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On the Combinatorial Classification of Nondegenerate Configurations in the Plane

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We classify nondegenerate plane configurations by attaching, to each such configuration of n points, a periodic sequence of permutations of $\{1, 2,..., n\}$ which satisfies some simple conditions; this classification turns out to be appropriate for questions involving convexity. In 1881 Perrin stated that every sequence satisfying these conditions was the image of some plane configuration. We show that this statement is incorrect by exhibiting a counterexample, for $n = 5$, and prove that for $n \leqslant 5$ every sequence essentially distinct from this one *is* realized geometrically by giving a complete classification of configurations in these cases; there is 1 combinatorial equivalence class for $n = 3$, 2 for $n = 4$, and 19 for $n = 5$. We develop some basic notions of the geometry of "allowable sequences" in the course of proving this classification theorem. Finally, we state some results and an open problem on the realizability question in the general case.

1. INTRODUCTION

An outstanding problem of combinatorial geometry has long been to classify, in a reasonable and effective way, nondegenerate configurations of n points in the plane—indeed in Euclidean space of any dimension—into finitely many "essentially distinct" classes. Any classification scheme can be described by mapping the set of nondegenerate configurations of n points into some finite set A and identifying configurations with the same image. The utility of such a scheme depends of course on (1) how faithfully and simply properties of interest are represented by the objects of A, and (2) how well we know the image of the map, i.e., which objects of A are

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geometrically realizable. What makes a property interesting, in turn, is determined by what kinds of problems one wants to resolve.

Our primary concern is with problems related to convexity, such as the Erdös–Szekeres conjecture [1, 2]. Suppose, for example, one wanted to ask a computer to test each nondegenerate plane configuration of 17 points to determine whether it contained the vertices of a convex hexagon. How could the computer, *even in principle*, generate finitely many 17-tuples of points such that any two were "essentially distinct," and any 17-tuple "essentially the same" as one of those generated? For this purpose, a natural classification scheme would be to associate with a configuration of points $P_1,..., P_n$ all those subsets $\{i_1,..., i_k\}$ of $[1, n]$, called semispaces, for which $P_{i_1},..., P_{i_k}$ lie on one side of a line and to consider two configurations equivalent if, after a possible relabeling of the points, they each have the same family of semispaces. Here A can be taken to be the family of subsets of $[1, n]$. While it is not hard to see that all questions pertaining to convexity are easily read off from a realizable element of A, it is far from clear which elements of A are realizable. We shall return to the question of the realizability of semispaces in a subsequent paper [3]. In this paper, which is the first in a series on the classification problem, we shall examine a finer classification scheme for which the question of realizability is somewhat more tractable and which also sheds light on the realizability question for semispaces.

This classification results from the assignment, to each nondegenerate configuration of n points in the Euclidean plane, a periodic sequence of permutations of the set $[1, n]$ which is determined by projecting the points of the configuration orthogonally onto a rotating directed line. The sequence determines the semispaces as initial or final segments of the various permutations and thus reflects the convexity properties of the configuration, as well as the classification by semispaces; we shall call two configurations "combinatorially equivalent" if—possibly after renumbering or reflecting—they give rise to the same sequence of permutations. Sequences obtained in this way satisfy a simple necessary condition (see Remark 2.3), and we call any sequence satisfying this condition an "allowable sequence of permutations." Our main concern in this paper is with the question of the geometric realizability of these allowable sequences.

In [5], Perrin, writing on the "problème des aspects" which had been proposed by Halphen, asserted—in reference to the sequnce of permutations associated to a nondegenerate configuration—"L'ordre dans lequel ces permutations se présenteront n'est pas complètement arbitraire, puisque, pour passer d'un aspect au suivant, on ne peut permuter que deux nombres contigus; *mais c'est la seule condition à remplir, comme il est facile de s'en assurer...*" [italics ours]. The condition he refers to is essentially what we call allowability, and so he asserts that all allowable sequences are realizable.

This assertion, which has apparently gone unchallenged for nearly 10(years, is in fact false. We shall give an example, for $n=5$, of an allowable sequence which is not realizable (Theorem 3.3), for an amusing geometric reason, and establish a further necessary condition for realizability (Corollary 3.2 of Theorem 3.1). On the other hand, all allowable sequences for $n \leqslant 5$ which are essentially distinct from this one *are* realizable, and are—it turns out—realized by precisely 1, 2, 19 combinatorially distinct configurations for $n=3, 4, 5$, respectively (Theorem 4.1). Along the way toward proving Theorems 3.3 and 4.1, we are led naturally to develop some notions of the geometry of nondegenerate configurations purely in terms of their associated sequences (Section 2); this offers an interesting subject for further investigation.

We would like to express our gratitude to William Sit for several valuable conversations while this work was in its formative stages, and to Thomas Zaslavsky for making us aware of another assault on the classification problem—one involving oriented matroids. Finally, we would like to thank Herman Hanisch for coming up with a proof (we now have four in all!) of the "amusing geometric fact" mentioned above, which we were later able to generalize to Theorem 3.1.

For an excellent bibliography on configurations of points, see [4], in particular the comments on p. 112.

Added in proof. The classification of nondegenerate configurations presented here can be extended to a classification of *arbitrary* plane configurations, and this turns out to be a key step in settling the conjecture of B. Grünbaum that every arrangement of eight pseudolines is stretchable; see our forthcoming paper "Proof of Grünbaum's conjecture on the stretchability of certain arrangements of pseudolines," to appear in *J. Combinatorial Theory, Ser. A.*

2. Some Geometry of Allowable Sequences

Definition 2.1. A *nondegenerate configuration of n points* is an ordered n-tuple of distinct points in the plane with no three points collinear and no two pairs of points lying on parallel lines. We shall think of the points of the configuration as labeled by the numbers $1, 2, \ldots, n$. Given a configuration C of n points and a directed line l which is not orthogonal to any line determined by two points of C, the orthogonal projection of C on l determines a permutation of $1, 2, \ldots, n$ in an obvious way. As the line l rotates counterclockwise about a fixed point we obtain a sequence of permutations of period $2 \cdot {}_nC_2 = n(n-1)$, which we shall call the *circular sequence of the configuration.*

EXAMPLE 2.2. Associated with the quadrilateral of Fig. 1 we have the circular sequence

$$\cdots 1432 \overset{32}{—} 1423 \overset{42}{—} 1243 \overset{12}{—} 2143 \overset{43}{—} 2134 \overset{13}{—} 2314 \overset{14}{—}$$
$$2341 \overset{23}{—} 3241 \overset{24}{—} 3421 \overset{21}{—} 3412 \overset{34}{—} 4312 \overset{31}{—} 4132 \overset{41}{—} 1432 \cdots. \tag{2.1}$$

Here we have indicated which "switches" take us from each permutation to the next. (We distinguish the ordered switch ij from the ordered switch ji in that the former indicates that i originally precedes j.) Note that the sequence of ordered switches in line (2.1) is nothing more than the sequence of vectors $\overrightarrow{ij}$ in Fig. 1 arranged in counterclockwise order.

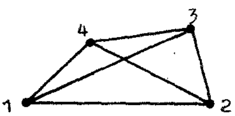

FIGURE 1

Remark 2.3. In the circular sequence of a configuration, (1) successive permutations differ only by having the order of two adjacent numbers switched, and (2) any ${}_nC_2$ consecutive permutations make use of all ${}_nC_2$ possible switches in passing from each to the next. Property (1) corresponds to the switching of the projections of i and j on l as l rotates through the direction orthogonal to the line through i and j, while property (2) stems from the fact that as the line l rotates through an angle of π it passes orthogonally to each of the ${}_nC_2$ lines determined by the points of C. It is an immediate consequence of (1) and (2) that the ordered switch occurring ${}_nC_2$ steps after ij must be ji, and that the permutation ${}_nC_2$ steps after a given one must be its reverse.

DEFINITION 2.4. A sequence $(\ldots, P_{-1}, P_0, P_1, \ldots)$ of permutations of the numbers $1, \ldots, n$ which satisfies properties (1) and (2) of Remark 2.3 (hence is automatically of period $n(n-1)$) is called an *allowable circular sequence*. We say that two allowable circular sequences $(\ldots, P_{-1}, P_0, P_1, \ldots)$ and $(\ldots, Q_{-1}, Q_0, Q_1, \ldots)$ are equal if, for some k, $Q_{i+k} = P_i$ for all i.

DEFINITION 2.5. The restriction of an allowable circular sequence S to a subset $i_1 < i_2 < \cdots < i_k$ of $[1, n]$ will mean the sequence of permutations obtained by (1) deleting the numbers not in this subset from each permutation of S, (2) omitting repeated permutations, and (3) renaming i_j $(j = 1, \ldots, k)$ simply j. This clearly yields an allowable sequence, and the result corresponds to the subconfiguration $\{i_1,\dots,i_k\}$ of $[1,n]$ if S is realizable.

Clearly the circular sequence of a configuration is allowable. Equally clearly, an allowable circular sequence determines its periodic sequence of ordered switches. Our first proposition shows that the converse is also true.

PROPOSITION 2.6. *An allowable circular sequence is determined by its sequence of ordered switches.*

Proof. We use induction on n. If $n=3$ (the cases $n=1$ and $n=2$ being trivial), there are only two allowable sequences:

$$\cdots 123 \overset{12}{—} 213 \overset{13}{—} 231 \overset{23}{—} 321 \overset{21}{—} 312 \overset{31}{—} 132 \overset{32}{—} 123 \cdots \tag{2.2}$$

and

$$\cdots 123 \overset{23}{—} 132 \overset{13}{—} 312 \overset{12}{—} 321 \overset{32}{—} 231 \overset{31}{—} 213 \overset{21}{—} 123 \cdots, \tag{2.3}$$

and their sequences of ordered switches are distinct. Suppose $n>3$ and consider any switch involving n, say in. If π is the preceding permutation, it is sufficient to reconstruct π, for the sequence of switches—applied one at a time—will then determine the rest of the sequence. But if we restrict the sequence of switches to $[1,n-1]$, the induction hypothesis allows us to determine the restricted sequence, hence the restriction π_0 of π. To get π, we just insert n after i in π_0.

Remark 2.7. If a nondegenerate configuration is reflected about a line, the sequence of switches of the new configuration is the reverse of that of the old; hence the same is true of the sequence of permutations. It is clear that this operation, henceforth called *reflection*, when applied to *any* allowable sequence, yields another allowable one.

Since we do not wish to distinguish between configurations which are merely numbered differently, nor between those which are reflections of each other, we choose not to make the corresponding distinctions among sequences either. Hence

DEFINITION 2.8. Two allowable circular sequences are *combinatorially equivalent* if, by a suitable permutation of the numbers $1,\dots,n$ or a reflection or both, one is transformed into the other. Two configurations are *combinatorially equivalent* if they have combinatorially equivalent circular sequences.

We now introduce some geometric language which will facilitate our discussion of allowable sequences; it is motivated, of course, by the corresponding terminology for the configurations from which some of our sequences arise.

DEFINITION 2.9. The numbers in our permutations will generally be referred to as *points*. An allowable sequence involving n points will often be called simply an *n-sequence*. If S is an n-sequence and $j, i_1,..., i_k \in [1, n]$, we say *$j$ is in the convex hull of* $i_1,..., i_k$, written $j \in \operatorname{conv}(i_1,..., i_k)$, if j is one of $i_1,..., i_k$ or if j is preceded, in each permutation of S, by one of $i_1,..., i_k$ (hence followed as well). We say j is an *extreme point* of S if j occupies position 1 in some permutation of S, i.e., if j is not in the convex hull of the other points of S. (Note that every sequence with $n \geqslant 3$ has at least three extreme points.) If the extreme points of an n-sequence S occupy position 1 in the (circular) order ..., $i_1,..., i_k, i_1,...$ we call that the *counterclockwise* order of the extreme points. The set of points in any initial (hence in any terminal) segment of a permutation of S is called a *semispace* of S. If $\{i_1, i_2\}$ is a semispace of S consisting of two extreme points, we call $\{i_1, i_2\}$ an *edge* of S. A *triangle*, (*convex*) *quadrilateral*,..., (*convex*) *k-gon* is an n-sequence with $n = 3, 4,..., k$, respectively, all of whose points are extreme; the points are called the *vertices*. The 2-point subsets of a k-gon which are not edges are called *diagonals*. If the vertices of a k-gon are $i_1,..., i_k$ in counterclockwise order, we say that the edge or diagonal $\{i_\alpha, i_\beta\}$ is *parallel* to the edge or diagonal $\{i_{\alpha-\nu}, i_{\beta+\nu}\}$ (with the indices read modulo k).

Remark 2.10. If $j \in \operatorname{conv}(i_1,..., i_k)$ in an n-sequence S, then the same is true in any restriction or extension of S to a set containing j and $i_1,..., i_k$. Hence if j is an extreme point of S, it is an extreme point of any restriction of S to a set containing j. Similarly, the intersection of any semispace of S with a subset $\{i_1,..., i_k\}$ gives a semispace of the restriction of S so that subset.

DEFINITION 2.11. If the ordered switches $i_1 j_1,..., i_k j_k$ occur in that order in S, *with $i_1 j_1$ not recurring before $i_k j_k$*, we shall write $(i_1 j_1 \prec \cdots \prec i_k j_k)$. Note that if this holds in a sequence S it holds in any restriction or extension of S to a subset containing the points in question.

DEFINITION 2.12. Suppose 1, 2, 3, 4 are the vertices of a quadrilateral, in counterclockwise order. We say that *lines* 12 *and* 43 *meet on the side of points* 2 *and* 3 if $(43 \prec 12 \prec 34)$; otherwise they *meet on the side of points* 1 *and* 4.

DEFINITION 2.13. If 1, 2, 3, 4 are the points of a 4-sequence, we say that *line* 34 *separates points* 1 *and* 2, or *cuts segment* 12, if the following semispaces exist: $\{1\}$, $\{1, 3\}$, $\{1, 4\}$, and $\{1, 3, 4\}$. (Since the complement of one semispace is another, this condition is obviously symmetric in the points 1 and 2.)

In the rest of this section we give some "geometric" properties of allowable sequences which are needed for the classification theorem in Section 4. The proofs of all but Proposition 2.18 follow immediately from the definitions.

Remark 2.14. Suppose the extreme points of an n-sequence S are $i_1,\dots, i_k$ in counterclockwise order. Since in order to get from position 1 to position n such a point must make $n-1$ interchanges, each with a point initially to its right, it is clear that each extreme point moves monotonically left-to-right until it is in position n, then returns monotonically to position 1. It therefore makes sense to talk about the direction that the extreme point is moving in at some stage when it is not at either end of a permutation; by convention we shall also say it is moving to the *left* when it occupies position 1 or position n. Because of the monotonic motion in each direction, it is clear that *no two extreme points moving in the same direction can change places*; hence $i_1,\dots, i_k$ follow each other in their back-and-forth motion, changing places with each other only when two are moving in opposite directions. (Note: This "monotonicity in a half-period" does not hold, of course, for the nonextreme points, since if a point is in position p at its leftmost extreme, with $p>1$, it must make $p-1$ moves to the left and $n-p$ moves to the right by the end of a half-period.)

PROPOSITION 2.15. *Suppose the extreme points of an n-sequence are $i_1,\dots, i_k$ in counterclockwise order. Then*

(a) $(i_1 i_2 \prec i_2 i_3 \prec \cdots \prec i_k i_1 \prec i_1 i_2)$.

(b) $(i_1 i_2 \prec i_1 i_3 \prec \cdots \prec i_1 i_k \prec i_2 i_1)$.

(c) Every i_j belongs to exactly two edges—$\{i_{j-1}, i_j\}$ and $\{i_j, i_{j+1}\}$.

PROPOSITION 2.16. (a) $(i_1 j_1 \prec \cdots \prec i_k j_k) \Rightarrow (j_1 i_1 \prec \cdots \prec j_k i_k)$.

(b) If $(i_1 j_1 \prec \cdots \prec i_k j_k \prec j_1 i_1)$, $(i_k j_k \prec i_{k+1} j_{k+1} \prec j_k i_k)$, and $(i_1 j_1 \prec i_{k+1} j_{k+1} \prec \cdots \prec i_m j_m \prec j_1 i_1)$, then $(i_1 j_1 \prec \cdots \prec i_m j_m \prec j_1 i_1)$.

(c) $(ij \prec jk \prec ji) \Rightarrow (ij \prec ik \prec jk \prec ji)$.

(d) $(i_1 j_1 \prec i_2 j_2 \prec \cdots \prec i_k j_k \prec j_1 i_1) \Rightarrow (i_2 j_2 \prec \cdots \prec i_k j_k \prec j_1 i_1 \prec j_2 i_2)$.

PROPOSITION 2.17. *If* 1, 2, 3 *are the extreme points, in counterclockwise order, of a* 4-*sequence, then the sequence must be*

$$\cdots - 1423 - 1243 - 2143 - 2413 - 2431 - 2341 - 3241 - \cdots. \qquad (2.4)$$

The next proposition says that the convex hull of an n-sequence may be "triangulated" by joining any one of the extreme points to each of the others; each nonextreme point will then belong to precisely one of the triangles formed.

PROPOSITION 2.18 ("Triangulation lemma"). *Suppose the extreme points of an n-sequence S are $i_1, \ldots, i_k$ in counterclockwise order, and suppose $1 \leqslant \alpha \leqslant k$. If j is any point other than $i_1, \ldots, i_k$, there is a unique β ($1 \leqslant \beta \leqslant k$, $\beta \neq \alpha$, $\beta + 1 \neq \alpha$) such that $j \in \operatorname{conv}(i_\alpha, i_\beta, i_{\beta+1})$. ($\beta + 1$ is understood modulo k, of course.)*

Proof. Let P be the permutation of S immediately preceding the switch $i_\alpha j$, and let i_β and $i_{\beta+1}$ be the successive left-moving extreme points (see Remark 2.14) which surround j in P; they exist, by virtue of our convention about the endpoints of P. (Note that i_α cannot be the initial point of P because of the switch that is about to take place: j is not an extreme point.) In other words, P has the form

$$\cdots \overleftarrow{i_\beta} \cdots \overrightarrow{i_\alpha} j \cdots \overleftarrow{i_{\beta+1}} \cdots, \tag{2.5}$$

where the arrow indicates the direction in which the corresponding extreme point is moving. We claim $j \in \operatorname{conv}(i_\alpha, i_\beta, i_{\beta+1})$. It is sufficient to show that one of i_α, i_β, $i_{\beta+1}$ is always to the left of j. Since the (circular) order in which these three points reach position 1 is i_α, i_β, $i_{\beta+1}$, we can argue as follows: For the half-sequence immediately preceding the switch $i_\alpha j$, i_α is to the left of j. From that switch until i_β returns to j and the switch $i_\beta j$ takes place, i_β is to the left of j. When the switch $i_\beta j$ occurs, $i_{\beta+1}$ *is still to the left of* j, since it is "following" i_β, and it remains so until the end of the half-sequence, for the position then is exactly the reverse of the one shown in (2.5). To prove uniqueness, consider an index $\gamma \neq \beta$. If j is always surrounded by two of i_α, i_γ, $i_{\gamma+1}$, then since i_α switches with j as we go from P to the next permutation, P must have the form

$$\cdots \overleftarrow{i_\gamma} \cdots \overleftarrow{i_\beta} \cdots \overrightarrow{i_\alpha} j \cdots \overleftarrow{i_{\beta+1}} \cdots \overleftarrow{i_{\gamma+1}} \cdots,$$

which is impossible since $i_{\gamma+1}$ is the successor of i_γ among the left-moving extreme points in P.

COROLLARY 2.19 (Carathéodory's theorem in the plane). *If $j \in \operatorname{conv}(i_1, \ldots, i_k)$ then for some α, β, γ we have $j \in \operatorname{conv}(i_\alpha, i_\beta, i_\gamma)$.*

Remark 2.20. It may be shown, using the above, that if a line passes through a point inside a triangle, it cuts one side of the triangle; that any line cutting one side cuts a second side; and that no line cuts all three sides. This is Pasch's axiom, and we mention this fact to show that some standard theorems of ordered geometry can be proven for our "generalized configurations."

3. Unrealizable Sequences

THEOREM 3.1. *If* $1, 2, \ldots, n$ *are the vertices of a convex n-gon (numbered modulo n) listed in counterclockwise order, and if* $a, k > 0$ *and* $a + 2k < n$, *then it is impossible that the diagonals (or edges)* $\overline{i, i+a}$ *and the "parallel" diagonals (or edges)* $\overline{i-k, i+a+k}$ *intersect on the side of points* $i+a$ *and* $i+a+k$ *for all* i (Fig. 2).

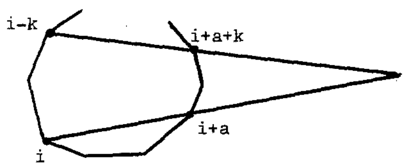

FIGURE 2

Proof. Let ω be a unit vector normal to the plane of the polygon, respecting the counterclockwise order of the vertices. Let us denote by v_i $(i = 1, \ldots, n)$ the vector $\overrightarrow{i-1, i}$. Suppose $i, i+a$ and $i-k, i+a+k$ meet on the side of points $i+a$ and $i+a+k$ for all i. Then $i-k$ is farther from line $i, i+a$ than $i+a+k$ is, i.e.,

$$\overrightarrow{i-k, i} \cdot \overrightarrow{i, i+a} \times \omega) > \overrightarrow{i+a+k, i+a} \cdot (\overrightarrow{i, i+a} \times \omega)$$

for $i = 1, \ldots, n$. Rewriting, we have

$$(\overrightarrow{i-k, i} \times \overrightarrow{i, i+a} + \overrightarrow{i+a, i+a+k} \times \overrightarrow{i, i+a}) \cdot \omega > 0,$$

or

$$\left(\sum_{i-k+1}^{i} v_p \times \sum_{i+1}^{i+a} v_p + \sum_{i+a-1}^{i+a+k} v_p \times \sum_{i+1}^{i+a} v_p \right) \cdot \omega > 0.$$

Rewriting the last two sums in descending order, we get

$$\left(\sum_{p=1}^{k} v_{p+i-k} \times \sum_{q=1}^{a} v_{q+i} + \sum_{p=1}^{k} v_{i+a+k+1-p} \times \sum_{q=1}^{a} v_{i+a+1-q} \right) \cdot \omega > 0,$$

which yields

$$\left(\sum_{i=1}^{n}\sum_{p=1}^{k}\sum_{q=1}^{a} v_{p+1-k} \times v_{q+i} + v_{i+a+k+1-p} \times v_{i+a+1-q}\right) \cdot \omega > 0$$

when we sum over i. If we interchange the order of summation we have

$$\sum_{p=1}^{k}\sum_{q=1}^{a}\left(\sum_{i=1}^{n} v_{p+i-k} \times v_{q+i} + \sum_{i=1}^{n} v_{i+a+k+1-p} \times v_{i+a+1-q}\right) \cdot \omega > 0.$$

Since the numbering is modulo n, we may add $p+q-a-k-1$ to each subscript in the second summation, getting

$$\sum_{p=1}^{k}\sum_{q=1}^{a}\left(\sum_{i=1}^{n} v_{p+i-k} \times v_{q+i} + \sum_{i=1}^{n} v_{q+i} \times v_{p+i-k}\right) \cdot \omega > 0,$$

which contradicts the fact that this sum obviously vanishes.

COROLLARY 3.2. *If an allowable circular sequence S is a convex n-gon with points $1,\ldots, n$ in counterclockwise order, and if—for each $i = 1,\ldots, n$—the diagonal (or edge) $\{i, i+a\}$ and the diagonal (or edge) $\{i-k,\ i+a+k\}$ intersect on the side of points $i+a$ and $i+a+k$ for all i, where $a, k > 0$ and $a+2k < n$, then S is not geometrically realizable.*

We can now give a counterexample, in the case $n = 5$, to Perrin's assertion [5, p. 119] that every allowable sequence is geometrically realizable:

THEOREM 3.3. *The allowable sequence*

$$\begin{aligned} \cdots\ 12534 \overset{12}{—} 21534 \overset{53}{—} 21354 \overset{54}{—} 21345 \overset{13}{—} 23145 \overset{23}{—} \\ 32145 \overset{14}{—} 32415 \overset{15}{—} 32451 \overset{24}{—} 34251 \overset{34}{—} 43251 \overset{25}{—} 43521 \cdots \end{aligned} \tag{3.1}$$

is not geometrically realizable.

Proof. Suppose (3.1) were realizable. Since each point is extreme, the configuration would be a convex pentagon. Moreover since $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 5\}$, and $\{5, 1\}$ are the edges of (3.1), they are the edges of the pentagon as well, and the vertices appear in the counterclockwise order 1, 2, 3, 4, 5 (in both senses). Finally, since $(12 \prec 53 \prec 21)$, $(23 \prec 14 \prec 32)$, $(34 \prec 25 \prec 43)$, $(45 \prec 31 \prec 54)$, and $(51 \prec 42 \prec 15)$, each side and its "parallel" diagonal must meet as in Fig. 3, i.e., 12 meets 53 on the side of 1 and 5,..., 51 meets 42 on the side of 5 and 4, again in both senses (see Definition 2.12). Corollary 3.2 therefore yields the result.

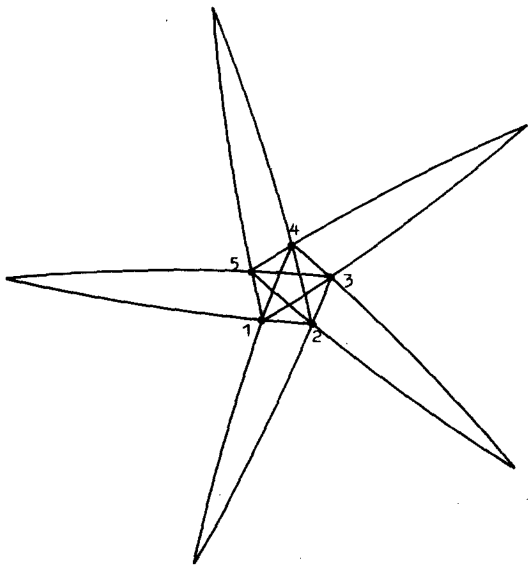

FIGURE 3

This gives us a great many unrealizable sequences for $n > 5$, since we have only to extend the "impossible pentagon" sequence in an arbitrary way to an allowable sequence, in order to get another unrealizable sequence:

COROLLARY 3.4. *If the restriction of an allowable N-sequence S to any subset of n points satisfies the conditions of Corollary* 3.2, *then S is not realizable.*

We believe it is not the case, however, that the possession, by an allowable N-sequence S, of an n-subsequence in which two systems of "parallel" diagonals and/or edges meet as in Corollary 3.2 is the *only* obstruction to the realizability of S. We therefore pose the following problem, whose solution seems essential, to us, for a thorough understanding of plane configurations:

PROBLEM 3.5. What further obstructions, if any, are there to the geometric realizability of an allowable sequence of permutations, besides those given by Corollary 3.4?

4. COMBINATORIAL CLASSIFICATION FOR $n \leqslant 5$

THEOREM 4.1. *For* $n = 1, 2, 3, 4, 5$ *there are precisely* 1, 1, 1, 2, 20 (*respectively*) *combinatorial equivalence classes of allowable sequences, of* *which* 1, 1, 1, 2, 19 (*respectively*) *are geometrically realizable*; *the realizations for* $n = 4$ *and* 5 *are as shown* (*schematically*) *in Fig.* 4.

Proof. In order to prove that for a certain n there are precisely $N(n)$ combinatorial equivalence classes, of which precisely $R(n)$ are realizable, we

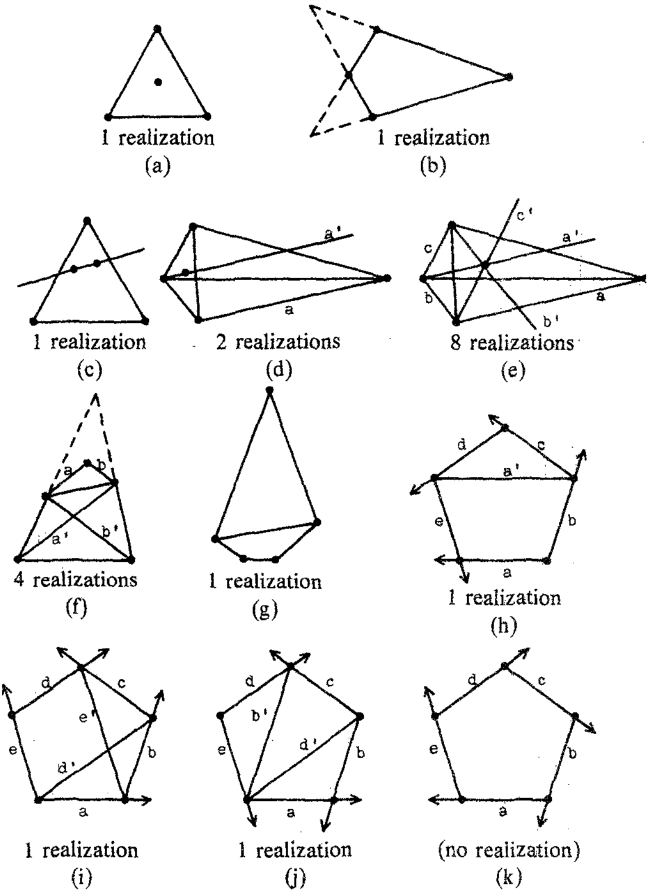

FIGURE 4

In (d), (e), and (f), lines x and x' can meet in either direction; in (h), (i), (j), and (k), each side meets the "parallel" diagonal in the direction indicated.

must do three things: (1) give $N(n)$ n-sequences and show that any n-sequence is equivalent to one of them, (2) show that they are pairwise inequivalent, and (3) show that at least $R(n)$ of them are realizable and at least $N(n) - R(n)$ unrealizable. Since in our case all but one of them (for $n = 5$) will be realizable and we will actually give the realizations, (2) will be immediate by inspection of the pictures. It is therefore enough to do (1) and (3).

For $n=1$ and 2 there is nothing to do, and for $n=3$ there are only the two (equivalent) sequences (2.2) and (2.3), each of which is realized by an appropriately numbered triangle.

For $n=4$, there may be either three or four extreme points. In the first case Proposition 2.17 shows there is only one equivalence class (see Fig. 4a), while in the second case—after renumbering so that 12 and 43 meet on the side of 2 and 3, and 23 and 14 on the side of 3 and 4 (see Definition 2.12)—Propositions 2.15 and 2.6 show that only the sequence realized in Fig. 5 is possible.

If $n=5$ there are either three, four, or five extreme points.

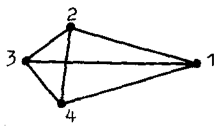

FIGURE 5

Case 1. *Three Extreme Points*

We have seen (Proposition 2.17) that—up to relabeling—there is only one possible restriction to those three points plus either "inside" point. Thus if 1, 2, 3 are extreme, the orders of all switches are determined with the exception of 45, and $i4$ vs $i5$ (and of course $4i$ vs $5i$). Clearly 45 can be inserted anywhere, and 54 at the corresponding position, as is shown by the fact that the line $\overline{45}$ in Fig. 6 can have any desired direction; by Proposition 2.16(c), this choice determines $i4$ vs $i5$ for all i. Now apply Proposition 2.6 to determine a sequence. This argument shows at the same time that every such sequence is realizable, and it is clear that by a cyclic relabeling of points 1, 2, 3, followed by a reflection if necessary, we may transform any realization into the one shown in Fig. 6. This gives one equivalence class in Case 1.

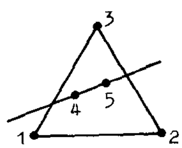

FIGURE 6

Case 2. *Four Extreme Points*

We have already seen that up to relabeling, any 4-sequence with four extreme points must be unique and is realized by Fig. 5 above. Then point 5 must be in one of the four regions the quadrilateral is cut into by its diagonals, in the following sense: By the triangulation lemma, $5 \in \operatorname{conv}(1, 2, 3)$ or $5 \in \operatorname{conv}(1, 3, 4)$; also $5 \in \operatorname{conv}(2, 3, 4)$ or $5 \in \operatorname{conv}(2, 4, 1)$. By reflecting and renumbering, if necessary, making use of the symmetry about the diagonal 13 apparent from Fig. 5, we may assume $5 \in \operatorname{conv}(1, 2, 3)$.

Case 2a. Suppose $5 \in \operatorname{conv}(2, 3, 4)$. Then the position of switch 5i in relation to the remaining switches is determined, by Proposition 2.17, except for 53 vs 14. It is clear from Fig. 5 that point 5 can be chosen so that 53 and 14 meet either on the side of 3 and 4 or on the side of 5 and 1 (for the former near line 23, for the latter near line 13); hence there are precisely two inequivalent sequences in this case, and both are realizable.

Case 2b. Suppose $5 \in \operatorname{conv}(2, 4, 1)$ (see Fig. 7). Then only the position of switch 51 is determined relative to the rest (namely, $(21 \prec 51 \prec 31)$), while 52 vs 43, 53 vs 14, and 54 vs 23 are all undetermined. What is not clear from Fig. 7, however, is that *all three pairs of alternatives may be*

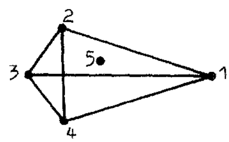

FIGURE 7

realized independently, i.e., that there are as many as *eight* inequivalent sequences in this case. That this is so, and that in fact all eight sequences are geometrically realizable, follows from Fig. 8, in which the four locations of point 5 with $(23 \prec 54 \prec 32)$ are indicated by the dots in Fig. 8(a), while the dots in 8(b) give the four locations for which $(54 \prec 23 \prec 45)$. (The dotted lines in Fig. 8 are parallels.)[1] Case 2 thus yields a total of 10 inequivalent sequences, all realizable.

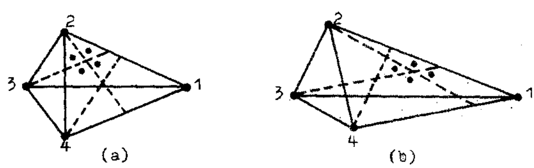

FIGURE 8

[1] Figure 8 also shows an interesting property of realizations, and one which makes the whole question of realizability so difficult: Figures 8(a) and (b) are each realizations of the same 4-sequence; yet one allows a fifth point to be placed so as to realize certain extensions of that sequence, while the other does not.

Case 3. *A Convex Pentagon*

Here we must distinguish a certain type of vertex. We shall call vertex 1 of pentagon 1, 2, 3, 4, 5 (numbered counterclockwise) *special* if edges 23 and 54 meet on the side of points 2 and 5.

The proof in this case proceeds by a breakdown into the number of special vertices; it is not hard to see, using Propositions 2.15 and 2.16, that a convex pentagon can have no more than two special vertices, and that if it has two they must be adjacent. The case of only one special vertex resolves itself into the four realizations shown schematically in Fig. 4f, that of two special vertices into the realization of Fig. 4g, and that of no special vertices into Fig. 4h—which is realized by the configuration $\{(0, 9), (1, 0), (10, 0), (10, 3), (2, 10)\}$, 4i—which is realized as in Fig. 9, 4j—which is realized as

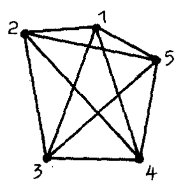

FIGURE 9

in Fig. 10, and 4k—which is the sequence of Theorem 3.3 and therefore has no realization. In every case, Propositions 2.15 and 2.16 turn out to be all that is needed to enumerate all the possibilities; we omit the details, which

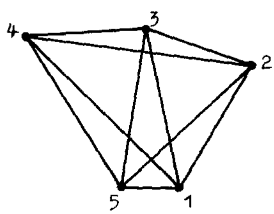

FIGURE 10

are similar to those of Cases 1 and 2 above. This gives a total of 20 sequences, of which all but one are realizable, and completes the proof of the theorem.

REFERENCES

1. P. ERDÖS AND G. SZEKERES, A combinatorial problem in geometry, *Comp. Math.* 2 (1935), 463–470.

2. P. ERDÖS AND G. SZEKERES, On some extremum problems in elementary geometry, *Ann. Univ. Sci. Budapest Eötvös Sect. Nat.* **3–4** (1960/1961), 53–62.
3. J. E. GOODMAN AND R. POLLACK, Semispaces of configurations, to appear.
4. B. GRÜNBAUM, "Arrangements and Spreads," CBMS Regional Conference Series in Mathematics, No. 10, Amer. Math. Soc., Providence, R.I., 1972.
5. R. PERRIN, Sur le problème des aspects, *Bull. Soc. Math. France* **10** (1881–1882), 103–127.