Lattice-ordered $2 \times 2$ triangular matrix algebras

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Abstract

We show that there are four different lattice orders on a $2 \times 2$ triangular matrix algebra over a totally ordered field to make it into a lattice-ordered algebra in which the identity matrix is positive. A general method is also given to construct lattice orders in which the identity matrix is not positive.

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1. Introduction

Let $K$ be a totally ordered field (fields are commutative herein), and let $T_n(K)$ denote the $n \times n$ ($n \geq 2$) upper triangular matrix algebra over $K$. Then $T_n(K)$ becomes a lattice-ordered algebra ($\ell$-algebra) if we define an upper triangular matrix to be positive exactly when each entry of the matrix is positive. This lattice order is called the usual lattice order on $T_n(K)$. The identity matrix 1 is positive in the usual lattice order.

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In 1983, S.A. Steinberg gave a characterization of lattice-ordered $2 \times 2$ triangular matrix algebras over a totally ordered field with the usual lattice order [4]. This seems to be the only result on lattice-ordered triangular matrix algebras in the literature.

In this paper we study lattice orders on $T_2(K)$. In Section 2, we show that there exist four different lattice orders (up to isomorphism or anti-isomorphism) on $T_2(K)$ to make $T_2(K)$ into $\ell$-algebras in which 1 is positive. Among them, three are Archimedean lattice orders. This is quite different from $2 \times 2$ full matrix algebras over $K$.

It was shown in [6,5] that there exists only one lattice order (up to isomorphism) on $2 \times 2$ full matrix algebra over $K$ to make it into an $\ell$-algebra in which 1 is positive. In Section 3 we produce lattice orders on $T_2(K)$ in which 1 is not positive.

First we collect some definitions and results that we are going to use later. The reader is referred to [1] for the general theory of lattice-ordered rings (\(\ell\)-rings). Throughout this paper, $K$ always denotes a totally ordered field.

An algebra $A$ over $K$ is called an $\ell$-algebra if $A$ is also a lattice, and the lattice operations and algebra operations on $A$ are compatible. Let $A$ be an $\ell$-algebra. The positive cone of $A$ is defined as $A^+ = \{a \in A : a \geq 0\}$ and elements in $A^+$ are called positive. Clearly, $A^+$ is closed under addition, multiplication, and positive scalar multiplication, and also $A^+ \cap -A^+ = \{0\}$. Conversely, if a subset $P$ of an algebra $A$ has the above properties, then the partial order defined by $a \geq b$ if and only if $a - b \in P$ makes $A$ into a partially ordered algebra (po-algebra) with positive cone $P$. If the partial order introduced in this way is also a lattice order, then $A$ becomes an $\ell$-algebra. In the following, we always use $(A, P)$ to denote an $\ell$-algebra $A$ with the positive cone $P$.

Let $G$ be a lattice-ordered group ($\ell$-group), and $a \in G$. Then the absolute value, positive part, and negative part of $a$ are $|a| = a \lor -a$, $a^+ = a \lor 0$, and $a^- = -a \lor 0$, respectively. An element $0 < a \in G$ is called basic if for any $x, y \in G^+$, $x, y \leq a$ implies $x$ and $y$ are comparable, that is, $x \leq y$ or $y \leq x$. Two elements $b, c \in G^+$ are called disjoint if $b \land c = 0$ and a subset $S \subseteq G^+$ is called disjoint if $a > 0$ for each $a \in S$ and any two distinct elements in $S$ are disjoint. It is well-known that a disjoint subset in a vector lattice is linearly independent.

Let $R$ be an $\ell$-ring. An element $a \in R^+$ is called an $f$-element if $b \land c = 0$ implies $ab \land c = ba \land c = 0$ for all $b, c \in R$. We define

$$F = F(R) = \{a \in R : |a| \text{ is an } f\text{-element of } R\}.$$  

Then $F$ is a convex $\ell$-subring of $R$, and $R$ is called an $f$-ring exactly when $F(R) = R$. An $\ell$-ring with identity element 1 is called $\ell$-unital if $1 > 0$. If $R$ is an $\ell$-unital $\ell$-ring, then the identity element 1 is in $F$ and $F$ is totally ordered if and only if 1 is a basic element in $R$. Let $M$ be a nonempty subset of $R$. The polar of $M$ is defined as

$$M^\perp = \{x \in R : \forall y \in M, x \land y = 0\}.$$  

\[1\] No proof for the characterization was provided in the paper, but Professor Steinberg privately communicated a proof with the second author.
If $F = F(R) \neq \emptyset$ is totally ordered, then, by [2, Lemma 6.2], $R$ has the following decomposition.

$$R = U(R) \cup (F \oplus F^\perp),$$

where $F \oplus F^\perp$ is a direct sum of convex $\ell$-subgroups,

$$U(R) = \{x \in R : |x| \geq y, \text{ for each } y \in F\},$$

and

$$U(R) \cap (F \oplus F^\perp) = \emptyset.$$ Since $F$ is totally ordered, $F^\perp = x^\perp$ for any $0 \neq x \in F$. In particular, if $R$ is $\ell$-unital, then $F^\perp = 1^\perp$.

An $\ell$-ring is called $\ell$-reduced if it contains no nonzero positive nilpotent element, and an $\ell$-ring is called an $\ell$-domain if $ab = 0$ for $a, b \in R^+$ implies $a = 0$ or $b = 0$. An $\ell$-algebra $A$ over $K$ is called Archimedean over $K$ if given $a, b \in A^+$, $aa \leq b$ for all $\alpha \in K^+$ implies $a = 0$. If $A$ is an $\ell$-unital Archimedean $\ell$-algebra over $K$ and $F(A)$ is totally ordered, then, since $U(A) = \emptyset$,

$$A = F(A) \oplus (F(A))^\perp$$
as a vector lattice.

Let $q \in T_2(K)$ be invertible. The inner automorphism determined by $q$ is denoted by $i_q$. Thus for each $p \in T_2(K)$, $i_q(p) = q^{-1}pq$. Each automorphism of algebra $T_2(K)$ is an inner automorphism determined by an invertible matrix in $T_2(K)$. Let $\varphi : T_2(K) \rightarrow T_2(K)$ be defined by

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mapsto \begin{pmatrix} z & y \\ 0 & x \end{pmatrix}.$$ Then any anti-automorphism of algebra $T_2(K)$ is equal to $\varphi i_q$ for some inner automorphism $i_q$.

2. 1 is positive

In this section we construct all the lattice orders on $T_2(K)$ in which $1$ is positive.

We use $P_0$ to denote the positive cone of the usual lattice order on $T_2(K)$, that is, $P_0 = T_2(K^+)$. Let

$$g = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in T_2(K).$$

It is clear that $\{1, g, h\}$ is a basis for vector space $T_2(K)$ over $K$ and we have the following multiplication table.

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If we define the positive cone
\[ P_1 = K^+ 1 + K^+ g + K^+ h, \]
then \((T_2(K), P_1)\) is an Archimedean \(\ell\)-unital \(\ell\)-algebra over \(K\) which is not \(\ell\)-reduced.
Clearly \([1, g, k]\) is also a basis for vector space \(T_2(K)\) over \(K\) and we have the following multiplication table.

\[
\begin{array}{c|ccc}
  & 1 & g & k \\
\hline
1 & 1 & g & k \\
g & g & g & g \\
k & k & k & k \\
\end{array}
\]

Define the positive cone
\[ P_2 = K^+ 1 + K^+ g + K^+ k, \]
Then \((T_2(K), P_2)\) is an Archimedean \(\ell\)-reduced \(\ell\)-unital \(\ell\)-algebra over \(K\). Clearly, \(P_2 \subseteq P_1 \subseteq P_0\).

We first state a lemma, which will be used later in proofs. For the rest of the paper, \(F = F(T_2(K))\).

**Lemma 2.1.** Let \(T_2(K)\) be an \(\ell\)-unital \(\ell\)-algebra.

(1) If \(T_2(K)\) is \(\ell\)-reduced, then \(T_2(K)\) is an \(\ell\)-domain, and hence \(F\) is totally ordered.

(2) If \(F\) is totally ordered and \(F^\perp\) contains nonzero positive nilpotent elements, then \(F = K1\).

**Proof.**
(1) Let \(0 \leq u, v \in T_2(K)\) with \(uv = 0\). Then \((uv)^2 = 0\), and hence \(vu = 0\) since \(T_2(K)\) is \(\ell\)-reduced. Thus
\[
(vzu)^2 = (uzv)^2 = 0,
\]
for any \(z \in T_2(K)^+\). Thus
\[
vzu = uzv = 0,
\]
for any \(z \in T_2(K)^+\) since \(T_2(K)\) is \(\ell\)-reduced. Therefore
\[
vT_2(K)u = uT_2(K)v = 0,
\]
since each element in \(T_2(K)\) is a difference of two positive elements. Then, by a direct calculation, we have that \(u\) is nilpotent or \(v\) is nilpotent, and hence \(u = 0\) or \(v = 0\). It is well-known that an \(f\)-ring without nonzero zero divisor is totally ordered. Thus \(F\) is totally ordered.

(2) Let \(0 < a \in F^\perp\) with \(a^2 = 0\). We notice that \(T_2(K)\) cannot be an \(f\)-ring since, for example, idempotent elements in an \(f\)-ring with identity are all central elements [3, 2.1], but \(T_2(K)\) contains idempotent elements which are not central. We claim that \(F\) cannot be two-dimensional. In fact, if \(F\) is two-dimensional, then \(T_2(K) = F \oplus F^\perp\) as a vector lattice and \(F^\perp = Ka\). Let
\[ d = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = b + c, \]

where \( b \in F \) and \( c \in F^\perp = Ka \). Then \( b = d - c \) is an idempotent element. By [3, 2.1], the only idempotent elements in a totally ordered ring with identity are 1 and 0, so \( b = 1 \) or \( b = 0 \), a contradiction. Thus \( F \) cannot be two-dimensional and hence \( F = K1 \). □

We also notice that if \( T_2(K) \) is an \( \ell \)-unital \( \ell \)-algebra over \( K \), then \( F \) contains no nilpotent element [1, Corollary 3, p. 63], and hence \( F \) is a finite direct sum of totally ordered algebras over \( K \) [1, Theorem 17]. Let \( 0 \leq e, f \leq 1 \) implies that \( e \) and \( f \) are \( f \)-elements. Thus \( \ell \)-ideal of \( T_2(K) \).

**Theorem 2.2.** Let \( T_2(K) \) be an Archimedean \( \ell \)-unital \( \ell \)-algebra over \( K \).

1. If \( T_2(K) \) is not \( \ell \)-reduced, then
   a. \( T_2(K) \) is isomorphic to \( (T_2(K), P_0) \) provided 1 is not a basic element;
   b. \( T_2(K) \) is isomorphic or anti-isomorphic to \( (T_2(K), P_1) \) provided 1 is a basic element.
2. If \( T_2(K) \) is \( \ell \)-reduced, then \( T_2(K) \) is isomorphic or anti-isomorphic to \( (T_2(K), P_2) \).

**Proof.** (1) Let
\[
I = \left\{ \begin{pmatrix} 0 & x \\
0 & 0 \end{pmatrix} : x \in K \right\}.
\]

Since \( T_2(K) \) is not \( \ell \)-reduced, there exists \( a > 0 \) which is nilpotent, so \( a \in I \), and hence \( I = Ka \) since \( I \) is a one-dimensional subspace of \( T_2(K) \) over \( K \). Clearly \( I = Ka \) is an \( \ell \)-ideal of \( T_2(K) \).

Since \( F \) is an Archimedean \( f \)-algebra over \( K \) with identity element, \( F \) contains no nilpotent element [1, Corollary 3, p. 63], and hence \( F \) is a finite direct sum of totally ordered algebras over \( K \) [1, Theorem 17]. Let \( 0 \leq b \in F \). Then \( a \wedge b = 0 \). Thus we have the direct sum \( F \oplus Ka \) as vector lattices. We consider the following two cases.

(a) Suppose 1 is not basic in \( T_2(K) \). Since 1 is not basic in \( T_2(K) \), \( F \) is a finite direct sum of at least two totally ordered algebras, and since \( T_2(K) \) is three-dimensional, \( F \) is a direct sum of exactly two totally ordered algebras. Thus \( F \) is two-dimensional and \( T_2(K) = F \oplus Ka \) as a vector lattice.

Now let \( 1 = e + f \), where \( e > 0, f > 0 \), and \( e \wedge f = 0 \). Then we have \( e^2 = e, f^2 = f \), and \( ef = fe = 0 \) since \( 0 \leq e, f \leq 1 \) implies that \( e \) and \( f \) are \( f \)-elements. Thus \( T_2(K) = Kf \oplus Ke \oplus Ka \) as a vector lattice. Without loss of generality, we may assume that
\[ f = \begin{pmatrix} 1 & u \\ 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & v \\ 0 & 1 \end{pmatrix} \]

with \( v = -u \in K \). Also, suppose that

\[ a = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \quad \text{where } 0 \neq r \in K, \]

and define

\[ q = \begin{pmatrix} 1 & u \\ 0 & r \end{pmatrix}. \]

Then the inner automorphism \( i_q \) is an isomorphism from \( \ell \)-algebra \((T_2(K), P_0)\) to \( \ell \)-algebra \( T_2(K) = Kf \oplus Ke \oplus Ka \).

(b) Suppose 1 is basic in \( T_2(K) \). Since 1 is basic, \( F \) is totally ordered, and hence \( T_2(K) = F \oplus F^\perp \), where

\[ F^\perp = 1^\perp = \{ w \in T_2(K) : |w| \wedge 1 = 0 \}. \]

Thus \( Ka \subseteq F^\perp \). By Lemma 2.1, \( F = K1 \), and hence \( F^\perp \) is two-dimensional. Let \( 0 < a_1 \in F^\perp \setminus Ka \). Since \( T_2(K) \) is Archimedean over \( K \), there exists \( 0 < a_2 \in Ka \) such that \( a_2 \not\approx a_1 \). Let \( a_1 \wedge a_2 = a_3 \). Then \( (a_1 - a_3) \wedge (a_2 - a_3) = 0 \), and \( 0 < (a_1 - a_3) \in F^\perp \setminus Ka \), \( 0 < (a_2 - a_3) \in Ka \). Let \( e_1 = a_1 - a_3 \) and \( f_1 = a_2 - a_3 \). Then \( 0 < e_1 \in F^\perp \setminus Ka \), \( 0 < f_1 \in Ka \), and \( e_1 \wedge f_1 = 0 \), so

\[ F^\perp = Ke_1 \oplus Kf_1, \]

as a vector lattice, and hence

\[ T_2(K) = K1 \oplus Ke_1 \oplus Kf_1, \]

as a vector lattice. Now we determine \( e_1 \). Let

\[ e_1 = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \quad \text{where } x, y, z \in K. \]

Since \( e_1 f_1 = x f_1 \) and \( f_1 e_1 = z f_1 \), \( x \geq 0 \) and \( z \geq 0 \). Since \( \{1, e_1, f_1\} \) is linearly independent, \( x \neq z \). Otherwise \( e_1 \) is a linear combination of 1 and \( f_1 \). Let

\[ e_1^2 = \begin{pmatrix} x^2 \\ 0 \\ z^2 \end{pmatrix}, \]

for some \( \alpha, \beta, \gamma \in K^+ \). Then we have

\[ x^2 - \beta x - \alpha = 0 \quad \text{and} \quad z^2 - \beta z - \alpha = 0, \]

and hence

\[ x + z = \beta \quad \text{and} \quad xz = -\alpha. \]

If \( x \) and \( z \) are both not zero, then one of them must be negative since \( xz = -\alpha \leq 0 \), which is a contradiction. Thus we have \( x = 0 \) or \( z = 0 \).
Suppose \( x = 0 \). Then \( z > 0 \) since \( e_1 \) is not nilpotent, \( \alpha = 0 \), and

\[
e_1^2 = \begin{pmatrix} 0 & zy \\ 0 & z^2 \end{pmatrix} = ze_1.
\]

Let

\[
i = z^{-1}e_1 = \begin{pmatrix} 0 & z^{-1}y \\ 0 & 1 \end{pmatrix}.
\]

Then \( T_2(K) = K1 \oplus Ki \oplus Kf_1 \) as a vector lattice. Now let

\[
f_1 = \begin{pmatrix} 0 & r_1 \\ 0 & 0 \end{pmatrix},
\]

where \( 0 \neq r_1 \in K \), and define

\[
q = \begin{pmatrix} 1 & -z^{-1}y \\ 0 & r_1 \end{pmatrix}.
\]

Then \( i q (g) = i, i q (h) = f_1 \). Thus \( i q \) is an isomorphism from \( \ell \)-algebra \((T_2(K), P_1)\) to \( \ell \)-algebra \( T_2(K) = K1 \oplus Ki \oplus Kf_1 \).

Suppose \( z = 0 \). Then \( x > 0, \alpha = 0, and

\[
e_1^2 = \begin{pmatrix} x^2 & xy \\ 0 & 0 \end{pmatrix} = xe_1.
\]

Let

\[
j = x^{-1}e_1 = \begin{pmatrix} 1 & x^{-1}y \\ 0 & 0 \end{pmatrix}.
\]

Then \( T_2(K) = K1 \oplus Kj \oplus Kf_1 \) as a vector lattice. Define

\[
q = \begin{pmatrix} 1 & -x^{-1}y \\ 0 & r_1 \end{pmatrix}.
\]

Then \( \phi i q (g) = j, \phi i q (h) = f_1 \). Thus \( \phi i q \) an anti-isomorphism from \( \ell \)-algebra \((T_2(K), P_1)\) to \( \ell \)-algebra \( T_2(K) = K1 \oplus Kj \oplus f_1 \).

(2) Since \( T_2(K) \) is \( \ell \)-reduced, by Lemma 2.1, \( F \) is totally ordered, and hence 1 is basic. Since \( F \) is totally ordered, \( T_2(K) = F \oplus F^\perp \) as a vector lattice.

Let

\[
a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^+ \quad \text{and} \quad b_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^-.
\]

Since \( T_2(K) \) is \( \ell \)-reduced, \( a_1 > 0 \) and \( b_1 > 0 \). Since \( a_1 \land b_1 = 0 \) \((a_1 \land 1) \land (b_1 \land 1) = 0 \) and hence \( a_1 \land 1 = 0 \) or \( b_1 \land 1 = 0 \) since 1 is basic. In the following we suppose \( a_1 \land 1 = 0 \). A similar argument may be used to prove the case that \( b_1 \land 1 = 0 \). Let

\[
a_1 = \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}.
\]
Then $a_1^2 = (x_1 + x_3)a_1 + (-x_1 x_3)1 \geq 0$ implies $x_1 + x_3 \geq 0$ and $-x_1 x_3 \geq 0$ since $a_1 \wedge 1 = 0$.

First we claim that $b_1$ is not an $f$-element. Suppose $b_1$ is an $f$-element. From

$$(a_1 - b_1)^2 = a_1^2 - a_1b_1 - b_1a_1 + b_1^2 = 0,$$

we have

$$(x_1 + x_3)a_1 + (-x_1 x_3)1 - a_1b_1 - b_1a_1 + b_1^2 = 0,$$

and hence

$$(−x_1 x_3)1 + b_1^2 = 0$$

since $−x_1 x_3)1 + b_1^2 \in F$ and $(x_1 + x_3)a_1 - a_1b_1 - b_1a_1 \in F^\perp$. It follows from

$$(−x_1 x_3)1 + b_1^2 = 0$$

that $b_1^2 = 0$, and hence $b_1 = 0$, which is a contradiction. Thus $b_1$ is not an $f$-element.

Since $T_2(K)$ is Archimedean over $K$, there exists $0 < \alpha \in K$ such that $\alpha 1 \notin b_1$. Let $b_1 \wedge \alpha 1 = c$. Then $c < \alpha 1$, and $c < b_1$ since $b_1$ is not an $f$-element. Thus

$$(b_1 - c) \wedge (\alpha 1 - c) = 0,$$

with $b_1 - c > 0$ and $\alpha 1 - c > 0$.

so $b_1 - c \in F^\perp$ since $0 < (\alpha 1 - c) \in F$. Let $d = b_1 - c$. Then $0 < a_1, d \in F^\perp$ and $a_1 \wedge d = 0$ since $d \leq b_1$. Thus $T_2(K) = K \oplus Ka_1 \oplus Kd$ as a vector lattice.

Now we determine $a_1$ and $d$. Recall that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = a_1 - b_1 = a_1 - d - c.$$

Since $a_1^2 = (x_1 + x_3)a_1 + (-x_1 x_3)1$, we have $−x_1 x_3 \geq 0$, and hence $x_1 \leq 0$ or $x_3 \leq 0$. Suppose $x_1 \leq 0$. From

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leq a_1,$$

we have

$$a_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leq a_1^2,$$

and hence

$$x_1a_1 - x_3d - x_1c \leq (x_1 + x_3)a_1 + (-x_1 x_3)1,$$

so

$$-x_1 d - x_1 c \leq x_3 a_1 + (-x_1 x_3)1.$$

Since $\{1, a_1, d\}$ is a disjoint set, we have $-x_1 d = 0$ and hence $x_1 = 0$. By a similar argument, if $x_3 \leq 0$ then $x_3 = 0$. Thus we have $x_1 = 0$ or $x_3 = 0$ but not both of them are zero since $a_1$ is not nilpotent.

Let

$$d = \begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix},$$

where $y_1, y_2, y_3 \in K$. 
Then \( d^2 = (y_1 + y_3)d + (-y_1y_3)1 \geq 0 \) implies that \( (y_1 + y_3) \geq 0 \) and \(-y_1y_3 \geq 0\), so \( y_1 \leq 0 \) or \( y_3 \leq 0 \). Suppose \( y_1 \leq 0 \). From
\[
-\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leq b_1 = d + c,
\]
we have
\[
-d \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leq d^2 + dc,
\]
and hence
\[
-y_1(a_1 - d - c) \leq (y_1 + y_3)d + (-y_1y_3)1 + dc,
\]
so
\[
-y_1a_1 \leq y_3d + (-y_1y_3)1 + dc - y_1c.
\]
Since \( c \) is an \( f \)-element, \( a_1 \) is disjoint with \( d, 1, dc \), and \( c \), so we have \(-y_1a_1 = 0\), and hence \( y_1 = 0 \). Similarly, if \( y_3 \leq 0 \), then \( y_3 = 0 \). Therefore, we have \( y_1 = 0 \) or \( y_3 = 0 \) but not both of them are zero.

If \( x_1 = 0 \) and \( y_3 = 0 \), then \( a_1d = 0 \), which is a contradiction by Lemma 2.1. Similarly, \( x_3 \) and \( y_1 \) cannot be both zero. Thus we have the following two cases.

(i) \( x_1 = 0 \) and \( y_1 = 0 \). Let
\[
 u = x_3^{-1}a_1 = \begin{pmatrix} 0 & x_3^{-1}x_2 \\ 0 & 1 \end{pmatrix}, \quad v = y_3^{-1}d = \begin{pmatrix} 0 & y_3^{-1}y_2 \\ 0 & 1 \end{pmatrix}.
\]
Then \( T_2(K) = K1 \oplus Ku \oplus Kv \) as a vector lattice. Define
\[
 q = \begin{pmatrix} 1 & -x_3^{-1}x_2 \\ 0 & y_3^{-1}y_2 - x_3^{-1}x_2 \end{pmatrix}.
\]
Then \( \varphi \iota_q (g) = u, \varphi \iota_q (k) = v \). Thus \( \iota_q \) is an isomorphism from \( \ell \)-algebra \( (T_2(K), P_2) \) to \( \ell \)-algebra \( T_2(K) = K1 \oplus Ku \oplus Kv \).

(ii) \( x_3 = 0 \) and \( y_3 = 0 \). Now let
\[
 u = x_1^{-1}a_1 = \begin{pmatrix} 1 & x_1^{-1}x_2 \\ 0 & 0 \end{pmatrix}, \quad v = y_1^{-1}d = \begin{pmatrix} 1 & y_1^{-1}y_2 \\ 0 & 0 \end{pmatrix}.
\]
Then we have \( T_2(K) = K1 \oplus Ku \oplus Kv \) as a vector lattice. Define
\[
 q = \begin{pmatrix} 1 & -x_1^{-1}x_2 \\ 0 & y_1^{-1}y_2 - x_1^{-1}x_2 \end{pmatrix}.
\]
Then \( \varphi \iota_q (g) = u, \varphi \iota_q (k) = v \). Therefore, \( \varphi \iota_q \) is an anti-isomorphism from \( \ell \)-algebra \( (T_2(K), P_2) \) to \( \ell \)-algebra \( T_2(K) = K1 \oplus Ku \oplus Kv \). □

Next we determine non-Archimedean lattice orders on \( T_2(K) \) in which 1 is positive. Let \( g \) and \( h \) be defined as before, that is,
\[
 g = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in T_2(K).
\]
Then $T_2(K)$ can be lattice-ordered as a vector lattice as follows.

$$T_2(K) = K1 \oplus (Kg \oplus Kh),$$

where $Kg \oplus Kh$ is the anti-lexicographic order, that is, $\alpha g + \beta h \geq 0$ if and only if $\alpha > 0$ or $\alpha = 0$ and $\beta \geq 0$. We denote the positive cone of this lattice order on $T_2(K)$ by $P_3$. Then

$$P_3 = \{\alpha1 + \beta g + \gamma h : \alpha \geq 0, \beta > 0, \text{ or } \alpha \geq 0, \beta = 0, \gamma \geq 0, \forall \alpha, \beta, \gamma \in K\}.$$ 

We omit the routine checking that $P_3$ is closed under the multiplication in $T_2(K)$. Thus $(T_2(K), P_3)$ becomes an $\ell$-unital $\ell$-algebra which is not Archimedean over $K$.

**Theorem 2.3.** Let $T_2(K)$ be an $\ell$-unital $\ell$-algebra which is not Archimedean over $K$. Then $T_2(K)$ is isomorphic or anti-isomorphic to $(T_2(K), P_3)$.

**Proof.** Since $T_2(K)$ is non-Archimedean over $K$, $T_2(K)$ is not $\ell$-reduced [1, Corollary 1, p. 51]. Let $a > 0$ and $a^2 = 0$.

We first claim that $a$ is not an $f$-element. Suppose that $a$ is an $f$-element. Then $a, 1 \in F$, and by [1, Lemma 5, p. 60], $aa < 1$ for each $a \in K$, so $a$ and 1 are linearly independent over $K$. Since $T_2(K)$ cannot be an $f$-ring, $F$ is two-dimensional and totally ordered. Thus $T_2(K) = F \oplus F^\perp$ as a vector lattice and $F^\perp$ is one-dimensional. Let $0 < b \in F^\perp$. Then $F^\perp = Kb$. Since $a$ is an $f$-element, $ab, ba \in F^\perp$, then we have $ab = ab$ and $ba = \beta b$, for some $a, \beta \in K^\perp$. On the other hand, $ab, ba \in Ka$ since $Ka$ is an $\ell$-ideal of $T_2(K)$. Then we have $b^2 = 0$, so $b \in Ka \subseteq F$, a contradiction. Thus, $a$ is not an $f$-element.

Since $a \wedge 1 \in Ka$ and $a$ is not an $f$-element, $a \wedge 1 = 0$, so $a \in F^\perp$. If $F$ is not totally ordered, then there are $0 < u, v \in F$ with $u \wedge v = 0$, and hence

$$T_2(K) = Ku \oplus Kv \oplus Ka,$$

as a vector lattice. Thus $T_2(K)$ is Archimedean over $K$, a contradiction. Therefore, $F$ is totally ordered. By Lemma 2.1, $F = K1$, and hence

$$T_2(K) = U \cup (F \oplus F^\perp),$$

where

$$U = \{w \in T_2(K) : |w| \geq \alpha 1, \forall \alpha \in K\}.$$

If $0 < f \in U$, then $a \leq f$ for all $a \in K$, so $aa \leq fa$ for all $a \in K$, which is a contradiction since $fa \in Ka$. Thus $U = \emptyset$, and $T_2(K) = F \oplus F^\perp$, so $F^\perp$ is two-dimensional.

Next we claim that $F^\perp$ is totally ordered. If $F^\perp$ is not totally ordered, then there exist $0 < s, t \in F^\perp$ such that $s \wedge t = 0$, so

$$F^\perp = Ks \oplus Kt, \quad \text{and} \quad T_2(K) = K1 \oplus Ks \oplus Kt,$$

as vector lattices. Thus, again, $T_2(K)$ is Archimedean over $K$, a contradiction. Therefore, $F^\perp$ is totally ordered. Let $0 < c \in F^\perp$ such that $a$ and $c$ are linearly independent over $K$. If $c \leq aa$ for some $a \in K$, then $c \in Ka$ since $Ka$ is an $\ell$-ideal, so $a$
and \( c \) are linearly dependent, a contradiction. Thus for all \( \alpha \in K \), we have \( \alpha a < c \).

Let 
\[
c = \begin{pmatrix} z_1 & z_2 \\ 0 & z_3 \end{pmatrix}, \quad \text{and} \quad a = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}.
\]

Then \( ac = z_3a \geq 0 \) and \( ca = z_1a \geq 0 \) implies that \( z_1 \geq 0 \) and \( z_3 \geq 0 \). Since
\[
c^2 = (z_1 + z_3)c + (-z_1z_3)1 \geq 0,
\]
we have \(-z_1z_3 \geq 0\), and hence \( z_1z_3 = 0 \), so \( z_1 = 0 \) or \( z_3 = 0 \).

Suppose \( z_1 = 0 \). Then \( z_3 > 0 \). Let
\[
d = z_3^{-1}c = \begin{pmatrix} 0 \\ 0 \\ z_3^{-1}z_2 \\ 1 \end{pmatrix}.
\]

Then \( T_2(K) = K1 \oplus (Kd \oplus Ka) \) as a vector lattice. Define
\[
q = \begin{pmatrix} 1 & -z_3^{-1}z_2 \\ 0 & x \end{pmatrix}.
\]

Then \( iq(g) = d, iq(h) = a \). Thus \( iq \) is an isomorphism from \( \ell\)-algebra \( (T_2(K), P_3) \) to \( \ell\)-algebra \( T_2(K) = K1 \oplus (Kd \oplus Ka) \).

Suppose \( z_3 = 0 \). Then \( z_1 > 0 \). Let
\[
e = z_1^{-1}c = \begin{pmatrix} 1 \\ 0 \\ z_1^{-1}z_2 \\ 0 \end{pmatrix}.
\]

Then \( T_2(K) = K1 \oplus (Ke \oplus Ka) \) as a vector lattice. Define
\[
q = \begin{pmatrix} 1 & -z_1^{-1}z_2 \\ 0 & x \end{pmatrix}.
\]

Then \( qi_q(g) = e, qi_q(h) = a \), and hence \( qi_q \) is an anti-isomorphism from \( \ell\)-algebra \( (T_2(K), P_3) \) to \( \ell\)-algebra \( T_2(K) = K1 \oplus (Ke \oplus Ka) \). □

By using inner automorphisms of \( T_2(K) \) and the above four lattice orders \( P_0, P_1, P_2, \) and \( P_3 \), we could construct all lattice orders on \( T_2(K) \) in which 1 is positive.

3. 1 is not positive

In this section we produce lattice orders on \( T_2(K) \) in which 1 is not positive. First we give a general method to construct new lattice orders from a given lattice order.

Let \( R \) be a unital \( \ell\)-ring with the positive cone \( R^+ \), and let \( v \in R^+ \) be a unit in \( R \), that is, \( v^{-1} \) exists in \( R \). Define the subset \( P_v \) of \( R \) as follows.
\[
P_v = vR^+ = \{ a \in R : a = vb \text{ for some } b \in R^+ \}.
\]

The following result shows that \( (R, P_v) \) is an \( \ell\)-ring with positive cone \( P_v \).
Theorem 3.1. Let \( R \) be a unital \( \ell \)-ring with the positive cone \( R^+ \), and let \( v \in R^+ \) be a unit in \( R \) and \( Pv = vR^+ \). Then \( (R, Pv) \) is an \( \ell \)-ring.

Proof. It is clear that \( Pv + Pv \subseteq Pv \) and \( Pv \cap -Pv = \{0\} \). Since \( v \in R^+ \), \( PvPv \subseteq Pv \). Thus \( (R, Pv) \) is a po-ring with the positive cone \( Pv \). To see \( (R, Pv) \) is an \( \ell \)-ring, let \( \varphi : R \to R \) be defined by for each \( a \in R \), \( \varphi(a) = va \). Since \( v^{-1} \in R \), \( \varphi \) is a group automorphism between the underlying group of \( R \). It is clear that \( a \in R^+ \) if and only if \( \varphi(a) \in Pv \), and hence \( Pv \) is a lattice order. Thus \( (R, Pv) \) is an \( \ell \)-ring with the positive cone \( Pv \). This completes the proof. \( \square \)

We make some remarks on this construction. Let \( Pv \) be defined as above.

1. Given \( a \in R \), then \( a = vb \) for some \( b \in R \). Let \( b^+ \) be the least upper bound of \( b \) and 0 with respect to \( R^+ \). Then \( a^+ = vb^+ \), where \( a^+ \) denotes the least upper bound of \( a \) and 0 with respect to \( Pv \).
2. \( Pv \subseteq R^+ \) and if \( (R, R^+) \) is Archimedean, then so is \( (R, Pv) \).
3. If \( A \) is an \( \ell \)-algebra over \( K \) and \( v \in A^+ \) is a unit, then \( (A, Pv) \) is also an \( \ell \)-algebra over \( K \).
4. If \( (R, R^+) \) is an \( f \)-ring, then \( vP = R^+ \) for any unit \( v \in R^+ \). Thus this technique would not produce new lattice order in an \( f \)-ring.

Now back to \( T_2(K) \). In section 2, we have found four different lattice orders \( P_0, P_1, P_2, \) and \( P_3 \) on \( T_2(K) \) in which 1 is positive. If we pick up nonsingular upper triangular matrices \( f_i \in P_i, \) \( i = 0, 1, 2, 3 \), such that \( f_i^{-1} \notin P_i \), then \( f_iP_i \) will be a lattice order on \( T_2(K) \) in which 1 is not positive. We close with a concrete example.

Example 3.2. Let

\[
\begin{align*}
e &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & g &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & h &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

Take \( f = 1 + g + h \). Then \( f^{-1} = 1 - \frac{1}{2}g - \frac{1}{2}h \). Clearly, \( f \in P_i \) and \( f^{-1} \notin P_i \) for \( i = 0, 1, 2, 3 \). Below we consider \( \ell \)-algebras \( T_2(K) \) with the positive cones \( fP_0, fP_1, fP_2, \) and \( fP_3 \), respectively.

(I) \((T_2(K), fP_0)\)

Let \( a_1 = fe, a_2 = fg, a_3 = fh \). Then \( T_2(K) = Ka_1 \oplus Ka_2 \oplus Ka_3 \) as a vector lattice with the following multiplication table, and \( 1 = a_1 + \frac{1}{2}a_2 - \frac{1}{2}a_3 \).

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(II) \((T_2(K), f P_1)\)

Let \(b_1 = f, b_2 = fg, b_3 = fh\). Then \(T_2(K) = Kb_1 \oplus Kb_2 \oplus Kb_3\) as a vector lattice with the following multiplication table, and \(1 = b_1 - \frac{1}{2}b_2 - \frac{1}{2}b_3\).

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</table>

(III) \((T_2(K), f P_2)\)

Let \(c_1 = f, c_2 = fg, c_3 = f(g + h)\). Then \(T_2(K) = Kc_1 \oplus Kc_2 \oplus Kc_3\) as a vector lattice with the following multiplication table, and \(1 = c_1 - \frac{1}{2}c_3\).

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(IV) \((T_2(K), f P_3)\)

Let \(d_1 = f, d_2 = fg, d_3 = fh\). Then

\[ T_2(K) = Kd_1 \oplus (Kd_2 \oplus Kd_3) \]

as a vector lattice with the same multiplication table as in (II), and \(1 = d_1 - \frac{1}{2}d_2 - \frac{1}{2}d_3\).

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References