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Noncommutative recurrence over locally compact Hausdorff groups

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Abstract

We extend previous results on noncommutative recurrence in unital $*$-algebras over the integers to the case where one works over locally compact Hausdorff groups. We derive a generalization of Khintchine's recurrence theorem, as well as a form of multiple recurrence. This is done using the mean ergodic theorem in Hilbert space, via the GNS construction.

Keywords: Recurrence; Unital $*$-algebras; Locally compact Hausdorff groups

1. Introduction

The simplest form of recurrence occurs in a dynamical system consisting of a measure space X with probability measure ν and a transformation $T : X \to X$ such that $\nu(T^{-1}(S)) = \nu(S)$ for all measurable $S \subset X$. If $\nu(S) > 0$ for some $S \subset X$, then there is an $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ such that $\nu(S \cap T^{-n}(S)) > 0$. This is essentially a pigeon hole principle for measure spaces and is usually referred to as Poincaré's recurrence theorem. Note that in this case the group over which we work is simply $\mathbb{Z}$. More precisely, we are working on the subset $\mathbb{N}$, since we only consider T^{-n} with $n \in \mathbb{N}$.

Recurrence theorems can also be studied in the noncommutative setting of states on unital $*$-algebras and C^*-algebras, as done in [2,4]. Typically the goal is to generalize existing mea-

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sure theoretic recurrence theorems to the $*$-algebraic setting. The measure theoretic case is then recovered by taking the algebra to be a suitable commutative algebra of measurable functions. The simplest way of doing this, is to use the unital $*$-algebra $B_\infty(\Sigma)$ of bounded complex-valued measurable functions on the measure space X, with Σ the σ-algebra of measurable sets of X. The state ω on the algebra is simply $\omega(f) := \int_X f\,d\nu$, so the state represents the measure, while T is represented on the algebra by the Koopman construction $\tau(f) = f \circ T$. Note that $\omega(\chi_S) = \nu(S)$ and $\tau(\chi_S) = \chi_{T^{-1}(S)}$. By following this type of recipe, the results in Sections 3 and 4 can be applied to the measure theoretic case.

In this paper we continue our work in [2]. We study recurrence in unital $*$-algebras as before, but instead of just working over the group $\mathbb{Z}$ as mentioned above, we will consider locally compact Hausdorff groups and suitable subsets thereof, namely subsemigroups (similar to the subset $\mathbb{N}$ of $\mathbb{Z}$). The main result is an extension of Khintchine's recurrence theorem in Section 3, which is subsequently used to prove a simple multiple recurrence result in Section 4. The latter result is inspired by the work of Furstenberg [3] on extensions of Poincaré recurrence to recurrence theorems of the form

$$\nu\big(S \cap T^n S \cap T^{2n} S \cap \cdots \cap T^{(k-1)n} S\big) > 0,$$

which Furstenberg used to give an alternative proof of Szemerédi's theorem in combinatorial number theory. Our result is of a different form than Furstenberg's, however.

The main tool we use is the mean ergodic theorem in Hilbert space, which we review in Section 2. A Hilbert space version of Khintchine's recurrence theorem is also proved in Section 2 and applied in the subsequent sections. In the recurrence theorems we need to make stronger assumptions than in the mean ergodic theorem, namely that the subsemigroup of the group over which we work is abelian or that the group itself is unimodular.

2. Recurrence in a Hilbert space setting

We start with a review of the mean ergodic theorem. This is based on Petersen [5] and also Bratteli and Robinson [1]. The former discusses the theorem over the group $G = \mathbb{Z}$ (see the more general form below), while the latter gives it in an abstract form involving no group.

First, consider a function $f : G \to \mathfrak{H}$ where G is a locally compact Hausdorff group with right Haar measure μ, and $\mathfrak{H}$ a complex Hilbert space, such that $G \ni g \mapsto \langle f(g), x\rangle$ is Borel measurable for every $x \in \mathfrak{H}$. We shall take the second slot in the inner product to be the linear one. If $\Lambda \subset G$ is Borel with $\mu(\Lambda) < \infty$ and f is bounded on Λ, say $\|f(g)\| \leqslant b$ for all $g \in \Lambda$ for some positive $b \in \mathbb{R}$, then we can define $\int_\Lambda f\,d\mu$ by $\langle \int_\Lambda f\,d\mu, x\rangle := \int_\Lambda \langle f(g), x\rangle\,d\mu(g)$ for all $x \in \mathfrak{H}$ using the Riesz representation theorem. So we also have $\langle x, \int_\Lambda f\,d\mu\rangle = \int_\Lambda \langle x, f(g)\rangle\,d\mu(g)$. We will also use the notation $\int_\Lambda f(g)\,dg = \int_\Lambda f\,d\mu$. One can then easily prove all the standard properties for this integral, like linearity and

$$\int_\Lambda f(gh)\,dg = \int_{\Lambda h} f\,d\mu, \tag{1}$$

$$\int_\Lambda Af(g)\,dg = A\int_\Lambda f\,d\mu, \tag{2}$$

$$\int_\Lambda x\,d\mu = x\int_\Lambda d\mu = \mu(\Lambda)x, \tag{3}$$

$$\int_{\Lambda_1 \cup \Lambda_2} f \, d\mu = \int_{\Lambda_1} f \, d\mu + \int_{\Lambda_2} f \, d\mu \tag{4}$$

and

$$\left\| \int_{\Lambda} f \, d\mu \right\| \leqslant b\mu(\Lambda) \tag{5}$$

for every $h \in G$, $A \in B(\mathfrak{H})$, $x \in \mathfrak{H}$ and Borel $\Lambda_1, \Lambda_2 \subset G$ of finite measure on which f is bounded, with $\mu(\Lambda_1 \cap \Lambda_2) = 0$, where $B(\mathfrak{H})$ denotes the algebra of bounded linear operators $\mathfrak{H} \to \mathfrak{H}$. We will use these properties in the sequel.

A *net* is family $\{\Lambda_\alpha\}$ of subsets of G indexed by a directed set. If a $K \subset G$ (with equality allowed) has the property that $gh \in K$ for all $g, h \in K$, we shall call K a *subsemigroup* of G. We call a net $\{\Lambda_\alpha\}$ of Borel subsets of G *space-filling in* K if $\Lambda_\alpha \subset K$, $\mu(\Lambda_\alpha) < \infty$, and

$$\lim_\alpha \frac{\mu(\Lambda_\alpha \Delta (\Lambda_\alpha g))}{\mu(\Lambda_\alpha)} = 0$$

for all $g \in K$, where we assume that $\mu(\Lambda_\alpha) > 0$ for α large enough (i.e., $\alpha \geqslant \alpha_0$ for some α_0). Here $A \Delta B := (A \cup B) \backslash (A \cap B)$. Now we can state

The mean ergodic theorem. *Let G be a locally compact Hausdorff group with right Haar measure μ, and consider a Borel measurable subsemigroup K of G. Let $U : K \to B(\mathfrak{H}) : g \mapsto U_g$ be such that $\|U_g\| \leqslant 1$, $U_g U_h = U_{gh}$ for all $g, h \in K$, and $K \ni g \mapsto \langle U_g x, y\rangle$ is Borel measurable for all $x, y \in \mathfrak{H}$. Take P to be the projection of $\mathfrak{H}$ onto $V := \{x \in \mathfrak{H}\colon U_g x = x \text{ for all } g \in K\}$. For any space-filling net $\{\Lambda_\alpha\}$ in K we then have*

$$\lim_\alpha \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} U_g x \, dg = Px$$

for all $x \in \mathfrak{H}$.

Proof. Set $N := \overline{\operatorname{span}\{x - U_g x\colon x \in \mathfrak{H},\ g \in K\}}$. For any g, a fixed point of U_g^* is a fixed point of U_g, and vice versa, since $\|U_g^*\| \leqslant 1$. From this it follows that $V = N^\perp$, which means in particular that V is a closed subspace of $\mathfrak{H}$. Set

$$I_\alpha(x) := \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} U_g x \, dg.$$

We now prove that $\lim_\alpha I_\alpha(x) = 0$ for $x \in N$. First let $x = y - U_h y$ for some $y \in \mathfrak{H}$ and $h \in K$, then we have from (1) and (4) that

$$\begin{aligned} I_\alpha(x) &= \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} U_g y \, dg - \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha h} U_g y \, dg \\ &= \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha \backslash (\Lambda_\alpha \cap (\Lambda_\alpha h))} U_g y \, dg - \frac{1}{\mu(\Lambda_\alpha)} \int_{(\Lambda_\alpha h) \backslash (\Lambda_\alpha \cap (\Lambda_\alpha h))} U_g y \, dg \end{aligned}$$

hence

$$\|I_\alpha(x)\| \leqslant \frac{1}{\mu(\Lambda_\alpha)}\left\| \int\limits_{\Lambda_\alpha\setminus(\Lambda_\alpha\cap(\Lambda_\alpha h))} U_g y\,dg\right\| + \frac{1}{\mu(\Lambda_\alpha)}\left\| \int\limits_{(\Lambda_\alpha h)\setminus(\Lambda_\alpha\cap(\Lambda_\alpha h))} U_g y\,dg\right\|$$
$$\leqslant \|y\|\frac{\mu(\Lambda_\alpha\Delta(\Lambda_\alpha h))}{\mu(\Lambda_\alpha)}$$

by (5), since $\|U_g\| \leqslant 1$, so $\lim_\alpha I_\alpha(x) = 0$. However, we need this for any $x \in N$, so set $N_0 := \{y - U_g y\colon\ y \in \mathfrak{H},\ g \in K\}$. Then for any $\varepsilon > 0$ there is a $y \in \operatorname{span} N_0$ such that $\|x - y\| < \varepsilon$, say $y = \sum_{j=1}^m x_j$ where $x_j \in N_0$. Therefore

$$\big|\|I_\alpha(x)\| - \|I_\alpha(y)\|\big| \leqslant \|I_\alpha(x) - I_\alpha(y)\| \leqslant \frac{1}{\mu(\Lambda_\alpha)}\|x-y\|\int\limits_{\Lambda_\alpha} d\mu < \varepsilon$$

while

$$\|I_\alpha(y)\| \leqslant \sum_{j=1}^m \|I_\alpha(x_j)\| \to 0$$

in the α limit, as shown above. Hence $\lim_\alpha I_\alpha(x) = 0$.

For any $x \in \mathfrak{H}$, write $x = x_0 + Px$, where $x_0 = (1-P)x \in V^\perp = N$, then

$$\|I_\alpha(x) - Px\| = \|I_\alpha(x_0) + I_\alpha(Px) - Px\| = \|I_\alpha(x_0)\| \to 0$$

in the α limit, since $I_a(Px) = \frac{1}{\mu(\Lambda_\alpha)}\int_{\Lambda_\alpha} Px\,d\mu = Px$ by the definition of P and (3). □

Using this theorem, we can prove a Hilbert space version of Khintchine's recurrence theorem:

Theorem 2.1. *Consider the situation given in the mean ergodic theorem above, but assume K is abelian, i.e., $gh = hg$ for all $g, h \in K$. Take any $x, y \in \mathfrak{H}$ and $\varepsilon > 0$. Then there is an α_0 such that*

$$\left|\frac{1}{\mu(\Lambda_{\alpha_0})}\int\limits_{\Lambda_{\alpha_0}h} \langle x, U_g y\rangle\,dg\right| > \big|\langle x, Py\rangle\big| - \varepsilon$$

for all $h \in K$. In particular, for every $h \in K$ there is a $g \in \Lambda_{\alpha_0}h$ such that

$$\big|\langle x, U_g y\rangle\big| > \big|\langle x, Py\rangle\big| - \varepsilon.$$

Proof. By the mean ergodic theorem there is an α_0 such that

$$\left\|\frac{1}{\mu(\Lambda_{\alpha_0})}\int\limits_{\Lambda_{\alpha_0}} U_g y\,dg - Py\right\| < \frac{\varepsilon}{\|x\|+1}$$

while by definition of P we have $U_h Py = Py$ for all $h \in K$. Using these two facts along with (1) and (2) and the fact that K is abelian, we get

$$\left\|\frac{1}{\mu(\Lambda_{\alpha_0})}\int\limits_{\Lambda_{\alpha_0}h} U_g y\,dg - Py\right| = \left\|\frac{1}{\mu(\Lambda_{\alpha_0})}\int\limits_{\Lambda_{\alpha_0}} U_{gh} y\,dg - Py\right\|$$
$$= \left\|\frac{1}{\mu(\Lambda_{\alpha_0})}U_h\int\limits_{\Lambda_{\alpha_0}} U_g y\,dg - U_h Py\right\| \leqslant \frac{\varepsilon}{\|x\|+1}$$

since $\|U_h\| \leqslant 1$. Hence

$$\left|\frac{1}{\mu(\Lambda_{\alpha_0})}\int_{\Lambda_{\alpha_0}h}\langle x, U_g y\rangle\,dg - \langle x, Py\rangle\right| = \left|\left\langle x, \frac{1}{\mu(\Lambda_{\alpha_0})}\int_{\Lambda_{\alpha_0}h} U_g y\,dg - Py\right\rangle\right| < \varepsilon$$

from which the result follows. □

In this result we assumed K to be abelian, but if we assume G is unimodular, i.e., its right Haar measure is also a left Haar measure, then essentially the same proof also works for nonabelian K to give

Theorem 2.2. *Consider the situation given in the mean ergodic theorem above, but assume G is unimodular. For any $x, y \in \mathfrak{H}$ and $\varepsilon > 0$ there then exists an α_0 such that*

$$\left|\frac{1}{\mu(\Lambda_{\alpha_0})}\int_{h\Lambda_{\alpha_0}}\langle x, U_g y\rangle\,dg\right| > \left|\langle x, Py\rangle\right| - \varepsilon$$

for all $h \in K$.

This works, since if μ is also a left Haar measure, one has $\int_\Lambda f(hg)\,dg = \int_{h\Lambda} f\,d\mu$ similar to (1).

3. ∗-Dynamical systems

Let $L(V)$ denote the space of all linear operators $V \to V$ with V a vector space.

Definition 3.1. Let ω be a state on a unital ∗-algebra $\mathfrak{A}$, let G be a locally compact Hausdorff group with right Haar measure μ, and K a Borel measurable subsemigroup of G. Consider a $\tau : K \to L(\mathfrak{A}) : g \mapsto \tau_g$ with

$$\tau_g \circ \tau_h = \tau_{gh}, \qquad \tau_g(1) = 1, \qquad \omega\big(\tau_g(A)^*\tau_g(A)\big) \leqslant \omega\big(A^*A\big)$$

for all $g, h \in K$ and $A \in \mathfrak{A}$, and $K \ni g \mapsto \omega(A^*\tau_g(B))$ Borel measurable for all $A, B \in \mathfrak{A}$. Then we shall call $(\mathfrak{A}, \omega, \tau, K)$ a *∗-dynamical system*.

Given a state ω on a unital ∗-algebra $\mathfrak{A}$, the GNS construction provides us with a cyclic representation $(\mathfrak{G}, \pi, \Omega)$ where $\mathfrak{G}$ is an inner product space, $\pi : \mathfrak{A} \to L(\mathfrak{G})$ is linear, and $\iota : \mathfrak{A} \to \mathfrak{G} : A \mapsto \pi(A)\Omega$ is surjective. Also, $\omega(A^*B) = \langle \iota(A), \iota(B)\rangle$ for all $A, B \in \mathfrak{A}$. Then for τ_g as above

$$U_g : \mathfrak{G} \to \mathfrak{G} : \iota(A) \mapsto \iota\big(\tau_g(A)\big)$$

is well defined, linear, and $\|U_g\| \leqslant 1$. We can therefore uniquely extend U_g to the completion $\mathfrak{H}$ of $\mathfrak{G}$, such that $\|U_g\| \leqslant 1$. Call $g \mapsto U_g$ the GNS representation of τ.

Proposition 3.2. *For a ∗-dynamical system $(\mathfrak{A}, \omega, \tau, K)$, the GNS representation U of τ on a Hilbert space $\mathfrak{H}$ has the following properties: $U_g U_h = U_{gh}$ for all $g, h \in K$, and $K \ni g \mapsto \langle x, U_g y\rangle$ is Borel measurable for all $x, y \in \mathfrak{H}$.*

Proof. For $A \in \mathfrak{A}$ one has $U_g U_h \iota(A) = U_g \iota(\tau_h(A)) = \iota(\tau_g(\tau_h(A))) = \iota(\tau_{gh}(A)) = U_{gh}\iota(A)$, and by continuity of U_g on $\mathfrak{H}$, this extends to $U_g U_h x = U_{gh} x$ for all $x \in \mathfrak{H}$. By the definition of a $*$-dynamical system, $g \mapsto \omega(A^* \tau_g(B)) = \langle \iota(A), U_g \iota(B)\rangle$ is Borel measurable, and since the pointwise limit of a sequence of measurable functions is measurable, we need only consider $\langle x_n, U_g y_n\rangle \to \langle x, U_g y\rangle$ where $x, y \in \mathfrak{H}$ and $x_n, y_n \in \mathfrak{G}$ (as defined above) such that $x_n \to x$ and $y_n \to y$, keeping in mind that $x_n = \iota(A_n)$ and $y_n = \iota(B_n)$ for some $A_n, B_n \in \mathfrak{A}$. □

Now we can state a recurrence theorem for $*$-dynamical systems, containing in particular the conventional form of the Khintchine recurrence theorem (which includes the measure theoretic version over $K = \mathbb{N}$, as a special case; see Petersen [5]):

Theorem 3.3. *Let $(\mathfrak{A}, \omega, \tau, K)$ be a $*$-dynamical system, but assume K is abelian. Let $\{\Lambda_\alpha\}$ be a space-filling net in K. Then for any $A, B \in \mathfrak{A}$ and $\varepsilon > 0$, there exists an α_0 such that*

$$\left| \frac{1}{\mu(\Lambda_{\alpha_0})} \int\limits_{\Lambda_{\alpha_0} h} \omega\big(A^* \tau_g(B)\big)\, dg \right| > \left| \lim_\alpha \frac{1}{\mu(\Lambda_\alpha)} \int\limits_{\Lambda_\alpha} \omega\big(A^* \tau_g(B)\big)\, dg \right| - \varepsilon$$

for all $h \in K$. In particular, if $A = B$, then for every $h \in K$,

$$\big|\omega\big(A^* \tau_g(A)\big)\big| > \big|\omega(A)\big|^2 - \varepsilon$$

for some $g \in \Lambda_{\alpha_0} h$.

Proof. We use the GNS construction discussed above to represent τ by U and set $x := \iota(A)$ and $y := \iota(B)$. By the mean ergodic theorem

$$\langle x, Py\rangle = \lim_\alpha \frac{1}{\mu(\Lambda_\alpha)} \int\limits_{\Lambda_\alpha} \langle x, U_g y\rangle\, dg = \lim_\alpha \frac{1}{\mu(\Lambda_\alpha)} \int\limits_{\Lambda_\alpha} \omega\big(A^* \tau_g(B)\big)\, dg.$$

The first part of the result now follows immediately from Theorem 2.1. The second part also follows, since $|\omega(A)| = |\omega(1^* A)| = |\langle \iota(1), \iota(A)\rangle| = |\langle \Omega, x\rangle| = |\langle P\Omega, x\rangle| = |\langle \Omega, Px\rangle| \leqslant \|\Omega\| \|Px\| = \|Px\| = \sqrt{\langle x, Px\rangle}$, where we have used the fact that $P\Omega = \Omega$, which follows from $U_g \Omega = U_g \iota(1) = \iota(\tau_g(1)) = \iota(1) = \Omega$. □

Even though K has to be abelian in this theorem, one could still have a $*$- dynamical system $(\mathfrak{A}, \omega, \tau, H)$ with H nonabelian, and then apply the theorem to various abelian $K \subset H$ with K a Borel measurable subsemigroup of the underlying group G.

Remarks on ergodicity. Call a $*$-dynamical system $(\mathfrak{A}, \omega, \tau, K)$ *ergodic* when

$$\lim_\alpha \frac{1}{\mu(\Lambda_\alpha)} \int\limits_{\Lambda_\alpha} \omega\big(A \tau_g(B)\big)\, dg = \omega(A)\omega(B)$$

for all $A, B \in \mathfrak{A}$ and some space-filling net $\{\Lambda_\alpha\}$ in K.

In the GNS representation and with P as in the mean ergodic theorem, the above definition of ergodicity is equivalent to P having a one-dimensional range, and in particular the definition is independent of which space-filling net in K is used. In fact, $P = \Omega \otimes \Omega$ in case of ergodicity, where $(x \otimes y)z := x\langle y, z\rangle$ for all $x, y, z \in \mathfrak{H}$. We see this as follows:

Since $P\Omega = \Omega$ as we saw in the proof of Theorem 3.3, it follows that P having one-dimensional range is equivalent to $P = \Omega \otimes \Omega$. Now, if $P = \Omega \otimes \Omega$, then the mean ergodic theorem tells us that

$$\begin{aligned}\lim_\alpha \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \omega\big(A\tau_g(B)\big)\,dg &= \lim_\alpha \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \big\langle \iota\big(A^*\big), U_g \iota(B)\big\rangle\,dg \\ &= \big\langle \iota\big(A^*\big), P\iota(B)\big\rangle \\ &= \big\langle \iota\big(A^*\big), (\Omega \otimes \Omega)\iota(B)\big\rangle \\ &= \big\langle \iota\big(A^*\big), \Omega\big\rangle\big\langle \Omega, \iota(B)\big\rangle \\ &= \omega(A)\omega(B).\end{aligned}$$

Conversely, if the system is ergodic, a similar argument shows that $\langle \iota(A^*), P\iota(B)\rangle = \omega(A)\omega(B) = \langle \iota(A^*), \Omega\rangle\langle \Omega, \iota(B)\rangle = \langle \iota(A^*), (\Omega \otimes \Omega)\iota(B)\rangle$, and since $\mathfrak{G}$ is dense in $\mathfrak{H}$, it follows that $P = \Omega \otimes \Omega$.

A corollary of Theorem 3.3 for ergodic systems is clearly

Corollary 3.4. *Assume* $(\mathfrak{A}, \omega, \tau, K)$ *given in Theorem* 3.3 *is ergodic. For any* $A, B \in \mathfrak{A}$ *and* $\varepsilon > 0$*, there then exists an* α_0 *such that*

$$\left| \frac{1}{\mu(\Lambda_{\alpha_0})} \int_{\Lambda_{\alpha_0} h} \omega\big(A\tau_g(B)\big)\,dg \right| > \omega(A)\omega(B) - \varepsilon$$

for all $h \in K$.

While ergodicity can be formulated and proven equivalent to P having one-dimensional range, even when K is not abelian, as we did above, Theorem 3.3 and Corollary 3.4 are only stated for abelian K, though G is allowed to be nonabelian. However, using Theorem 2.2, Theorem 3.3 can be modified to

Theorem 3.5. *Let* $(\mathfrak{A}, \omega, \tau, K)$ *be a* $*$*-dynamical system, but assume that the underlying group* G *is unimodular. Let* $\{\Lambda_\alpha\}$ *be a space-filling net in* K*. Then for any* $A, B \in \mathfrak{A}$ *and* $\varepsilon > 0$*, there exists an* α_0 *such that*

$$\left| \frac{1}{\mu(\Lambda_{\alpha_0})} \int_{h\Lambda_{\alpha_0}} \omega\big(A^*\tau_g(B)\big)\,dg \right| > \left| \lim_\alpha \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \omega\big(A^*\tau_g(B)\big)\,dg \right| - \varepsilon$$

for all $h \in K$.

We can modify Corollary 3.4 in a corresponding way.

4. Towards multiple recurrence

In this section we study a form of multiple recurrence, inspired by Furstenberg's work, as mentioned in the introduction. Also refer to Petersen [5] for a discussion of multiple recurrence in the measure theoretic setting over the group $G = \mathbb{Z}$. We will formulate our results for an abelian subsemigroup K of a locally compact Hausdorff group G, but as with Theorem 3.5, the results in this section can easily be modified to the case where G is unimodular, but K not necessarily abelian.

Let $\mathfrak{A}\otimes\mathfrak{B}$ denote the algebraic tensor product of the $*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$. First we state a result about simultaneous recurrence in more than one system:

Proposition 4.1. *Let $(\mathfrak{A}_j,\omega_j,\tau_j,K)$ be a $*$-dynamical system such that $\omega(\tau_{j,g}(A)^*\tau_{j,g}(B))=\omega(A^*B)$ for all $A,B\in\mathfrak{A}_j$ and $g\in K$, for $j=1,\dots,q$, and assume K is abelian. (Here we use the notation $\tau_j:K\to L(\mathfrak{A}):g\mapsto\tau_{j,g}$.) Let $\{\Lambda_\alpha\}$ be a space-filling net in K. Then for any $A_j,B_j\in\mathfrak{A}_j$ and $\varepsilon>0$, there exists an α_0 such that*

$$\left|\frac{1}{\mu(\Lambda_{\alpha_0})}\int_{\Lambda_{\alpha_0}h}\omega_1\big(A_1^*\tau_{1,g}(B_1)\big)\dots\omega_q\big(A_q^*\tau_{q,g}(B_q)\big)\,dg\right|$$
$$>\left|\lim_\alpha\frac{1}{\mu(\Lambda_\alpha)}\int_{\Lambda_\alpha}\omega_1\big(A_1^*\tau_{1,g}(B_1)\big)\dots\omega_q\big(A_q^*\tau_{q,g}(B_q)\big)\,dg\right|-\varepsilon$$

for all $h\in K$. In particular, if $A_j=B_j$, then for every $h\in K$,

$$\left|\omega_1\big(A_1^*\tau_{1,g}(A_1)\big)\dots\omega_q\big(A_q^*\tau_{q,g}(A_q)\big)\right|>\left|\omega_1(A_1)\dots\omega_q(A_q)\right|^2-\varepsilon$$

for some $g\in\Lambda_{\alpha_0}h$.

Proof. Set $\mathfrak{A}:=\mathfrak{A}_1\otimes\cdots\otimes\mathfrak{A}_q$, $\omega:=\omega_1\otimes\cdots\otimes\omega_q$ and $\tau_g:=\tau_{1,g}\otimes\cdots\otimes\tau_{q,g}$. We first show that $(\mathfrak{A},\omega,\tau,K)$ is a $*$-dynamical system. (It is in this step that we require the additional condition $\omega(\tau_{j,g}(A)^*\tau_{j,g}(B))=\omega(A^*B)$.) First consider $A=A_1\otimes A_2$ and $B=B_1\otimes B_2$ where $A_j,B_j\in\mathfrak{A}_j$. Then $(\tau_{1,g}\otimes\tau_{2,g})((\tau_{1,h}\otimes\tau_{2,h})(A))=(\tau_{1,g}\otimes\tau_{2,g})(\tau_{1,h}(A_1)\otimes\tau_{2,h}(A_2))=(\tau_{1,gh}\otimes\tau_{2,gh})(A)$, $(\tau_{1,g}\otimes\tau_{2,g})(1)=(\tau_{1,g}\otimes\tau_{2,g})(1\otimes1)=1\otimes1=1$, and

$$\begin{aligned}
&\omega_1\otimes\omega_2\big(\big[(\tau_{1,g}\otimes\tau_{2,g})(A)\big]^*(\tau_{1,g}\otimes\tau_{2,g})(B)\big)\\
&\quad=\omega_1\big(\tau_{1,g}(A_1)^*\tau_{1,g}(B_1)\big)\omega_2\big(\tau_{2,g}(A_2)^*\tau_{2,g}(B_2)\big)\\
&\quad=\omega_1\big(A_1^*B_1\big)\omega_2\big(A_2^*B_2\big)\\
&\quad=\omega_1\otimes\omega_2\big(A^*B\big).
\end{aligned}$$

Furthermore,

$$K\ni g\mapsto\omega_1\otimes\omega_2\big(A^*(\tau_{1,g}\otimes\tau_{2,g})(B)\big)=\omega_1\big(A_1^*\tau_{1,g}(B_1)\big)\omega_2\big(A_2^*\tau_{2,g}(B_2)\big)$$

is Borel measurable. All these facts then also hold for any $A,B\in\mathfrak{A}_1\otimes\mathfrak{A}_2$, since these elements have the form $A=\sum_{k=1}^M A_{1,k}\otimes A_{2,k}$ where $A_{j,k}\in\mathfrak{A}_j$. By induction this can be extended to obtain $\tau_g\circ\tau_h=\tau_{gh}$, $\tau_g(1)=1$, $\omega(A^*\tau_g(B))=\omega(A^*B)$, and $K\ni g\mapsto\omega(A^*\tau_g(B))$ Borel measurable. In particular, $(\mathfrak{A},\omega,\tau,K)$ is a $*$-dynamical system. Applying Theorem 3.3 to $A:=A_1\otimes\cdots\otimes A_q$ and $B:=B_1\otimes\cdots\otimes B_q$, the proposition is proved. □

We are now going to apply this result to prove a form of multiple recurrence. Given a $\tau:K\to L(\mathfrak{A})$ such that $\tau_g\circ\tau_h=\tau_{gh}$, and we want to construct a $\sigma:K\to L(\mathfrak{A}):g\mapsto\tau_{\varphi(g)}$ such that we also have $\sigma_g\circ\sigma_h=\sigma_{gh}$, then it seems sensible to take $\varphi:K\to K$ such that $\varphi(g)\varphi(h)=\varphi(gh)$. Such φ's will determine the pattern of the multiple recurrence (see Corollary 4.3 and the discussion following it).

Theorem 4.2. *Consider a $*$-dynamical system $(\mathfrak{A}, \omega, \tau, K)$ for which K is abelian and $\omega(\tau_g(A)^*\tau_g(B)) = \omega(A^*B)$ holds for all $A, B \in \mathfrak{A}$ and $g \in K$. Let $\varphi_j : K \to K$ be a Borel measurable function such that $\varphi_j(gh) = \varphi_j(g)\varphi_j(h)$ for all $g, h \in K$, for $j = 1, \ldots, q$. Let $\{\Lambda_\alpha\}$ be a space-filling net in K. For $A, B \in \mathfrak{A}$ and $\varepsilon > 0$ there then exists an α_0 such that*

$$\left|\frac{1}{\mu(\Lambda_{\alpha_0})}\int_{\Lambda_{\alpha_0}h} \omega\big(A^*\tau_{\varphi_1(g)}(B)\big)\ldots\omega\big(A^*\tau_{\varphi_q(g)}(B)\big)\,dg\right|$$
$$> \left|\lim_\alpha \frac{1}{\mu(\Lambda_\alpha)}\int_{\Lambda_\alpha} \omega\big(A^*\tau_{\varphi_1(g)}(B)\big)\ldots\omega\big(A^*\tau_{\varphi_q(g)}(B)\big)\,dg\right| - \varepsilon$$

for all $h \in K$. In particular, if $A = B$, then for every $h \in K$,

$$\left|\omega\big(A^*\tau_{\varphi_1(g)}(A)\big)\ldots\omega\big(A^*\tau_{\varphi_q(g)}(A)\big)\right| > \left|\omega(A)\right|^{2q} - \varepsilon$$

for some $g \in \Lambda_{\alpha_0}h$.

Proof. We will apply Proposition 4.1 to $\mathfrak{A}_j := \mathfrak{A}$, $\omega_j := \omega$ and $\tau_{j,g} := \tau_{\varphi_j(g)}$. It is given that $F : K \to \mathbb{C} : g \mapsto \omega(A\tau_g(B))$ is Borel measurable, hence $F \circ \varphi_j : K \to \mathbb{C} : g \mapsto \omega(A\tau_{j,g}(B))$ is also Borel measurable, while $\tau_{j,g} \circ \tau_{j,h} = \tau_{\varphi_j(g)} \circ \tau_{\varphi_j(h)} = \tau_{\varphi_j(g)\varphi_j(h)} = \tau_{\varphi_j(gh)} = \tau_{j,gh}$. Clearly $\omega_j(\tau_{j,g}(A)^*\tau_{j,g}(B)) = \omega_j(A^*B)$ for all $A, B \in \mathfrak{A}_j$ and $\tau_{j,g}(1) = 1$, hence $(\mathfrak{A}_j, \omega_j, \tau_j, K)$ is a $*$-dynamical system with the properties required in Proposition 4.1. With $A_j := A$ and $B_j := B$, the result now follows from Proposition 4.1. □

Corollary 4.3. *If $\omega(A) > 0$, then for every $h \in K$ there is a $g \in \Lambda_{\alpha_0}h$ such that*

$$\left|\omega\big(A^*\tau_{\varphi_j(g)}(A)\big)\right| > 0$$

for $j = 1, \ldots, q$. (*Just take $\varepsilon < |\omega(A)|^{2q}$ in Theorem* 4.2.)

For example, since K is abelian, we can take $\varphi_j(g) = g^{n_j}$ where $n_j \in \mathbb{N}$. If G is abelian (or if we just use abelian notation for K), then this says $\varphi_j(g) = n_j g$, and Corollary 4.3 reduces to the form

$$\left|\omega\big(A^*\tau_{n_j g}(A)\big)\right| > 0$$

for $j = 1, \ldots, q$.

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