Some applications of a model theoretic fact to (semi-) algebraic geometry

by Lou van den Dries

Department of Mathematics, Stanford University, Stanford, California 94305, U.S.A.

Communicated by Prof. A.S. Troelstra at the meeting of March 29, 1982

Logicians are familiar with Tarski’s quantifier elimination for the theories of algebraically closed and real closed fields. Sometimes this result is not precise enough: the problem may be that a certain set has to be shown definable, not just by a q.f. (= quantifier free) formula, but by a positive q.f. formula (i.e., by a conjunction of equations in the case of algebraic geometry, by a disjunction of systems of weak inequalities $f_1 \geq 0, \ldots, f_k \geq 0$ in the case of semi-algebraic geometry).

In this paper we indicate a simple model theoretic way to obtain representations by positive q.f. formulas: we prove a Lyndon-type theorem characterizing those formulas which are equivalent to a positive q.f. formula (relative to a given theory); in the case of interest one just applies the place extension theorem for (ordered) integral domains to verify the hypothesis of our model theoretic result.

As this rather general method does not seem generally known, Professor Kreisel urged me to write an account of it. (I found this trick in 1977 but thought it would be well known as the model theory and algebra involved are perfectly standard.)

NOTATIONAL CONVENTIONS. We write $A \subseteq B$ to indicate that $A$ is a substructure of $B$ in the sense of model theory. $\phi(C)$ indicates a formula whose parameters (= free variables) are among $C_1, \ldots, C_m$, where $C = (C_1, \ldots, C_m)$. 
The Lyndon-type (or maybe Lyndon-Robinson-type) result referred to above is the following.

**Lemma.** Let $T$ be a first-order theory and $\phi(C)$ an $L(T)$-formula. Suppose that for each model $K$ of $T$ and each homomorphism $f : A \to L$ where $A \subseteq K$ and $L \models T$ we have:

if $c \in A^m$ and $K \models \phi(c)$, then $L \models \phi(f(c))$.

Then there is a positive q.f. formula $\psi(C)$ such that $T \vdash \phi(C) \leftrightarrow \psi(C)$.

**Proof.** Consider $C_1, \ldots, C_m$ as new constant symbols which we add to $L(T)$ and let $T'$ be the theory

$$T \cup \{ \neg \psi(C) : \psi \text{ is a positive q.f. formula and } T \vdash \psi \rightarrow \phi \}.$$

**Claim.** $T' \vdash \neg \phi(C)$.

If this were not the case, then there is a model $(K, c)$ of $T'$ with $K \models \phi(c)$. Let $A$ be the substructure of $K$ generated by $c$.

Then the hypothesis of the lemma gives that

$$T \cup \text{Positive Diagram } (A, c) \models \phi(C).$$

So by the completeness theorem there is a finite conjunction $\psi(C)$ of sentences in Positive Diagram $(A, c)$ such that $T \vdash \psi(C) \rightarrow \phi(C)$. But then $\neg \psi(C) \in T'$ contradicting $(A, c) \subseteq (K, c) \models T'$.

From $T' \vdash \neg \phi(C)$ it follows that $T \vdash \phi(C) \leftrightarrow \psi_1(C) \lor \ldots \lor \psi_k(C)$ for certain positive q.f. formulas $\psi_i$ such that $T \vdash \psi_i(C) \rightarrow \phi(C)$, $i = 1, \ldots, k$. Hence

$$T \vdash \phi(C) \leftrightarrow \psi_1(C) \lor \ldots \lor \psi_m(C).$$

**Place Extension Theorems**

1. **(Chevalley).** If $K$ is a field and $f : A \to L$ a homomorphism of a subring $A$ of $K$ into an algebraically closed field $L$, then $f$ can be extended to a homomorphism $V \to L$, $V$ a valuation ring of $K$ (see [4, p. 251]).

For the ordered rings version, let us make the convention that an ordered ring is a pair $(R, \leq)$, $R$ a commutative ring with unit, $\leq$ a total order on $R$ compatible with the ring operations (so the language of ordered rings is $\{0, 1, +, \cdot, -, \leq\}$).

2. **(S. Lang).** If $K$ is an ordered field and $f : A \to L$ a homomorphism of an ordered subring $A$ of $K$ into a real closed field $L$, then $f$ can be extended to a homomorphism $V \to L'$, $V$ a convex valuation ring of $K$, $L'$ a real closed extension of $L$ (see [1, 7.7.4 on p. 152]).

3. I know a similar place extension theorem for (Krull) valued fields. There are

---

1 In case $m = 0$, we assume that $L(T)$ contains a constant symbol.
results in the literature on extensions of differential homomorphisms defined on differential rings ....

APPLICATIONS

(a) The main theorem of classical elimination theory (= "the completeness of projective varieties") says: let \( p_1(C, X), ..., p_k(C, X) \in \mathbb{Z}[C, X] \) be homogeneous in \( X = (X_0, ..., X_n) \) (where \( C = (C_1, ..., C_m) \) as before). Then there are polynomials \( q_1(C), ..., q_k(C) \) in \( \mathbb{Z}[C] \) such that for each algebraically closed field \( K \) and \( c \in K^m \) the system \( p_1(c, X) = ... = p_k(c, X) = 0 \) has a non-trivial solution in \( K \) if and only if \( q_1(c) = ... = q_k(c) = 0 \).

PROOF. Let \( \phi(C) \) be the formula \( \exists X(p_1(C, X) = ... = p_k(C, X) = 0) \). Let \( T \) be the theory of algebraically closed fields. We have to show that \( \phi(C) \) is equivalent to a positive q.f. formula, relative to \( T \). By the model theoretic lemma we are reduced to showing: if \( f : A \rightarrow L \) is a morphism of a subring \( A \) of an algebraically closed field \( K \) into an algebraically closed field \( L \) and if the system \( p_1(c, X) = ... = p_k(c, X) = 0 \), where \( c \in A^m \), has a non-trivial solution in \( K \), then the system \( p_1(f(c), X) = ... = p_k(f(c), X) = 0 \) has a non-trivial solution in \( L \). Now by Chevalley's place extension theorem we may as well assume that \( A \) is a valuation ring of \( K \). Multiplying a non-trivial solution in \( K \) by a suitable constant we may as well assume that we have \( x \) in \( A^n+1 \) with at least one coordinate invertible in \( A \).

Then, applying \( f \) to \( p_1(c, X) = ... = p_k(c, X) = 0 \) we get a solution \( f(x) \) of \( p_1(f(c), X) = ... = p_k(f(c), X) = 0 \) in \( L \), and \( f \) maps the invertible coordinate of \( x \) into a non-zero element of \( L \).

(b) An analog of (a) in semi-algebraic geometry says (with the \( p_i(C, X) \) as before): There is a positive q.f. formula \( \psi(C) \) such that for each real closed field \( K \) and \( c \in K^m \) the system \( p_1(c, X) \geq 0, ..., p_k(c, X) \geq 0 \) has a non-trivial solution in \( K \) if and only if \( K \models \psi(c) \).

The proof is quite similar to that for (a). Of course a non-trivial solution in \( K \) is now multiplied by a constant \( >0 \) to get a solution in \( A \) with at least one invertible coordinate. In slight contrast with the proof of (a), it seems necessary to use the fact that a real closed field \( L \) is existentially closed in its extension \( L' \).

Using spherical homogeneous coordinates, that is, identifying \( (x_0, ..., x_n) \) and \( (y_0, ..., y_n) \) if there is \( a>0 \) with \( ax_i = y_i, i = 0, ..., n \), the result above can be interpreted as the "completeness of closed semi-algebraic spherical sets." (A semi-algebraic substitute for the compactness of real spheres.)

(c) The "Finiteness" theorem ([1, p. 164], [2, 2.2 and 3(b)]).

Let \( R \) be a real closed field and \( S \) a closed semi-algebraic subset of \( R^m \). Then \( S \) is definable by a positive q.f. formula \( \psi(C) \).

399
Remark

1. The topology on $\mathbb{R}^m$ is the product topology coming from the interval topology on $\mathbb{R}$. In the proof we also mention the ‘norm’

$$|x| = \max (|x_1|, \ldots, |x_m|), \quad x \in \mathbb{R}^m.$$  

2. The coefficients of the polynomials in $\psi$ are supposed to be from $\mathbb{R}$, but, in fact, they can be taken as piecewise $\mathbb{Q}$-rational functions in the constants occurring in the defining formula for $S$. Readers familiar with elementary model theory—to be specific, with the trick of adding new constant symbols—can easily make this statement precise and adapt the proof below to give this extra information.

Proof. Put $T = (\text{theory of real closed fields}) \cup \text{Diagram (R)}$. Let $\phi = \phi(C)$ be a formula defining $S$ in $\mathbb{R}$. It suffices to show existence of a positive q.f. formula $\psi = \psi(C)$ such that $T \vdash \phi \leftrightarrow \psi$ ($\phi, \psi$ in the language of $T$). By the lemma and the place extension theorem for ordered domains we are led to the following situation:

$K$ is real closed field $\supset R$ (i.e., $K \models T$), $A$ a convex valuation ring of $K$, $A \supset R$ and $f : A \to L$ a morphism (order preserving!) into a real closed field $L$; further we have $c \in A^m$ with $K \models \phi(c)$. We just have to derive from these assumptions that $L \models \phi(f(c))$.

Localizing $A$ at kernel$(f)$ (and extending $f$ appropriately) we reduce to the case that kernel$(f) = \text{maximal ideal of } A$. A second reduction: by Zorn we can extend $R$ to a subfield of $A$ which is mapped onto the real closed field $f(A)$ by $f$; we may as well assume that $R$ is already such a field of representatives of $A$, since, by model completeness, $\phi$ still defines a closed set in each real closed extension of $R$. So now there is $d \in R^m$ on $R$-infinitesimal distance from $c \in A^m$, in other words, all coordinates of $c - d$ are in the maximal ideal of $A$. We claim that $d \in S$, i.e., $R \models \phi(d)$. If not, then there is an $\varepsilon$-ball, $0 < \varepsilon \in R$, centered at $d$, included in the complement of $S$. As $R \subseteq K$, the same $\varepsilon$ also defines in $K$ a ball (around $d$) included in the set defined by $\neg \phi$. Contradiction, since $K \models \phi(c)$ and $|c - d| < \varepsilon$. The claim is proved.

Now $R \models f(A)$ under $f$ and $R \models \phi(d)$, hence $f(A) \models \phi(f(d))$. Since $f(c) = f(d)$ and $f(A) \subseteq L$ we get $L \models \phi(f(c))$. \hfill \Box

(d) For those interested in representation theory of finite dimensional algebras over fields we mention that our method leads to a positive answer to a question which came up in [3, p. 93]: finite representation type is open over each prime field (this sharpens a result due to P. Gabriel).
REFERENCES


POSTSCRIPT. My attention was drawn to the following publications which also contain proofs of the “Finiteness Theorem,” c.f. (c) above. The proofs were obtained independently and each is based on a different principle.