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Some Results on λ-Designs

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ABSTRACT

A λ-design as introduced by Ryser [3] is a (0, 1)-square matrix with constant column inner products but *not* all column sums equal. Ryser has shown such a matrix to have two row sums and he constructs an infinite family of λ-designs called H-designs. This paper does three things: (*1*) generalizes Ryser's H-design construction to an arbitrary (v, k, λ)-configuration, (*2*) establishes some additional general properties of λ-designs, and (*3*) determines all 4-designs.

I. INTRODUCTION

A λ-design is a (0, 1)-matrix A of size n by n such that

$$A^tA = \lambda J + \operatorname{diag}[k_1 - \lambda, ..., k_n - \lambda] \tag{1.1}$$

where A^t denotes the transpose of A, J is the $n \times n$ matrix of ones, $k_j > \lambda > 0$, and not all the k_j's are equal.

First definitively studied by de Bruijn and Erdös with $\lambda = 1$ [1], they have received new interest with the following theorem of Ryser [3] and Woodall [4]:

A (0, 1)-square matrix A satisfying (1.1) with $k_j > \lambda > 0$ either has all its row and column sums equal or has precisely two row sums r_1 and r_2 with $r_1 + r_2 = n + 1$.

Along with this result Ryser established that there is precisely one 2-design. This design, of order 7, is of a class of λ-designs called H-designs, constructed from the symmetric block design [2] with parameters $(4\lambda - 1, 2\lambda, \lambda)$.

In the present paper, we do three things: (*1*) generalize Ryser's H-design construction to an arbitrary (v, k, λ)-configuration; (*2*) establish some additional general properties of λ-designs; and (*3*) determine all 4-designs.

2. Type-1 λ-Designs

THEOREM 2.1. *If there exists a* (v, k, λ') *configuration* [*not of the form* $(4\lambda - 1, 2\lambda - 1, \lambda - 1)$], *then there exists a* λ*-design with* $\lambda = k - \lambda'$ *and row sums* $v - k$ *and* $k + 1$.

PROOF: Let B be the incidence matrix of the (v, k, λ') configuration written so that column one has its k ones in rows 1 through k, i.e.,

$$B = \left[\begin{array}{c|c} 1 & \\ \vdots & A_1 \\ 1 & \\ \hline 0 & \\ \vdots & A_2 \\ 0 & \end{array}\right].$$

Let A_1' denote the complement of the matrix A_1, and it is trivial to verify that the matrix A given by

$$A = \left[\begin{array}{c|c} 0 & \\ \vdots & A_1' \\ 0 & \\ \hline 1 & \\ \vdots & A_2 \\ 1 & \end{array}\right]$$

is the desired λ-design.

We call a λ-design derived in this way a *type*-1 λ*-design*. Note that Ryser's H-designs are type-1 designs derived from a $(4\lambda - 1, 2\lambda, \lambda)$-configuration.

3. Some Properties of λ-Designs

Let $A = (a_{ij})$ be a λ-design. We follow Ryser and denote the row sums of A:

$$r_1 > \frac{n+1}{r} \qquad \text{and} \qquad r_2 < \frac{n+1}{2}.$$

Let the first e_1 rows of A have sum r_1 and the remaining e_2 have sum r_2. Further, let k_j' denote the sum of those entries of column j in rows 1 through e_1, k_j denote the full j-th column sum, and $k_j^* = k_j - k_j'$. With $\rho = (r_1 - 1)/(r_2 - 1)$ we have

$$k_j^* = \lambda - \rho(k_j' - \lambda). \tag{3.1}$$

With

$$u = -\lambda + e_1\left(\frac{r_1 - 1}{n - 1}\right)^2 + e_2\left(\frac{r_2 - 1}{n - 1}\right)^2$$

we have from Ryser [3]:

$$\sum_{j=1}^{n} \frac{1}{k_j - \lambda} = -\frac{1}{\lambda} - \frac{1}{u} = \frac{\lambda(1+\rho)^2 - \rho}{\lambda\rho}. \tag{3.2}$$

If $x_i = (s_i - 1)/(n - 1)$ where s_i is the i-th row sum of A, then

$$\sum_{j=1}^{n} \frac{a_{ij}a_{lj}}{k_j - \lambda} = \delta_{il} - \frac{x_i x_l}{u}, \tag{3.3}$$

where δ_{il} is Kronecker's delta. Note that, if $\hat{x}_1 = (r_1 - 1)/(n - 1)$ and $\hat{x}_2 = (r_2 - 1)/(n - 1)$, then

$$\frac{\hat{x}_1\hat{x}_2}{u} = -1, \qquad \frac{-\hat{x}_1^2}{u} = \rho, \qquad \frac{-\hat{x}_2^2}{u} = \frac{1}{\rho}. \tag{3.4}$$

So the right side of (3.3) is one of the five values $1 + \rho$, ρ, $1 + 1/\rho$, $1/\rho$, 1. We also have

$$r_1 - 1 = \frac{\rho(n - 1)}{\rho + 1}, \qquad r_2 - 1 = \frac{n - 1}{\rho + 1}, \tag{3.5}$$

so that the relation $e_1 r_1(r_1 - 1) + e_2 r_2(r_2 - 1) = \lambda n(n - 1)$ can be written as

$$e_1 = \frac{\lambda(1+\rho)^2 - (\rho + n)}{\rho^2 - 1}. \tag{3.6}$$

Finally, if $\Delta = \det A$, Δ is integral and

$$\Delta^2 = \left\{1 + \lambda \sum_{j=1}^{n} \frac{1}{k_j - \lambda}\right\} \prod_{j=1}^{n} (k_j - \lambda). \tag{3.7}$$

THEOREM 3.1. *A λ-design with $e_1 = 1$ has $\lambda = 1$.*

PROOF: With $e_1 = 1$, the matrix A has two column types from (3.1):

$$\begin{aligned} k_1' &= 1, \qquad k_1^* = \lambda\rho - \rho + \lambda, \\ k_2' &= 0, \qquad k_2^* = \lambda(1 + \rho), \end{aligned} \tag{3.8}$$

and (3.6) yields

$$(n - 1) = (\rho + 1)(\lambda\rho - \rho + \lambda), \qquad (3.9)$$

so we may compute from (3.5)

$$r_2 = \lambda(1 + \rho) - \rho + 1 = k_1 . \qquad (3.10)$$

Also note that from (3.8) $\rho = k_2^* - k_1^*$ is integral. Normalize the matrix A to the form

$$A = \begin{array}{c|c} 1 \cdots 1 & 0 \cdots 0 \\ \hline B & C \end{array}$$

Then (3.3) with $i = 1$ and $l > 1$ shows that the matrix B has constant row sums $k_1 - \lambda$. Since $r_2 = k_1$ (3.10), this means C has row sums λ. We now further normalize within the matrices B and C to bring A to the form

$$\begin{array}{c} \begin{array}{cc|cc} \multicolumn{2}{c|}{1 \cdots 1} & \multicolumn{2}{c}{0 \cdots 0} \end{array} \\ \left.\begin{array}{c|c|c|c} \hline 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ \hline & & \vdots & \\ & & 1 & \\ \hline B_1 & & C_1 & \end{array}\right\} k_2 = \lambda(1 + \rho), \end{array} \qquad (3.11)$$

where C_1 has an initial zero column. *We suppose C_1 is not vacuous.* Let σ denote the sum of row 1 of B_1, τ the sum of row 1 of C_1. Then (3.3) with $i = 2$ and $l = k_2 + 2$ becomes

$$\frac{\sigma}{\lambda\rho - \rho + 1} + \frac{\tau}{\lambda\rho} = \frac{1}{\rho}, \qquad (3.12)$$

which may be written

$$\lambda\rho(\sigma + \tau) = \lambda^2\rho + (\rho - 1)(\tau - \lambda).$$

Since $\rho > 1$ and $\tau < \lambda$,

$$\sigma + \tau < \lambda. \qquad (3.13)$$

Now (3.12) can also be written

$$\rho\{\lambda^2 - \lambda(\sigma + \tau + 1) + \tau\} = \tau - \lambda < 0.$$

Hence, $\lambda^2 - \lambda(\sigma + \tau + 1) + \tau < 0$, or $\lambda^2 + \tau < \lambda(\sigma + \tau + 1) \leqslant \lambda^2$ in view of (3.13). Thus, we must conclude that C_1 *is vacuous*, and (3.11), (3.8), and (3.9) imply $k_2 = n - 1$ and $\lambda = 1$ as asserted.

THEOREM 3.2. *A λ-design has $e_1 \neq 2$.*

PROOF: From (3.5) and (3.6) with $e_1 = 2$ we have

$$\begin{aligned} n &= (\lambda - 2)\,\rho^2 + (2\lambda - 1)\,\rho + (\lambda + 2), \\ r_1 &= (\lambda - 2)\,\rho + (\lambda + 2), \\ r_2 &= (\lambda - 2)\,\rho^2 + (\lambda + 1)\,\rho + 1. \end{aligned} \tag{3.14}$$

The possibilities for k_j' are 0, 1, 2 and the corresponding column types are displayed:

k_j'	0	1	2
k_j^*	$\lambda + \lambda\rho$	$\lambda + \lambda\rho - \rho$	$\lambda + \lambda\rho - 2\lambda$
k_j	$\lambda + \lambda\rho$	$\lambda + \lambda\rho - \rho + 1$	$\lambda + \lambda\rho - 2\rho + 2$
Number of columns	f_0	f_1	f_2

We have the relations

$$\begin{aligned} f_0 + f_1 + f_2 &= (\lambda - 2)\,\rho^2 + (2\lambda - 1)\,\rho + \lambda + 2, \\ f_1 + 2f_2 &= 2(\lambda - 2)\,\rho^2 + 2(\lambda + 1)\,\rho + 2. \end{aligned} \tag{3.15}$$

From $\Sigma f_i = n$ and $\Sigma k_j' = e_1 r_1$ and (3.14). Now (3.3) with $i = 1$, $l = 2$ yields

$$f_2 = (\lambda - 2)\,\rho^2 + 2\rho. \tag{3.16}$$

Hence from (3.15)

$$\begin{aligned} f_1 &= 2(\lambda - 1)\,\rho + 2, \\ f_0 &= \lambda - \rho. \end{aligned} \tag{3.17}$$

Thus, ρ is integral and $\rho \leqslant \lambda$.

Now write A in the form:

$$\begin{array}{c|c|c|c} 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ \hline 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ \hline A_1 & A_2 & A_3 & A_4 \end{array}$$

and let σ_i denote the sum of row one of A_i. Then use (3.3) with $i = 1$, $l = 3$ and again with $i = 2$, $l = 3$. The resulting equations force $\sigma_2 = \sigma_3$ and

$$\frac{\sigma_1}{\lambda\rho - 2\rho + 2} + \frac{\sigma_2}{\lambda\rho - \rho + 1} = 1. \tag{3.18}$$

Now (3.3) with $i = l = 3$ is

$$\frac{\sigma_1}{\lambda\rho - 2\rho + 2} + \frac{2\sigma_2}{\lambda\rho - \rho + 1} + \frac{\sigma_4}{\lambda\rho} = 1 + \frac{1}{\rho} \tag{3.19}$$

so that (3.18) and (3.19) imply

$$\sigma_1 + \sigma_2 = \lambda\rho + 2\rho + 3 - \left(\frac{\lambda + (\rho - 1)\,\sigma_4}{\lambda\rho}\right).$$

Hence $m = [\lambda + (\rho - 1)\,\sigma_4]/\lambda\rho$ is a positive integer, but (3.17) implies $\sigma_4 < \lambda$ whence $m < 1$. This contradiction denies the existence of a λ-design with $e_1 = 2$.

We remark that the corresponding statements to Theorems 3.1 and 3.2 for the parameter e_2 are almost immediate.

The next three lemmas will be used in the study of 4-designs and we sketch briefly the arguments establishing their validity.

LEMMA 3.3. *Let* $\lambda > 1$.

(1) *A* λ*-design with a column with* $k_j' = 2\lambda - 1$ *has* $\rho = \lambda/(\lambda - 1)$.

(2) *A* λ*-design with* $\rho = \lambda/\lambda - 1$ *is an H-design.*

PROOF: (1) The corresponding k_j^* is $\lambda - \rho(\lambda - 1)$; hence, $\rho \leqslant \lambda/(\lambda - 1)$, but $\rho(\lambda - 1)$ is integral and $\rho < 1$, so $\rho = \lambda/(\lambda - 1)$.

(2) From (3.5) we deduce that $2\lambda - 1$ divides $n - 1$ and have for a positive integer t

$$n - 1 = t(2\lambda - 1), \quad r_1 - 1 = \lambda t, \quad r_2 - 1 = t(\lambda - 1). \tag{3.18}$$

Then (3.6) becomes

$$e_1 = -t(\lambda - 1)^2 - (\lambda - 1) + \lambda(2\lambda - 1). \tag{3.19}$$

The two preceding theorems insure $e_1 \geqslant 3$, which forces $t = 1, 2$. If $t = 1$, (3.19) implies $e_1 = \lambda^2$, while (3.18) gives $n = 2\lambda$ and $e_1 < n$ forces $\lambda = 1$. Hence, $t = 2$ and (3.18) shows we have the replication numbers of an H-design which Ryser [3] has shown to be sufficient.

LEMMA 3.4. *Let A be a λ-design with two column sums k_1 and k_2. Suppose further that k_1 occurs precisely once. Then A is a type-1 λ-design.*

PROOF: If A has two column sums, write

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right],$$

where $[A_1 A_2]$ has e_1 rows with sum r_1. Then (3.3) with $i = l \leqslant e_1$ shows A_1 has constant row sums, and similarly one shows A_3 does also. In the present case, A_1 and A_3 are column vectors, and the only possibility is that one is a zero vector, the other a vector of ones. Then surely $k_2' = k_2^* = \lambda$, and it is clear A is a type-1 design.

LEMMA 3.5. *A λ-design with $e_1 = \lambda$ has $\rho \leqslant \lambda$ with $(2\lambda - 1)\rho$ an integer* ($\lambda > 1$).

PROOF: Let x denote the number of columns with $k_j' = k_j^* = \lambda$; then $x \leqslant (n - \lambda)/\lambda$. From (3.6) we deduce

$$n - 1 = (2\lambda - 1)(\rho + 1) \tag{3.20}$$

and (3.5) yields $r_2 = 2\lambda$ so that $r_1 = n + 1 - 2\lambda$. Hence, the first λ rows of A contain $\lambda(2\lambda - 1)$ zeros, and, if $n \geqslant \lambda(2\lambda - 1)$, then $x \geqslant n - \lambda(2\lambda - 1)$. This forces $n \leqslant \lambda(2\lambda + 1)$. Hence, in any case $n \leqslant \lambda(2\lambda + 1)$ and (3.20) gives $\rho = (n - 2\lambda)/(2\lambda - 1) \leqslant \lambda$.

Before proceeding to 4-designs, we note that Ryser remarks that for fixed λ there are at most a finite number of λ-designs with some $k_j < 2\lambda$. We note that (3.6) written as $n = (\lambda - e_1)\rho^2 + (2\lambda - 1)\rho + (e_1 + \lambda)$ makes it clear that for fixed λ there are at most a finite number of λ-designs with $e_1 > \lambda$, while Lemma 3.5 extends this to $e_1 \geqslant \lambda$.[1]

4. 4-DESIGNS

THEOREM 4.1. *All 4-designs are type-1.*[2]

[1] D. Woodall [4] has obtained $\rho \leqslant \lambda$ so that Theorem 3.2 implies $n \leqslant \lambda^3 - \lambda^2 + 3$ regardless of the value of e_1.

[2] The corresponding result for 3-designs has been obtained by E. Kramer and the author and appears in this journal.

PROOF: We first list the parameters of the type-1 4-designs with those of the (v, k, λ)-configurations from which they are derived (Table I). From (3.1) it is clear that $k_j' \leqslant 2\lambda - 1$. With some $k_j' = 2\lambda - 1$. Lemma 3.3 tells us we have a type-1 design (here, number 5 in Table I).

TABLE I

PARAMETERS FOR TYPE-1 4-DESIGNS

	(v, k, λ)	n	r_1	e_1	ρ
1	(21,5,1)	21	16	5	3
2	(21, 16, 12)	21	17	5	4
3	(16, 6, 2)	16	10	6	3/2
4	(16, 10, 6)	16	11	6	2
5	(15, 8, 4)	15	9	7	4/3

We suppose a 4-*design has some* $k_j' = 6$. Then $k_j^* = 4 - 2\rho$ so $\rho = 2$ or $\rho = 3/2$. If $\rho = 3/2$ since $n - 3 \geqslant e_1 \geqslant 6$, we have $13 \leqslant n \leqslant 16$. Further, $5r_1 = 3n + 2$ from (3.5), so $n = 16$, $e_1 = 6$, $r_1 = 10$. Here, $k_j' = 6$ implies $k_j^* = 1$; hence, all remaining columns have $k_j' = 3, 4$; $k_j' = 3$ is not possible since ρ is not integral. Thus, Lemma 3.4 applies. (We note here that in what follows we will stop once we have established that if the design exists it is type-1 without remarking, as we might in the preceding, that the design does not exist.) If $\rho = 2$, $k_j' = 6$ means $k_j^* = 0$ and $6 \leqslant e_1 \leqslant n - 3$ forces $11 \leqslant n \leqslant 16$, and, since $3r_1 = 2n + 1$, $n = 13$ or $n = 16$. With $n = 13$, $e_1 = 7$, so we have just one $k_j' = 6$ with remaining k_j''s either 4 or 5; in fact, $r_1 = 9$, so we have 9 columns with $k_j' = 5$ and three with $k_j' = 4$. But then (3.7) gives $\Delta^2 = 2^8 \cdot 3^{11}$; hence, no such design exists. With $n = 16$, $e_1 = 6$ and we obviously have the design from line 4 of Table I.

Next, suppose a 4-*design has some* $k_j' = 5$. Then ρ is 2, 3, or 4. Proceeding as above using $5 \leqslant e_1 \leqslant n - 3$ with each possible ρ value, we produce the candidates in Table II for a 4-design with some $k_j' = 5$. In each case the column structure can be uniquely determined. There are only three cases in which an admissible column structure exists and produces Δ^2 an integral square; III, V, and VI. The design VI is clearly type-1, namely, line 2 of Table I. Designs III and V are similar and we illustrate with case III. Let f_i denote the number of columns with $k_j' = i$. We clearly have $f_5 = 1$ and $f_2 + f_3 + f_4 = 18$, $2f_2 + 3f_3 + 4f_4 = 60$, $\frac{1}{3} + \frac{1}{6}f_2 + \frac{1}{5}f_3 + \frac{1}{4}f_4 = \frac{17}{4}$. This yields the solution $f_2 = 1$, $f_3 = 10$,

TABLE II

Case	ρ	n	r_1	e_1
I	2	13	9	7
II	2	16	11	6
III	2	19	13	5
IV	3	13	10	6
V	3	21	16	5
VI	4	21	17	5

$f_4 = 7$, and $\Delta^2 = 2^{16}3^4 5^{10}$ does not exclude the design. We look at a row of A with sum $r_1 = 13$ and a zero in the column with $k_j' = 2$. Let τ be the number of ones in this row in columns with $k_j' = 3$ and use (3.3) with $i = l$ obtaining

$$\frac{1}{3} + \frac{\tau}{5} + \frac{12 - \tau}{4} = 3,$$

which implies τ is not integral. Case V is similarly eliminated.

We are thus left to consider 4-designs with all $k_j' \leqslant 4$ and we have the column Table III.

TABLE III

k_j'	0	1	2	3	4
k_j^*	$4 + 4\rho$	$4 + 3\rho$	$4 + 2\rho$	$4 + \rho$	4
k_j	$4 + 4\rho$	$5 + 3\rho$	$6 + 2\rho$	$7 + \rho$	8

Suppose $e_1 \geqslant 7$. Then from (3.6) and the fact that $n \geqslant 12$ we may deduce

$$\rho < 2\,\frac{13}{72}$$

with some $i\rho$ integral for $i = 1, 2, 3, 4$ and n such that r_1 will be integral, all of which reduces to three easily eliminated possibilities: (*1*) $n = 13$, $\rho = 2$, $r_1 = 9$, $e_1 = 7$—but $e_2 = 6$ forces $f_3 = 1$, $f_4 = 12$; hence, Lemma 3.4 applies. (*2*) $n = 12$, $\rho = 7/4$, $e_2 = 4$—from Table III, only $k_j' = k_j^* = 4$ is possible. (*3*) $n = 15$, $\rho = 4/3$—here, Lemma 3.3 applies.

We next take the case $e_1 = 6$. Here, (3.6) becomes $n = -2\rho^2 + 7\rho + 10$ so that $\rho < 3\frac{1}{5}$ and 2ρ is integral. This yields four possibilities: (*1*) $\rho = 2$,

$n = 16$, $r_1 = 11$—here, we would need some $k_j' > 4$ since $e_1 r_1 = 66$. (*2*) $\rho = 3$, $n = 13$, $r_1 = 10$—Table III shows $f_3 = 1$, $f_4 = 12$, and Lemma 3.4 applies. (*3*) $\rho = 3/2$, $n = 16$, $r_1 = 10$—here we deduce $f_0 = 1, f_4 = 15$. This is line 3 of Table I. (*4*) $\rho = 5/2$, $n = 15$, $r_1 = 11$—$e_1 r_1 = 66$ forces some $k_j' > 4$.

The case $e_1 = 5$ proceeds in the same manner. One obtains ρ integral $\rho \leqslant 6$. There are then five possible designs corresponding to these ρ values. The column structure of each can be determined and the design eliminated with the exception of $\rho = 3$, which yields the design (1) of Table I.

For $e_1 = 4$, Lemma 3.5 applies and $\rho \leqslant 4$. Also, 7ρ is integral as well as one of ρ, 2ρ, 3ρ, and 4ρ so that ρ is integral and we obtain just three candidates: (*1*) $\rho = 4$, $n = 36$, $r_1 = 29$; (*2*) $\rho = 3$, $n = 29$, $r_1 = 22$; and (*3*) $\rho = 2$, $n = 22$, $r_1 = 15$. In each case the column structure can be determined and the design eliminated.

This then leaves only the case $e_1 = 3$. We have here from (3.6) and (3.5) $n = \rho^2 + 7\rho + 7$, $r_1 = \rho^2 + 6\rho + 1$, and $r_2 = \rho + 7$. Since $k_j' \leqslant 3$, we have three equations in f_0, f_1, f_2, f_3:

$$\frac{f_0}{4\rho} + \frac{f_1}{3\rho + 1} + \frac{f_2}{2\rho + 2} + \frac{f_3}{\rho + 3} = \frac{4\rho^2 + 7\rho + 4}{4\rho},$$

$$f_0 + f_1 + f_2 + f_3 = \rho^2 + 7\rho + 7,$$

and

$$f_1 + 2f_2 + 3f_3 = 3\rho^2 + 18\rho + 3.$$

It is easily verified that this is a rank-3 system and has the following one-parameter solution:

$$f_0 = \frac{4\rho^3 + 7\rho^2 - (11 + 4f_3)\rho + 12}{\rho + 3}, \tag{4.1}$$

$$f_1 = \frac{-3(3\rho + 1)(\rho^2 + 2\rho - 3 - f_3)}{\rho + 3}, \tag{4.2}$$

$$f_2 = \frac{6(\rho + 1)(\rho^2 + 3\rho - f_3)}{\rho + 3}. \tag{4.3}$$

From (4.2), $\rho^2 + 2\rho - 3 - f_3 \leqslant 0$ so that

$$(11 + 4f_3)\rho \geqslant 4\rho^3 + 8\rho^2 - \rho. \tag{4.4}$$

On the other hand, from (4.1)

$$(11 + 4f_3)\rho \leqslant 4\rho^3 + 7\rho^2 + 12. \tag{4.5}$$

Thus, (4.4) and (4.5) yield

$$\rho^2 - \rho - 12 \leqslant 0, \qquad \text{whence} \quad \rho \leqslant 4.$$

This means there are three designs to consider corresponding to $\rho = 2, 3, 4$. Using equations (4.1), (4.2), and (4.3) one can determine the column structures of these candidates and systematically eliminate them.

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