggestive notation).

\[ (\text{xix}) \text{If} (V, f, \varphi) \in \mathcal{A} \text{ and } T_D \vdash \varphi \Rightarrow \exists \dot{y}\psi \text{ with } z \in \mathcal{X}_0, z \notin V, z \text{ not an } H\text{-variable, then} (V \cup \{z\}, f \mid_{V \cup \{z\}}, \varphi \land \psi) \in \mathcal{A}. \]

Because \( \varphi \equiv \exists \dot{z}(\varphi \land \psi) \), and \( \circ \exists \dot{z}\varphi \equiv \circ \exists \dot{z}(\varphi \land \psi) \). This is because the “Barcan formula for \( \circ \) and \( \exists \)" is (automatically) true, expressing as it does the fact that preserves \( \exists \). Thus, the assumption \((V, f, \varphi) \in \mathcal{A} \) gives us a witness \( x \in [M] \) for the leading existential quantifier \( \exists x \) in the last formula; this \( x \) works as desired.

\[ (\text{x}xii) \text{If} (V, f, \varphi) \in \mathcal{A}, A \in A \text{ and } a \in M(A), \text{then for any } a \in \mathcal{X}_0 \text{ such that} a: A, a \notin V, \text{we have} (V \cup \{a\}, f \mid_{V \cup \{a\}}, \varphi) \in \mathcal{A}. \]

This is because \( \circ \exists \dot{z}\varphi \equiv \circ \exists \dot{z}(\varphi \land \psi) \), that is, \( \circ [\dot{x}; \exists \dot{z}\varphi] = \pi^* [\dot{x}: \exists \dot{z}\varphi] = \pi^* [\dot{x}, A \to X \exists \dot{z}\varphi] \), with \( \pi: X \times A \to X \) the projection; this is the substitutivity of \( \circ \) along \( \pi \), which is a consequence of the Barcan property of \( A \).

An approximation \((V', f', \varphi')\) extends another, \((V, f, \varphi)\), if \( V \subset V', f \subset f' \) and \( T_D \vdash \varphi' \supset \varphi \). Using (xviii)-(xix), we can easily construct an \( \omega \)-type sequence of approximations \( \langle x_n = (V_n, f_n, \varphi_n) \rangle_{n < \omega} \) such that for \( m < n, x_n \) extends \( x_m \), and such that the sequence is closed under the (possibly non-deterministic) closure conditions implicit in (xvii)-(xxi); call these facts (xvii)*-(xxi)*. Let \( \mathcal{X} = \bigcup_{n < \omega} V_n, \Theta \) the set of
formulas $\theta$ over $L_D \cup \mathcal{A}$ such that $T_D \vdash \varphi_n \Rightarrow \theta$ for some $n < \omega$. $\Theta$ is closed under logical consequence; it is consistent; if $\psi_1 \lor \psi_2 \in \Theta$, then $\psi_i \in \Theta$ for $i = 1$ or $i = 2$ (by (xviii)*); if $\exists u \psi \in \Theta$, then $\psi^{*} \in \Theta$ for a suitable $v$ (by (xix)* and (xx)*). Also using that $D$ is Boolean, it follows that $(\mathcal{A}, \Theta)$ is a consistent and complete extension of $T_D$, with a model $(N, \tilde{u})_{u \in \mathcal{A}}$ all whose elements are denotations of constants in $\mathcal{A}$.

Let $h: [N]^{H} \rightarrow [M]^{H}$ be given by $h(\tilde{u}) = f_n(u)$, with some (any) $n < \omega$ such that $u \in \text{dom}(f)$. It is easy to see that $h$ is well-defined, and it respects sorts. Fact (xv)* says that $h$ is $A$-surjective. To see that $h$ is an $H$-homomorphism, consider an object $Y$ in $H$, $\Psi \in \mathcal{S}(Y)$, and assume $\tilde{y} \in N(\Psi)$ to show that $h_\Psi \tilde{y} = f_\Psi y \in M(\Psi)$. Let $n < \omega$ be a number such that, with $(\pi, f_n, \varphi_n) = (V, f, \varphi)$, $y \in V$, and $\tilde{y}$ is a consequence of $\varphi$, $T_D \vdash \varphi \Rightarrow \Psi y$. Let us use the notation introduced when spelling out the condition for $(V, f, \varphi)$ to be an approximation. It follows that $T_D \vdash \exists \tilde{z} \varphi \Rightarrow \Psi y$, hence, with $\Phi = [\tilde{x}: \exists \tilde{z} \varphi]$, and $\pi: X \rightarrow Y$ the projection, $\Phi \preceq_X \pi^*(\Psi)$. It follows that $\circ \Phi \preceq_X \pi^*(\Psi)$. The fact that $(V, f, \varphi)$ is an approximation gives $f\tilde{x} \in M(X)$; thus, $f\tilde{x} \in M(\pi^*(\Psi))$, which is the same as $fy \in M(\Psi)$ as desired.

Fact (xvii)* amounts to saying that $h_\Psi (\tilde{u}) = u$. The proof of (9.10) is complete.

Theorem 9.3 and (9.10) give the following theorem.

Theorem 9.4. Let $\psi: H \rightarrow D$ be a countable biS4 category, $A$ a set of Barcan objects in $H$. Then the canonical map (9.9), with $L$ the category of countable $D$-models with $A$-surjective $H$-homomorphisms, is a conservative biS4 morphism.

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References

[40] H. Zolfaghari, These de doctorat, Université de Montréal (1992).