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WHAT IS AN Ω-KRULL RING?

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0. Introduction

In the past decade, a lot of results have been obtained on generalizations of Dedekind domains to the noncommutative case, e.g. Asano orders, Dedekind prime rings and hereditary Noetherian prime rings. Now, if one wants to produce a noncommutative analogue of a Krull domain, a very rich gamma of possible definitions is available. A commutative Krull domain may be characterized in two different ways: on the one hand a Krull domain is a completely integrally closed ring which satisfies the ascending chain condition on divisorial ideals; on the other hand, a Krull domain is a ring which is equal to the intersection of the localizations at its height one prime ideals, and each such localization is a discrete valuation ring. This duality may be exploited in the noncommutative case too. For instance, Chamarie defines a Krull ring to be a maximal order (the noncommutative analogue of a completely integrally closed ring) satisfying the ascending chain condition on certain left and right ideals (cf. [2], [3]). If the ring is written as an intersection of localizations, one may consider independent local conditions on the localized rings but also on the type of localization. Marubayashi defines a Krull ring (cf. [13], [15]) as an intersection of local, Noetherian, Asano orders and a simple, Noetherian ring, which are left and right localizations of the ring with respect to an idempotent kernel functor satisfying property (T) (cf. [5] and [21] for details on localization). On the other hand, an Ω-Krull ring is defined as an intersection of quasi-local Ω-rings (see Section 1 for the definition) which are symmetric localizations of the ring (i.e. the associated filter has a basis of ideals, cf. [21]). The difference between a Marubayashi-Krull ring and an Ω-Krull ring is that we only assume conditions on twosided ideals: the first problem of this kind is to find a symmetric analogue of the Goldie theorems, i.e. to give necessary and sufficient conditions such that a ring R may be embedded in a symmetric localization $Q_{\mathrm{sym}}(R)$ which is a simple ring. However, to find an intrinsic characterization in terms of elements or ideals of R,

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turns out to be rather difficult. In order to bypass this problem, we will limit ourselves to prime rings satisfying Formanek's condition, i.e. every nonzero ideal intersects the center nontrivially. Therefore Ω-Krull rings are closely related to the center. It is clear from the imposed condition that P.I.-Ω-Krull rings will constitute the class of Ω-Krull rings for the greater part. In this case, all definitions of noncommutative Krull rings coincide and a Krull P.I.-ring is just a maximal order over a Krull domain.

This note surveys some recent results. In Section 2, we establish an intrinsic characterization of central Ω-Krull rings. It is worth to note that the proof of this theorem uses arithmetical pseudovaluations, from the theory of primes, as developed by J. Van Geel (cf. [20]). Localization of Ω-Krull rings is being treated in Section 3 and symmetric maximal orders equivalent to a geometrical Ω-Krull ring are being studied in Section 4. Also the class group is introduced (there are several possible definitions!). We will limit ourselves to the central class group (cf. Section 5). To construct a wide class of examples, it was necessary to develop the notion of a Gr-Ω-Krull ring, i.e. the graded analogue of an Ω-Krull ring (cf. Section 6). This made it possible to handle generalized Rees rings. This class of rings includes (twisted)(semi) group rings (cf. Section 7). Finally, it is also possible to show when a skew polynomial ring is an Ω-Krull ring (cf. Section 8).

For the curiosity of the reader, we mention that the prefix 'Ω' has first been used by E. Nauwelaerts and F. Van Oystaeyen in [18] where the notion of an Ω-ring is studied. An Ω-ring is a noncommutative version of a Dedekind domain; in fact, the set of maximal ideals of a ring R is sometimes denoted by $\Omega(R)$. Now it is clear from the definition of an Ω-Krull ring why 'Ω' is used.

1. Definitions, examples and properties

All rings will be associative and have a unit element. Ideal always means a two-sided ideal. Let R be a prime ring satisfying Formanek's condition, i.e. every nonzero ideal of R intersects C, the center of R, nontrivially. In this case, the overring

$$Q_{\mathrm{sym}}(R)=Q=\{c^{-1}r=rc^{-1}\mid 0\neq c\in C, r\in R\}$$

is a simple ring. Moreover, $Q_{\mathrm{sym}}(R)\cong Q^{\mathrm{l}}_{R\setminus 0}(R)\cong Q^{\mathrm{r}}_{R\setminus 0}(R)$, where $Q^{\mathrm{l}}_{R\setminus 0}(R)$ denotes the localization of the left R-module R with respect to the symmetric filter

$$\mathscr{L}(R\setminus 0)=\{I\mid I \text{ a left ideal of } R \text{ containing a nonzero twosided ideal}\}.$$

Definition. A prime ring R satisfying Formanek's condition is said to be an *Ω-Krull ring* (cf. [6]) if

(1) There exist multiplicatively closed filters of ideals, $\mathscr{L}^2(\sigma_i)$ $(i\in\Lambda)$ such that

$$R_i=Q^{\mathrm{l}}_{\sigma_i}(R)=\{q\in Q_{\mathrm{sym}}(R)\mid \exists I\in\mathscr{L}^2(\sigma_i)\ \ Iq\subset R\}$$
$$=Q^{\mathrm{r}}_{\sigma_i}(R)=\{q\in Q_{\mathrm{sym}}(R)\mid \exists I\in\mathscr{L}^2(\sigma_i)\ \ qI\subset R\}.$$

(2) Each ring R_i is a quasi-local Ω-ring, i.e. every nonzero ideal of R_i is a power of the unique maximal ideal P_i' of R_i.

(3) $R=\bigcap_{i\in\Lambda} R_i$.

(4) For every $i\in\Lambda$ and for all $I\in\mathscr{L}^2(\sigma_i)$, $R_iI=IR_i=R_i$.

(5) For all $r\in R$ there are only finitely many $i\in\Lambda$ such that $RrR=(r)\notin\mathscr{L}^2(\sigma_i)$.

The next proposition provides some elementary properties of Ω-Krull rings.

Proposition 1.1 (cf. [6], [24]). *If R is an Ω-Krull ring, then*

(1) *$\forall i\in\Lambda$ σ_i is an idempotent kernel functor (in the sense of Goldman,* cf. [5]).

(2) *$P_i=P_i'\cap R\in X^1(R)$ ($X^1(R)$ stands for the set of height one prime ideals of R).*

(3) *$\forall i\in\Lambda$ $\mathscr{L}^2(\sigma_i)=\mathscr{L}^2(R\setminus P_i)=\{I\mid I$ an ideal of R such that $I\not\subset P_i\}$.*

(4) *$R=\bigcap_{P_i\in X^1(R)} Q_{R\setminus P_i}(R)$ (and $Q_{R\setminus P_i}(R)=\{q\in Q_{\mathrm{sym}}(R)\mid Iq\subset R$ and $qI\subset R$ for some ideal I of R, $I\not\subset P_i\}$).*

Note that the abstract theory of localization of noncommutative rings is not needed in the definition of an Ω-Krull ring, but nevertheless, the overrings R_i turn out to be localizations of R by Proposition 1.1.

We will often deal with some special types of Ω-Krull rings. A ring R is said to be a *geometrical Ω-Krull ring* if it is an Ω-Krull ring such that for all $i\in\Lambda$ σ_i is a geometric kernel functor, i.e. for every ideal I of R and for all $i\in\Lambda$ R_iI and IR_i are ideals of R_i (and hence $R_iI=IR_i$). An Ω-Krull ring R is a *central Ω-Krull ring* if for all $i\in\Lambda$ $I\in\mathscr{L}^2(\sigma_i)$ iff $R(I\cap C)\in\mathscr{L}^2(\sigma_i)$. In this case

$$Q_{\sigma_i}(R)=Q_{R\setminus P_i}(R)=Q_{C\setminus p_i}(R)=\{c^{-1}r\mid 0\neq c\in C\setminus p_i, r\in R\}$$

and $p_i=P_i\cap C$. Moreover

$$R=\bigcap_{p_i\in X^1(C)} Q_{C\setminus p_i}(R).$$

Clearly, a central Ω-Krull ring is also a geometrical Ω-Krull ring.

We define a *left (resp. right) fractional R-ideal I* to be a nonzero left (resp. right) R-submodule of $Q_{\mathrm{sym}}(R)$ such that $cI\subset R$ for some $0\neq c\in C$. A fractional ideal is a left and right fractional ideal. If R is an Ω-Krull ring, then it is easy to see that the set of fractional R_i-ideals forms an infinite cyclic group with generator P_i'.

Note that if R is commutative, this notion of an Ω-Krull ring reduces to a commutative Krull domain. More generally, if R is an Ω-Krull ring, define for all $i\in\Lambda$ $v_i:K^*\to\mathbb{Z}:a\mapsto n$ where $P_i'^n=R_ia$ (K is the field of fractions of C). It is straightforward to check that each v_i is a discrete valuation and that C is the intersection of the associated valuation rings. So we have proved

Proposition 1.2. *The center of an Ω-Krull ring is a Krull domain.*

Elementary examples of Ω-Krull rings are complete matrix rings $M_n(R)$ or

Azumaya algebras over a Krull domain. Polynomial rings over any simple ring which is not a Goldie ring yield examples of Ω-Krull rings which are neither Marubayashi–Krull rings nor Chamarie–Krull rings.

A prime Formanek ring R is said to be a *symmetric maximal order* if there does not exist an overring S such that $R \subsetneq S \subset Q_{\text{sym}}(R)$ and $cS \subset R$ for some $0 \neq c \in C$. This is equivalent to saying that for every ideal I of R $(I :_{\text{l}} I) = (I :_{\text{r}} I) = R$. If A and B are subsets of $Q_{\text{sym}}(R)$, then

$$(A :_{\text{l}} B) = \{q \in Q \mid qB \subset A\} \quad \text{and} \quad (A :_{\text{r}} B) = \{q \in Q \mid Bq \subset A\}.$$

It is straightforward to check that if R is a geometrical Ω-Krull ring, then R is a symmetric maximal order (cf. [8]).

Let R be an Ω-Krull ring and I a fractional R-ideal. Consider the twosided R-module $\bar{I} = \bigcap_i R_i I R_i$. Clearly, $\bar{I}$ is a fractional R-ideal and $I \subset \bar{I}$. If $I = \bar{I}$, I is said to be a *divisorial R-ideal.* Moreover, if R is also a symmetric maximal order (e.g. if R is a central Ω-Krull ring), then I is divisorial if and only if $I = I^*$ where $I^* = (R : (R : I))$. The set of divisorial ideals of R, denoted by $\mathbb{D}(R)$, is a semigroup under $*$ where $\bar{A} * \bar{B} = (\overline{\bar{A}\bar{B}})$ and $\bar{A}, \bar{B} \in \mathbb{D}(R)$. In case R is an Ω-Krull ring, we have

Theorem 1.3 (cf. [7]). *$\mathbb{D}(R)$ is a free abelian group generated by $X^1(R)$.*

Theorem 1.4. *Let R be a prime P.I.-ring. Then the following are equivalent:*

(1) *R is an Ω-Krull ring.*

(2) *R is a Marubayashi–Krull ring.*

(3) *R is a Chamarie–Krull ring.*

(4) *R is a (symmetric) maximal order and $Z(R)$ is a Krull domain.*

Note also that if R is a P.I.-Ω-Krull ring, then R is a central Ω-Krull ring (cf. [3], [9]).

2. A characterization of central Ω-Krull rings

It is well known that a commutative ring R is a Krull domain if and only if R is completely integrally closed and R satisfies the ascending chain condition on divisorial ideals contained in R. In the noncommutative case, we have the following generalization

Theorem 2.1 (cf. [8]). *A ring R is a central Ω-Krull ring if and only if*

(1) *R is a symmetric maximal order;*

(2) *R satisfies the ascending chain condition on divisorial ideals in R;*

(3) *For each $P \in X^1(R)$ and for any ideal I of R we have $I \subset P$ if and only if $(I \cap C) \subset (P \cap C)$.*

Proof. If R is a central Ω-Krull ring, it is quite obvious that the above mentioned conditions are satisfied. We will sketch the proof of the converse. Since R is a symmetric maximal order, the set of divisorial ideals $\mathbb{D}(R)=\{I \mid I$ a fractional R-ideal such that $I^*=(R:(R:I))=I\}$ is a commutative group. Moreover, $\mathbb{D}(R)$ is a free abelian group with basis $X^1(R)$, because R satisfies the ascending chain condition on divisorial ideals contained in R (cf. [16]). Therefore, if $I\in\mathbb{D}(R)$, we may write $I=P_1^{n_1}*\cdots*P_k^{n_k}$ in a unique way and $P_i\in X^1(R)$, $n_i\in\mathbb{Z}$ $(1\le i\le k)$. Define for all i

$$v_i:\mathbb{D}(R)\to\mathbb{Z}:P_1^{n_1}*\cdots*P_k^{n_k}\mapsto n_i.$$

Then v_i is an arithmetical pseudovaluation in the sense that

(1) $\forall I,J\in\mathbb{D}(R)$ $\quad v_i(I*J)=v_i(I)+v_i(J)$,

(2) $\forall I,J\in\mathbb{D}(R)$ $\quad v_i((I+J)^*)\ge\min\{v_i(I),v_i(J)\}$,

(3) $\forall I,J\in\mathbb{D}(R)$ $\quad$ if $I\subset J$, then $v_i(I)\ge v_i(J)$,

(4) $v_i(R)=0$.

Denote by $R_i=\{x\in Q_{\mathrm{sym}}(R)\mid v_i((RxR)^*)\ge 0\}$. Then $R_i=\{x\in Q_{\mathrm{sym}}(R)\mid xI\subset R$ and $Ix\subset R$ for some ideal I of R not contained in $P_i\}$. By condition (3) it follows that R_i is a quasi-local Ω-ring with unique maximal ideal $P_i'=\{x\mid v_i((RxR)^*)>0\}$. Finally, it is clear that $R=\bigcap_i R_i$, which establishes the theorem.

It will become apparent from the following sections that this characterization theorem is very useful. In nearly all cases where one wants to show that some ring is a central Ω-Krull ring, we will check the conditions of the preceding theorem.

3. Localization of Ω-Krull rings

In this section we will restrict to geometrical Ω-Krull rings. Nevertheless, in some cases, more general results hold. For full detail, we refer to [8]. So, let R be a geometrical Ω-Krull ring and write $R=\bigcap_{i\in\Lambda}R_i$. If Λ_0 is a subset of Λ, the ring $S=\bigcap_{i\in\Lambda_0}R_i$ is said to be a *subintersection* of R. We may state

Proposition 3.1 (cf. [8]). *If $R=\bigcap_{i\in\Lambda}R_i$ is a geometrical (resp. central) Ω-Krull ring and Λ_0 is a subset of Λ, then the subintersection $S=\bigcap_{i\in\Lambda_0}R_i$ is a geometrical (resp. central) Ω-Krull ring.*

Proposition 3.2 (cf. [8]). *Let R be a geometrical Ω-Krull ring and $\mathscr{L}^2(\kappa)$ a multiplicatively closed filter of ideals. Then the ring*

$$Q_\kappa(R)=\{q\in Q_{\mathrm{sym}}(R)\mid Iq\subset R \textit{ for some } I\in\mathscr{L}^2(\kappa)\}$$

is a subintersection of R and hence a geometrical Ω-Krull ring.

4. Symmetric maximal orders equivalent to a geometrical Ω-Krull ring

If R is a bounded Marubayashi–Krull ring, then any maximal order equivalent to R is again a bounded Marubayashi–Krull ring (cf. [14]). This result was generalized by Chamarie who proved that a maximal order equivalent to a Chamarie–Krull ring is again a Chamarie–Krull ring (cf. [2]). In case R is an Ω-Krull ring, this does not hold anymore. The reason for this is that a quasi-local Ω-ring need not be hereditary, while a local Noetherian Asano order is a left and a right hereditary ring.

Let R be a prime, Formanek ring. A ring $S\subset Q_{\mathrm{sym}}(R)=Q$ is said to be equivalent to R if there exist $0\neq c$, $d\in Z(Q)$ such that $cR\subset S$ and $dS\subset R$. I is said to be an $R-S$ ideal if I is a left R-ideal and a right S-ideal.

Theorem 4.1 (cf. [8]). *Let R be a geometrical Ω-Krull ring with property* (*P*)*, this means that for every symmetric maximal order $S(P)$ equivalent to $Q_{R\setminus P}(R)$ all $Q_{R\setminus P}(R)-S(P)$ ideals are projective left $Q_{R\setminus P}(R)$-modules and projective right $S(P)$-modules, then every symmetric maximal order equivalent to R is a geometrical Ω-Krull ring. Conversely, if the symmetric maximal orders equivalent to all geometrical Ω-Krull rings are again geometric Ω-Krull rings, then every geometric Ω-Krull ring satisfies property* (P).

5. Class groups of Ω-Krull rings

In commutative algebra and number theory, the class group of a Krull domain plays an important role. Roughly speaking, it measures the lack of unique factorization in the ring. In the noncommutative case, several possibilities arise to define the class group. We will introduce the normalizing class group and the central class group. Let R be an Ω-Krull ring. Define

$$\mathbb{P}^n(R)=\{Rn=nR \mid n\in Q_{\mathrm{sym}}(R)\} \quad \text{and} \quad \mathbb{P}^c(R)=\{Rc \mid c\in Z(Q)\}.$$

Then $\mathrm{Cl}^n(R)=\mathbb{D}(R)/\mathbb{P}^n(R)$ is said to be the *normalizing class group* of R and $\mathrm{Cl}^c(R)=\mathbb{D}(R)/\mathbb{P}^c(R)$ is the *central class group* of R. Some results on the normalizing class group may be found in [11] and [12]. In the sequel, we will restrict to the central class group and we will simply speak about the class group and write $\mathrm{Cl}(R)$. Since Ω-Krull rings as well as the class group are strongly related to the center, it is not surprising that the class group of the center is contained in the class group of the ring. However, if R is an Ω-Krull ring, $\mathrm{Cl}(C)$ need not to be equal to $\mathrm{Cl}(R)$ as proves the following example: consider $\mathbb{C}[X,\bar{\ }]$ where $\mathbb{C}$ is the field of complex numbers and $\bar{\ }$ denotes the complex conjugation. It is easy to verify that $\mathrm{Cl}(\mathbb{C}[X,\bar{\ }])\cong\mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Cl}(Z(\mathbb{C}[X,\bar{\ }]))=0$ since $Z(\mathbb{C}[X,\bar{\ }])=\mathbb{R}[X^2]$.

Let $R\subset S$ be rings. S is said to be a (ring) extension of R if $S=R\cdot Z_S(R)$ where $Z_S(R)=\{s\in S \mid \forall r\in R: sr=rs\}$. Then we have the following generalization of Nagata's theorem.

Theorem 5.1 (cf. [9]). *Suppose that R is a central Ω-Krull ring and let B be a subintersection of R such that B is an extension of R, say $R=\bigcap_{i\in\Lambda} R_i$ and $B=\bigcap_{i\in\Lambda_0} R_i$ ($\Lambda_0\subset\Lambda$). Then $\psi:\mathrm{Cl}(R)\to\mathrm{Cl}(B):[I]\mapsto[(BI)^*]$ is a surjective homomorphism and* ker ψ *is generated by the classes of prime ideals $P_i\in X^1(R)$ such that $i\in\Lambda\setminus\Lambda_0$.*

In particular, we have

Corollary 5.2 (cf. [9]). *Let R be a central Ω-Krull ring and σ a kernel functor satisfying property (T) and suppose that $Q_\sigma(R)$ is a ringextension of R. Put $\psi:\mathrm{Cl}(R)\to\mathrm{Cl}(Q_\sigma(R))$. Then* ker ψ *is generated by the classes of those prime ideals $P\in X^1(R)$ such that $P\in\mathscr{L}^2(\sigma)$.*

6. Graded Ω-Krull rings

In the following sections, a wide class of Ω-Krull rings will be constructed. All these rings are graded rings. Therefore it will be necessary to introduce a general technique to handle graded rings. It turns out that a graded ring R is an Ω-Krull ring if R satisfies certain 'graded Krull properties', some overring $Q^g(R)$ is an Ω-Krull ring and a third technical condition needs to be satisfied. This has motivated us to introduce the notion of a Gr-Ω-Krull ring. To be more precise, let R be a graded ring, graded by an abelian monoid S. R is said to be gr-prime, if $xRy=0$ implies $x=0$ or $y=0$ for $x,y\in h(R)$ ($h(R)$ is the set of homogeneous elements of R); R is gr-Formanek if every gr-ideal contains a nonzero (homogeneous) central element. Then the overring

$$Q^g=Q^g(R)=Q^g_{\mathrm{sym}}(R)=\{c^{-1}r=rc^{-1}\mid 0\neq c\in h(C), r\in R\}$$

is a gr-simple ring. R is said to be a Gr-Ω-Krull ring if R can be written as an intersection of rings R_i, such that there exist multiplicatively closed filters $\mathscr{L}^2(\kappa_i)$ with $I\in\mathscr{L}^2(\kappa_i)$ iff $R(I\cap h(R))\in\mathscr{L}^2(\kappa_i)$, $R_i=Q_{\kappa_i}(R)$, and each graded ideal of R_i is a power of the unique maximal gr-ideal of R_i. Also the graded analogues of the other conditions of an Ω-Krull ring need to hold. Similarly as in the ungraded case, one may define geometrical and central Gr-Ω-Krull rings. The interesting point is the relationship between an Ω-Krull ring and a Gr-Ω-Krull ring. The following result is quite obvious.

Proposition 6.1 (cf. [24]). *If R is a graded ring and a central Ω-Krull ring, then R is a central Gr-Ω-Krull ring.*

The converse of this result does not hold, for if R is a commutative Krull domain and G an arbitrary abelian group, then $R[G]$ is a Gr-Ω-Krull ring but $R[G]$ need not be a Krull domain (cf. [1]). Moreover, $R[G]$ is not prime in general. Therefore, we impose that the grading semigroup S is cancellative and torsion free abelian (this means that the quotient group of S is torsion free abelian). Then we may state

Theorem 6.2 (cf. [25]). *A graded ring R is a central Ω-Krull ring if and only if*

(1) *R is a central Gr-Ω-Krull ring.*

(2) *$Q^{g}_{sym}(R)$ is a central Ω-Krull ring.*

(3) *For each $P \in X^1(R)$ $Z(R/P)$ has the intersection property with respect to $C/P \cap C$ (i.e. that any nonzero ideal of $Z(R/P)$ has a nontrivial intersection with $C/P \cap C$).*

The investigation whether a graded ring is a central Ω-Krull ring is therefore split up in three parts. First, one has to examine whether the graded ring is a Gr-Ω-Krull ring. Usually, this follows from the appropriate conditions on the part of degree one of the ring and on the grading semigroup (cf. Section 7). In general, $Q^{g}_{sym}(R)$ is a well-known ring, e.g. a group ring over a simple ring. In the P.I.-case, $Q^{g}_{sym}(R)$ is always an Azumaya algebra over its center, which is a graded field (cf. [23]). Finally, the third condition is merely a technical one to make the localizations central. Moreover, it follows from a result of W. Schelter (cf. [19]) that this condition is always satisfied if R is a P.I.-ring.

As in the commutative case, we may define the graded class group of a graded Ω-Krull ring R. If $\mathbb{D}_g(R)$ is the free abelian group of graded divisorial ideals and $\mathbb{P}_g(R) = \{Rc \mid c \in h(Z(Q^g))\}$, then $\mathrm{Cl}_g(R) = \mathbb{D}_g(R)/\mathbb{P}_g(R)$. We have the following useful result:

Theorem 6.3 (cf. [9]). *If R is a graded ring and a central Ω-Krull ring, then the sequence $1 \to \mathrm{Cl}_g(R) \to \mathrm{Cl}(R) \to \mathrm{Cl}(Q^g) \to 1$ is exact.*

In particular, if R is a P.I.-ring, $\mathrm{Cl}(Q^g) = 1$ whence $\mathrm{Cl}_g(R) = \mathrm{Cl}(R)$.

7. Generalized Rees rings

If R is a commutative ring and I is an ideal of R, a Rees ring is defined to be the subring $R + IX + I^2X^2 + \cdots + I^nX^n + \cdots$ of the polynomial ring $R[X]$ and is used in the proof of the Artin–Rees Lemma. Afterwards, this construction has been generalized in several steps, mainly by F. Van Oystaeyen (cf. [22], [17], [12]). The construction we will give here contains the class of (twisted) semigroup rings.

Let R be a prime Formanek ring, and suppose that R is a symmetric maximal order, in particular R has a group of divisorial ideals $\mathbb{D}(R)$. Let S be a torsion free abelian cancellative monoid and $f: S \to \mathbb{D}(R)$ a monoid homomorphism (so $f(1) = R$). If $\gamma : S \times S \to \mathscr{U}(Z(R))$ is a two-cocycle, we define the *generalized Rees ring* $\check{R} = \check{R}(f, \gamma, S)$ associated to f and γ to be the subring of $Q_{sym}(R)^t[S]$ given by

$$\check{R}(f, \gamma, S) = \bigoplus_{s \in S} I_s \bar{s} \quad \text{where } I_s = f(s).$$

The identity element of $\check{R}$ is $1=\gamma(1,1)^{-1}\bar{1}$ and without loss of generality we will assume that $1=\bar{1}$. Note that if $I_s=R$ for all $s\in S$, $\check{R}(f,\gamma,S)$ is a twisted semigroup ring $R^t[S]$.

We will use Theorem 6.2 to check when a generalized Rees ring is a central Ω-Krull ring. First, $Q^g_{sym}(\check{R})=Q^t[G]$ where $Q=Q_{sym}(R)$, $G=\langle S\rangle$ is the quotient group of S and $\gamma: G\times G\to \mathscr{U}(Z(R))$ extends the given map γ from $S\times S$ to $\mathscr{U}(Z(R))$. Denote by

$$S_f=\{s\in S \mid \forall t\in S: \gamma(s,t)=\gamma(t,s)\}$$

and by G_f the quotient group of S_f. We have to assume that G/G_f is finite in order to treat $Q^g_{sym}(\check{R})$. Now, it turns out that $\check{R}(f,\gamma,S)$ is a Gr-Ω-Krull ring exactly when R is an Ω-Krull ring and S satisfies the same 'Krull properties', i.e. S is a Krull semigroup. The notion of a Krull semigroup has been introduced by Chouinard in [4].

Theorem 7.1 (cf. [25]). *Let R be an S-graded ring, S a torsion free abelian cancellative monoid. If*

(1) *R is a central Ω-Krull ring,*
(2) *S is a Krull semigroup,*
(3) *$Z(R/P)$ is algebraic over $C/P\cap C$ for all $P\in X^1(R)$,*
(4) *G/G_f is a finite group,*
(5) *G satisfies the* ACC *on cyclic subgroups,*

then the generalized Rees ring $\check{R}(f,\gamma,S)$ is a central Ω-Krull ring.

Corollary 7.2 (cf. [25]). *Suppose that S is a torsion free abelian cancellative monoid and $G=\langle S\rangle$ is such that G/G_f is finite. If R is a central Ω-Krull ring, S is a Krull semigroup, G satisfies the* ACC *on cyclic subgroups and $Z(R/P)$ is algebraic over $C/P\cap C$ for all $P\in X^1(R)$, then $R^t[S]$ is a central Ω-Krull ring. If the two-cocycle is trivial, then the converse also holds.*

Note that a (torsion free) abelian group is always a Krull semigroup.

Proposition 7.3 (cf. [25]). *Under the conditions of Theorem* 7.1 *we have the following exact sequence*

$$1\to \mathrm{Cl}(R)/(f(\mathscr{U}(S_f))*\mathbb{P}(R)/\mathbb{P}(R))\to \mathrm{Cl}(\check{R})\to \mathrm{Cl}(Q^t[S])\to 1$$

where $\mathscr{U}(S_f)$ denotes the group of units of S_f.

8. Skew polynomial rings

One of the main results of [10] gives sufficient conditions for a skew semigroup ring to be an Ω-Krull ring. These conditions are also necessary in a lot of cases. Unfortunately, this is a very technical matter. So, for the sake of simplicity, we will restrict to skew polynomial rings. Let R be an arbitrary ring and σ a ringautomorphism of R. It is clear that if $R[X,\sigma]$ is an Ω-Krull ring, this will imply conditions on σ-stable ideals I of R (i.e. $\sigma(I)\subset I$). This leads us to the notion of a σ-Ω-Krull ring, i.e. the conditions will depend on the automorphism σ. Roughly speaking, the Formanek condition is replaced by the assumption that every nonzero σ-stable ideal contains a σ-invariant element a such that for some $n>0$ and for all $r\in R$ $ra=a\sigma^n(r)$ (we denote this set by $\mathcal{N}(R)^\sigma$). Then R has an overring $Q^\sigma_{\text{sym}}(R)$ which is a σ-simple ring, i.e. all σ-stable ideals are trivial. Under these hypotheses, R is said to be a *normalizing σ-Ω-Krull ring* if R can be written as an intersection of rings R_i, such that there exist multiplicatively closed filters of ideals $\mathcal{L}^2(\kappa_i)$ such that $I\in\mathcal{L}^2(\kappa_i)$ iff $R(I\cap\mathcal{N}(R)^\sigma)\in\mathcal{L}^2(\kappa_i)$, $R_i=Q_{\kappa_i}(R)$ and each σ-stable ideal of R_i is a power of the unique maximal σ-stable ideal P_i' of R_i. Also some other technical conditions need to be satisfied (cf. [10]). Then we have

Theorem 8.1 (cf. [10]). *The following conditions imply that $R[X,\sigma]$ is a central Ω-Krull ring.*

(1) *R is a normalizing σ-Ω-Krull ring.*

(2) *R is a prime ring which has the intersection property with respect to $Z(R)^\sigma$.*

(3) *For every $P\in X^1_\sigma(R)\cup\{0\}$ (i.e. the σ-analogue of $X^1(R)$) there exists a $n>0$ such that $Z(R[X,\sigma])\setminus P[X,\sigma]\neq\emptyset$.*

(4) *For all $P_i\in X^1(R)$ $Z(R_i/P_i')^\sigma$ is an algebraic field extension of $Z(R_i)^\sigma/Z(R_i)^\sigma\cap P_i'$.*

Moreover, if all σ-stable ideals I of R are σ-invariant (i.e. $I=\sigma(I)$), then the converse is also true.

In particular, we have

Corollary 8.2 (cf. [10]). *Let R be a central Ω-Krull ring such that*

(1) *R has the intersection property with respect to $Z(R)^\sigma$.*

(2) *For all $P\in X^1(R)$ there exists an element $a\in R\setminus P$ and a nonzero natural number $n>0$ such that $aX^n\in Z(R[X,\sigma])$.*

(3) *For all $P\in X^1_\sigma(R)$ $Q(Z(R/P)^\sigma)$ is algebraic over $Q(Z(R)^\sigma/Z(R)^\sigma\cap P)$; then $R[X,\sigma]$ is a central Ω-Krull ring.*

Corollary 8.3 (cf. [7]). *A polynomial ring $R[X]$ is a central Ω-Krull ring if and only if R is a central Ω-Krull ring and $Z(R/P)$ is algebraic over $C/P\cap C$ for all $P\in X^1(R)$.*

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