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# Some lower bounds for the spectral radius of matrices using traces

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Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let $m$ be an integer satisfying $\operatorname{rank}(\mathbf{A}) \leqslant m \leqslant n$. If $\mathbf{A}$ is real, the best possible lower bound for its spectral radius in terms of $m$, $\operatorname{tr} \mathbf{A}$ and $\operatorname{tr} \mathbf{A}^2$ is obtained. If $\mathbf{A}$ is any complex matrix, two lower bounds for $\sum_{j=1}^{n} |\lambda_j|^2$ are compared, and furthermore a new lower bound for the spectral radius is given only in terms of $\operatorname{tr} \mathbf{A}$, $\operatorname{tr} \mathbf{A}^2$, $\|\mathbf{A}\|$, $\|\mathbf{A}^*\mathbf{A} - \mathbf{A}\mathbf{A}^*\|$, $n$ and $m$.

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## 1. Introduction

It is an interesting problem to characterize distribution of eigenvalues of a matrix in a simple way. Wolkowicz and Styan [10] extended the Samuelson inequality and thereby [11,12] initially proposed many important bounds for the matrix spectrum using traces. More results [3–8] follow them. Here

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our interest focuses on finding lower bounds for the spectral radius from some special traces which are simply available.

The following notations are used throughout the paper:

| | |
|---|---|
| $\mathbf{A}$ | an $n \times n$ matrix |
| $\mathbf{A}^*$ | the conjugate transpose of $\mathbf{A}$ |
| $\lambda_j (j = 1, 2, \ldots, n)$ | eigenvalues of $\mathbf{A}$ |
| $\rho(\mathbf{A})$ | the spectral radius of $\mathbf{A}$ |
| $a = \operatorname{tr} \mathbf{A}, b = \operatorname{tr} \mathbf{A}^2$ | respectively traces of $\mathbf{A}$ and $\mathbf{A}^2$ |
| $\|\mathbf{A}\| = \sqrt{\operatorname{tr}(\mathbf{A}^*\mathbf{A})}$ | the Euclidean (Frobenius) norm of $\mathbf{A}$ |
| $m$ | an integer satisfying $\operatorname{rank}(\mathbf{A}) \leqslant m \leqslant n$ |
| $\Re z$ | the real part of a complex number $z$ |
| $\bar{z}$ | the complex conjugate of a complex number $z$ |
| $[x], x \in \mathbb{R}$ | the largest integer not greater than $x$ |

A lower bound for the spectral radius following [12, Theorem 3.1, (3.2a)] is

$$\rho(\mathbf{A}) \geqslant \frac{|a|}{n} + \frac{1}{\sqrt{n(n-1)}} \max\left\{0, \|\mathbf{A}\|^2 - \sqrt{\frac{n^3-n}{12}}\|\mathbf{A}^*\mathbf{A} - \mathbf{A}\mathbf{A}^*\| - \frac{|a|^2}{n}\right\}^{1/2}.$$

However, there exist counterexamples, e.g.

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}.$$

The inequality above is not true for the block diagonal matrix $\mathbf{A} \oplus \cdots \oplus \mathbf{A}$ either.

If $\mathbf{A}$ is a real matrix such that $\operatorname{rank}(\mathbf{A}) \geqslant 3$, Horne [3, Theorems 1 and 3] gives a lower bound for $\rho(\mathbf{A})$ as below

$$\rho(\mathbf{A}) \geqslant L_H(a, b, m)$$
$$= \begin{cases} |a|/m + \sqrt{(m^2-m)^{-1}(b - a^2/m)}, & \text{if } b \geqslant a^2/m, \\ (a^2 - b)(m-1)^{-1}|a|^{-1}, & \text{if } -(m-3)a^2(2m)^{-1} \leqslant b \leqslant a^2/m, \\ \sqrt{3(m+3)a^2(4m^3 - 4m^2)^{-1} - 3b(m^2-m)^{-1}}, & \text{if } b < -(m-3)a^2(2m)^{-1}. \end{cases}$$

Besides, Merikoski and Virtanen [7] use $n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2$ to give the best possible lower bound for the Perron root of the nonnegative matrix $\mathbf{A}$.

The main goal of this paper is to find new lower bounds for the spectral radius $\rho(\mathbf{A})$ using efficiently computable quantities like $\operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2, \|\mathbf{A}\|$ and $\|\mathbf{A}^*\mathbf{A} - \mathbf{A}\mathbf{A}^*\|$. First, we give the best possible lower bound for $\rho(\mathbf{A})$ of real matrix $\mathbf{A}$ only using $\operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2$ and $m$. Then we compare two lower bounds for $\sum_{j=1}^n |\lambda_j|^2$, and show a new lower bound for the spectral radius of any complex matrix $\mathbf{A}$ involving $\operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2, \|\mathbf{A}\|, \|\mathbf{A}^*\mathbf{A} - \mathbf{A}\mathbf{A}^*\|, n$ and $m$. The paper concludes by leaving some relative problems in future.

## 2. The sharp lower bound for the spectral radius of a real matrix

Throughout this section suppose $\mathbf{A}$ to be a real matrix and $\operatorname{rank}(\mathbf{A}) \geqslant 3$. Then $a, b \in \mathbb{R}$. For $m \geqslant 3$ we define a set

$$S_m(a,b) = \Bigg\{ (z_j) \in \mathbb{C}^m : \sum_j z_j = a; \sum_j z_j^2 = b;$$
$$\exists k \in \{0, 1, \ldots, [m/2]\}, z_{2j-1} = \overline{z_{2j}}, 1 \leqslant j \leqslant k; z_l \in \mathbb{R}, 2k+1 \leqslant l \leqslant m \Bigg\}.$$

Observe that $S_m(a, b)$ is a closed set in $\mathbb{C}^m$.

For any closed set $S$ in the coordinate space $\mathbb{C}^m$, we define

$$L(S) = \min_{(z_j)\in S} \max_j |z_j|.$$

Here the minimum exists. A tuple $(z_j) \in S$ is called *an optimal point of S* if $\max_j |z_j| = L(S)$.

Let $p, q$ be positive integers such that $p < q$. Through the map

$$S_p(a,b) \to S_q(a,b),$$

$$(z_j)_{j=1}^p \mapsto (z_1, z_2, \ldots, z_p, \overbrace{0, \ldots, 0}^{q-p}),$$

the set $S_p(a,b)$ is embedded to a closed subset of $S_q(a,b)$. Hence it follows that

**Proposition 2.1.** *Let $a, b \in \mathbb{R}$. If $p, q \in \mathbb{Z}$ and $3 \leqslant p < q$, then*

$$L(S_p(a,b)) \geqslant L(S_q(a,b)).$$

On one hand, for any real matrix $\mathbf{A}$ such that $\operatorname{rank}(\mathbf{A}) \leqslant m$, $\operatorname{tr}\mathbf{A} = a$ and $\operatorname{tr}\mathbf{A}^2 = b$, the tuple $(\lambda_j)_{j=1}^m$ of its eigenvalues in a proper sort (possibly excluding some zero eigenvalues) belongs to $S_m(a,b)$. Thus, we get

**Theorem 2.2.** *Let $\mathbf{A}$ be a real matrix of order n, and m an integer satisfying* $\operatorname{rank}(\mathbf{A}) \leqslant m \leqslant n$. *Then*

$$\rho(\mathbf{A}) \geqslant L(S_m(\operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2)).$$

On the other hand, for any real numbers $a, b$ and an integer $m \geqslant 3$, there exists a real matrix $\mathbf{A}$ such that $3 \leqslant \operatorname{rank}(\mathbf{A}) \leqslant m$, $\operatorname{tr}\mathbf{A} = a$, $\operatorname{tr}\mathbf{A}^2 = b$ and $\rho(\mathbf{A}) = L(S_m(a,b))$. Given an optimal point of $S_m(a,b)$ as $(x_1 + \mathbf{i}y_1, x_1 - \mathbf{i}y_1, \ldots, x_k + \mathbf{i}y_k, x_k - \mathbf{i}y_k, w_{2k+1}, \ldots, w_m)$ where $x_j, y_j \in \mathbb{R}, 1 \leqslant j \leqslant k$ and $w_l \in \mathbb{R}, 2k+1 \leqslant l \leqslant m$, one of such real matrices is constructed as

$$\mathbf{A} = \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} x_k & y_k \\ -y_k & x_k \end{pmatrix} \oplus (w_{2k+1}) \oplus \cdots \oplus (w_m).$$

Hence, due to facts above we call $L(S_m(\operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2))$ to be *the sharp lower bound for the spectral radius of real matrix $\mathbf{A}$ in terms of $m, \operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2$.*

The explicit expression of $L(S_m(a,b))$ is given below.

**Theorem 2.3.** *Let $a, b \in \mathbb{R}$ and $3 \leqslant m \in \mathbb{Z}$. Then*

$$L(S_m(a,b)) = \begin{cases} D^{-1}\left(\sqrt{bD-(m-1)a^2} + |a|(m-2k-1)\right), & \text{if } b > a^2/m; \\ m^{-1}\sqrt{2a^2 - mb}, & \text{if } b \leqslant a^2/m, m \text{ is even}; \\ (m^2-3m)^{-1}\left(\sqrt{2a^2(m-1)(m-2) - bm(m-1)(m-3)} - 2|a|\right), & \\ \quad \text{if } -2a^2(m-3)(m+1)^{-2} \leqslant b \leqslant a^2/m, m > 3, m \text{ is odd}; & \\ \sqrt{(m-1)^{-1}(2a^2(m+1)^{-1} - b)}, & \\ \quad \text{if } b < -2a^2(m-3)(m+1)^{-2}, m > 3, m \text{ is odd}; & \\ (a^2 - b)(2|a|)^{-1}, & \text{if } 0 < b \leqslant a^2/3, m = 3; \\ 2^{-1}\sqrt{a^2 + 2|b|}, & \text{if } b \leqslant 0, m = 3; \end{cases}$$

*where* $k = \left[\frac{1}{2}\left(m - \sqrt{ma^2/b}\right)\right]$ *and* $D = m - 1 + (m-2k-1)^2$.

We leave the proof of Theorem 2.3 to the appendix at the end, which itself is not closely related to the theme here.

Simple but tedious calculation shows that

**Proposition 2.4.** *For any* $a, b \in \mathbb{R}$ *and* $3 \leqslant m \in \mathbb{Z}$,

$$L(S_m(a,b)) \geqslant L_H(a,b,m).$$

Therefore, $L(S_m(a,b))$ is a better lower bound for $\rho(\mathbf{A})$ than $L_H(a,b,m)$.

**Example 2.5.**

$$\mathbf{A} = \begin{pmatrix} -0.187772 & -0.362303 & -0.706214 & 0.107561 \\ -0.201396 & -0.306535 & -0.890974 & 0.055703 \\ -0.037347 & -0.884669 & 0.310718 & -0.081558 \\ 0.093463 & -0.334546 & -0.296899 & 0.443462 \end{pmatrix}.$$

Here $a = \operatorname{tr}\mathbf{A} = 0.25987, b = \operatorname{tr}\mathbf{A}^2 = 2.2288, n = 4$. The lower bound by [3] is $L_H(a,b,n) = 0.49430$, the sharp lower bound is $L(S_n(a,b)) = 0.80290$, and the true value of $\rho(\mathbf{A})$is 1.0781.

Because of Proposition 2.1, precise estimation of the spectral radius is possible for singular real matrices if the integer $m$ approaches closer to the rank. Below is an example.

**Example 2.6.**

$$\mathbf{A} = \begin{pmatrix} -2.95921 & 1.81492 & -0.22063 & 4.22845 & 0.29100 \\ 4.04665 & -2.89876 & 2.30456 & 4.05921 & 1.27605 \\ 2.36225 & 3.18211 & -2.03943 & -0.67160 & 4.10834 \\ -3.25901 & 4.90758 & -0.95127 & 2.33214 & 2.63481 \\ -1.95169 & 5.68627 & -1.51541 & 0.20106 & 3.83484 \end{pmatrix}.$$

Here $a = \operatorname{tr}\mathbf{A} = -1.7304, b = \operatorname{tr}\mathbf{A}^2 = 85.319, n = 5, m = 4$. Using $n, a, b$ we get $L_H(a,b,n) = 2.4042$ and $L(S_n(a,b)) = 4.5367$; the lower bounds in terms of $m, a, b$ are $L_H(a,b,m) = 3.0873$ and $L(S_m(a,b)) = 4.9898$. The true value of $\rho(\mathbf{A})$ is 7.0627.

**Remark.** When $\mathbf{A}$ is nonnegative, the sharp lower bound in terms of $n, a, b$, i.e. $L(S_n(a,b))$, is exactly *the optimal lower bound* given by [7, Theorem 10].

**Example 2.7.**

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 7 & 4 & 1 & 4 \\ 5 & 2 & 1 & 4 \\ 5 & 3 & 0 & 3 \end{pmatrix}.$$

Here $a = \operatorname{tr}\mathbf{A} = 8, b = \operatorname{tr}\mathbf{A}^2 = 102, n = 4$. The minimum column sum is 3 and the minimum row sum is 4. The lower bound by [7, Corollary 5, 3, Theorem 1] is 4.6771. The sharp lower bound $L(S_n(a,b)) = 5.6742$, equals to the lower bound by [7, Theorem 10]. The spectral radius is $\rho(\mathbf{A}) = 9.8888$.

In Example 2.7, the sharp lower bound is better than the minimum column (row) sum. However, in some other examples, the minimum column (row) sum is a sharper lower bound for Perron root.

**Example 2.8.**

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 6 & 7 \\ 3 & 1 & 6 & 7 \\ 6 & 2 & 6 & 4 \\ 4 & 3 & 6 & 7 \end{pmatrix}.$$

Here $a = \operatorname{tr}\mathbf{A} = 15, b = \operatorname{tr}\mathbf{A}^2 = 329, n = 4$. The lower bound by [7, Corollary 5, 3, Theorem 1] is 8.5175. The sharp lower bound is $L(S_n(a,b)) = 10.079$, the same as the lower bound by [7, Theorem 10]. The minimum column sum is 6 but the minimum row sum is 14. The spectral radius is $\rho(\mathbf{A}) = 17.690$.

**Remark.** In fact, Theorem 2.2 holds also for matrices whose nonreal eigenvalues occur in conjugate pairs, and real matrices are the most natural family of such matrices.

## 3. Lower bounds for $\sum_{j=1}^{n} |\lambda_j|^2$

It is also interesting to give lower bounds for $\sum_{j=1}^{n} |\lambda_j|^2$, which is used further to locate eigenvalues [11,12].

Considering $\operatorname{rank}(\mathbf{A}) \leqslant m \leqslant n$, without loss of generality we assume $\lambda_j = 0, j = m+1, \ldots, n$. Since

$$\left|b - a^2/m\right| = \left|\sum_{j=1}^{m} (\lambda_j - a/m)^2\right| \leqslant \sum_{j=1}^{m} |\lambda_j - a/m|^2 = \sum_{j=1}^{n} |\lambda_j|^2 - |a|^2/m,$$

we get a lower bound for $\sum_{j=1}^{n} |\lambda_j|^2$ as below

$$\sum_{j=1}^{n} |\lambda_j|^2 \geqslant \tau_1(\mathbf{A}) = \left|b - \frac{a^2}{m}\right| + \frac{|a|^2}{m}. \tag{1}$$

Another lower bound for $\sum_{j=1}^{n} |\lambda_j|^2$ by [2, Theorem 1] is

$$\sum_{j=1}^{n} |\lambda_j|^2 \geqslant \tau_2(\mathbf{A}) = \|\mathbf{A}\|^2 - \sqrt{\frac{n^3 - n}{12}}\|\mathbf{A}^*\mathbf{A} - \mathbf{A}\mathbf{A}^*\|. \tag{2}$$

Here neither lower bound for $\sum_{j=1}^{n} |\lambda_j|^2$ has comparative dominance, i.e. $\tau_1(\mathbf{A}) \geqslant \tau_2(\mathbf{A})$ (or $\tau_1(\mathbf{A}) \leqslant \tau_2(\mathbf{A})$) does not hold for all matrices. Below are some numerical examples.

**Example 3.1.**

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Here $n = m = 3$ and $\tau_1(\mathbf{A}) = 5 > \tau_2(\mathbf{A}) = 7 - 2\sqrt{11}$.

**Example 3.2.**

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Here $n = m = 4$ and $\tau_1(\mathbf{A}) = 0 < \tau_2(\mathbf{A}) = 8$.

## 4. A new lower bound for the spectral radius

In this section let $\mathbf{A}$ be any complex matrix and $\operatorname{rank}(\mathbf{A}) \geqslant 2$. To obtain a new lower bound of $\rho(\mathbf{A})$, we need the Brunk inequality.

**Lemma 4.1** [1, Brunk inequality]. *For real numbers $x_1, x_2, \ldots, x_m$, it holds that*

$$\max_j x_j \geqslant m^{-1}\sum_{j=1}^{m} x_j + \frac{1}{\sqrt{m(m-1)}}\left(\sum_{j=1}^{m} x_j^2 - \frac{1}{m}\left(\sum_{j=1}^{m} x_j\right)^2\right)^{1/2}.$$

From now on we denote

$$\eta(\mathbf{A}) = \max\left\{\left|b - \frac{a^2}{m}\right|, \|\mathbf{A}\|^2 - \sqrt{\frac{n^3-n}{12}}\|\mathbf{A}^*\mathbf{A} - \mathbf{A}\mathbf{A}^*\| - |a|^2/m\right\}.$$

**Theorem 4.2.** *If $b = a^2/m$, then*

$$\rho(\mathbf{A}) \geqslant |a|/m + \sqrt{\eta(\mathbf{A})(2m^2-2m)^{-1}}. \tag{3}$$

*If $a = 0$, then*

$$\rho(\mathbf{A}) \geqslant \sqrt{(\eta(\mathbf{A}) + |b|)(2m^2-2m)^{-1}}. \tag{4}$$

*Otherwise,*

$$\rho(\mathbf{A}) \geqslant \max_{-1\leqslant x\leqslant 1}\left\{p_1\sqrt{1+x} + p_2\sqrt{1-x} + p_3\sqrt{p_4+x}\right\}, \tag{5}$$

*where* $\gamma = \arg a - 2^{-1}\arg(b - a^2/m)$, $p_1 = |a\cos\gamma|\left(\sqrt{2}m\right)^{-1}$, $p_2 = |a\sin\gamma|\left(\sqrt{2}m\right)^{-1}$, $p_3 = \sqrt{|b-a^2/m|(2m^2-2m)^{-1}}$, *and* $p_4 = \eta(\mathbf{A})|b-a^2/m|^{-1}$.

**Proof.** Assume $\lambda_j = 0, m < j \leqslant n$. Applying inequalities (1) and (2), we have

$$\begin{aligned}
&2\left(\sum_{j=1}^m(\Re(\exp(\mathbf{i}\theta)\lambda_j))^2 - \frac{1}{m}\left(\sum_{j=1}^m\Re(\exp(\mathbf{i}\theta)\lambda_j)\right)^2\right)\\
&= \sum_{j=1}^m|\lambda_j|^2 + \Re\left(\exp(2\mathbf{i}\theta)\sum_{j=1}^m\lambda_j^2\right) - \frac{1}{m}\left|\sum_{j=1}^m\lambda_j\right|^2 - \frac{1}{m}\Re\left(\exp(2\mathbf{i}\theta)\left(\sum_{j=1}^m\lambda_j\right)^2\right)\\
&\geqslant \eta(\mathbf{A}) + \Re(\exp(2\mathbf{i}\theta)(b - a^2/m)).
\end{aligned}$$

Define the function

$$g_{\mathbf{A}}(\theta) = \frac{\Re(a\exp(\mathbf{i}\theta))}{m} + \frac{1}{\sqrt{2m(m-1)}}\left(\eta(\mathbf{A}) + \Re\left(\exp(2\mathbf{i}\theta)\left(b - \frac{a^2}{m}\right)\right)\right)^{1/2}.$$

We apply Brunk inequality to $\Re(\exp(\mathbf{i}\theta)\lambda_j)$ ($j = 1, \ldots, m$), and obtain

$$\max_j \Re(\exp(\mathbf{i}\theta)\lambda_j) \geqslant g_{\mathbf{A}}(\theta).$$

Since $\rho(\mathbf{A}) = \rho(\exp(\mathbf{i}\theta)\mathbf{A}) \geqslant \max_j \Re(\exp(\mathbf{i}\theta)\lambda_j)$ for any $\theta$, we have

$$\rho(\mathbf{A}) \geqslant \max_\theta g_{\mathbf{A}}(\theta). \tag{6}$$

Here the maximum exists because the function $g_{\mathbf{A}}(\theta)$ is periodic and continuous.

If $b = a^2/m$ or $a = 0$, the inequalities (3) and (4) follow from (6) obviously.

Suppose $b \neq a^2/m$ and $a \neq 0$. Due to the fact that trigonometric functions are periodic and symmetric, the inequality (6) yields

$$\begin{aligned}
\rho(\mathbf{A}) &\geqslant \max_\theta g_{\mathbf{A}}\left(\theta - 2^{-1}\arg(b - a^2/m)\right)\\
&= \max_\theta\Bigg\{\cos\theta\cos\gamma|a|/m - \sin\theta\sin\gamma|a|/m\\
&\quad + (2m^2-2m)^{-1/2}\sqrt{\eta(\mathbf{A}) + |b - a^2/m|\cos 2\theta}\Bigg\}
\end{aligned}$$

$$= \max_{0 \leqslant \theta \leqslant \pi/2} \left\{ \sqrt{2} p_1 \cos\theta + \sqrt{2} p_2 \sin\theta + p_3 \sqrt{p_4 + \cos 2\theta} \right\}.$$

Let $x = \cos 2\theta$. Then we get the inequality (5). $\square$

**Corollary 4.3.** *Let $a, b \in \mathbb{R}$. If $b - a^2/m \geqslant -|a|\sqrt{(2m)^{-1}(m-1)(\eta(\mathbf{A}) + b - a^2/m)}$, then*

$$\rho(\mathbf{A}) \geqslant |a|/m + \sqrt{(2m^2 - 2m)^{-1}\left(\eta(\mathbf{A}) + b - a^2/m\right)}.$$

*Otherwise,*

$$\rho(\mathbf{A}) \geqslant \sqrt{(2m-2)^{-1}(a^2 - mb)^{-1}(a^2 - b)(\eta(\mathbf{A}) - (b - a^2/m))}.$$

**Proof.** For convenience we use notations $\gamma, p_1, p_2, p_3, p_4$ as in Theorem 4.2. It only remains to find the maximum of the function

$$h(x) = p_1\sqrt{1+x} + p_2\sqrt{1-x} + p_3\sqrt{p_4 + x}, -1 \leqslant x \leqslant 1.$$

If $b > a^2/m$, then $\sin\gamma = 0$. Thus $h(x)$ reaches its maximum $\sqrt{2}p_1 + p_3\sqrt{p_4 + 1}$ at $x = 1$.

If $b < a^2/m$, then $\cos\gamma = 0$. Denote $x_0 = (p_3^2 - p_2^2 p_4)(p_2^2 + p_3^2)^{-1}$. If $x_0 \geqslant -1$, then $h(x)$ reaches its maximum $\sqrt{(p_4+1)(p_2^2 + p_3^2)}$ at $x = x_0$. If $x_0 \leqslant -1$, then $h(x)$ has its maximum $\sqrt{2}p_2 + p_3\sqrt{p_4 - 1}$ at $x = -1$. $\square$

Particularly, Corollary 4.3 is used to estimate spectral radii of real matrices. As more information than $\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{A}^2$ is applied, sometimes the lower bound by Corollary 4.3 is better than $L(S_m(\operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2))$ and $L_H(a, b, m)$. Below we show a numerical example.

**Example 4.4.**

$$\mathbf{A} = \begin{pmatrix} 0.626243 & -0.539359 & -0.016912 & 0.590066 \\ 0.475256 & 0.244961 & -0.718191 & -0.364156 \\ 0.463113 & 0.697466 & 0.432590 & 0.296332 \\ 0.446128 & -0.311304 & 0.522169 & -0.727854 \end{pmatrix}.$$

Here $a = \operatorname{tr}\mathbf{A} = 0.57594$, $b = \operatorname{tr}\mathbf{A}^2 = 0.70162$, and $n = 4$. The lower bound by [3] is $L_H(a, b, n) = 0.37105$, the sharp lower bound is $L(S_n(a, b)) = 0.48045$, and the lower bound by Corollary 4.3 is 0.58063. Actually, $\rho(\mathbf{A}) = 1.0500$.

Anyhow, below is an example where the lower bound by Corollary 4.3 is less than $L_H(a, b, n)$.

**Example 4.5.**

$$\mathbf{A} = \begin{pmatrix} 11.95393 & -37.46259 & -2.43785 & -6.49614 \\ 9.30982 & -33.06045 & 9.07286 & -0.27081 \\ -9.16662 & -25.31161 & -17.39773 & -1.60148 \\ 38.35426 & -4.88824 & -3.00578 & -17.67963 \end{pmatrix}.$$

Here $a = \operatorname{tr}\mathbf{A} = -56.184$, $b = \operatorname{tr}\mathbf{A}^2 = 252.96$, and $n = 4$. The lower bound for $\rho(\mathbf{A})$ by Corollary 4.3 is 15.555, the lower bound by [3, Theorem 3] is 17.227, the sharp lower bound is $L(S_n(a, b)) = 18.203$, and the actual value of $\rho(\mathbf{A})$ is 28.827.

## 5. Summary and future problems

We acquire the sharp lower bound for the spectral radius of real matrix $\mathbf{A}$ in terms of $m$, $\operatorname{tr}\mathbf{A}$ and $\operatorname{tr}\mathbf{A}^2$. After discussing two lower bounds for $\sum_{j=1}^{n}|\lambda_j|^2$, we give a new lower bound for the spectral radius of complex matrix $\mathbf{A}$ using $\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{A}^2$, $\|\mathbf{A}\|$, $\|\mathbf{A}^*\mathbf{A} - \mathbf{A}\mathbf{A}^*\|$, $n$ and $m$. It is natural to leave questions as below:

1. What is the best possible lower bound for the spectral radius of a complex matrix $\mathbf{A}$ using $m$, tr $\mathbf{A}$ and tr $\mathbf{A}^2$?
2. What is the best possible lower bound for $\sum_{j=1}^{n}|\lambda_j|^2$ using tr $\mathbf{A}$, tr $\mathbf{A}^2$, $\|\mathbf{A}\|$, $\|\mathbf{A}^*\mathbf{A}-\mathbf{A}\mathbf{A}^*\|$, $n$ and $m$? If a lower bound better than (1) and (2) is applied to Theorem 4.2 and Corollary 4.3, the lower bounds for spectral radii will be improved.

## Appendix. Proof of Theorem 2.3

**Lemma 5.1.** *An optimal point $(z_j)_{j=1}^m$ of $S_m(a,b)$ satisfies $|z_j| = L(S_m(a,b))$ for at least $m-1$ indices $j$.*

**Proof.** Suppose there exist two indices $p,q$ such that $\max_j|z_j| > |z_p| \geqslant |z_q|$ for a tuple $(z_j)_{j=1}^m \in S_m(a,b)$. We can choose $p,q$ such that either $z_p=\overline{z_q}$ or $z_p,z_q\in\mathbb{R}$. Let $\widetilde{z_p},\widetilde{z_q}$ be roots of the polynomial

$$z^2 - z\left(a-\delta\sum_{j\neq p,q} z_j\right) + \frac{1}{2}\left(\left(a-\delta\sum_{j\neq p,q} z_j\right)^2 - \left(b-\delta^2\sum_{j\neq p,q} z_j^2\right)\right) = 0,$$

where $\delta\in\mathbb{R}$. When $\delta=1$, the roots are $z_p,z_q$.

Due to continuous dependence of the roots of a polynomial on the coefficients of the polynomial [9], there exists a positive number $\delta<1$ such that $|\widetilde{z_p}|<\max_j|z_j|$ and $|\widetilde{z_q}|<\max_j|z_j|$. Hence,

$$\max_j |z_j| > \max_{j\neq p,q}\{|\widetilde{z_p}|,|\widetilde{z_q}|,\delta z_j\}.$$

Notice that the numbers $\delta z_j,\widetilde{z_p},\widetilde{z_q}$ in a proper sort constitute a tuple in $S_m(a,b)$. Thus, $\max_j|z_j| > L(S_m(a,b))$ and then $(z_j)_{j=1}^m$ is not an optimal point. □

**Lemma 5.2.** *If $b>a^2/m$, then an optimal point $(z_j)_{j=1}^m$ of $S_m(a,b)$ satisfies $(z_j)_{j=1}^m\in\mathbb{R}^m$.*

**Proof.** Assume $(z_j)_{j=1}^m\in S_m(a,b)$ and $(z_j)_{j=1}^m\notin\mathbb{R}^m$. Let

$$x_j = \frac{a}{m} + \left(\Re z_j - \frac{a}{m}\right)\sqrt{\frac{b-a^2/m}{\sum_{j=1}^m(\Re z_j)^2 - a^2/m}}.$$

On one hand, $\sum_{j=1}^m x_j = a$ and $\sum_{j=1}^m x_j^2 = b$. On the other hand, we have $\max_j|z_j|\geqslant\max_j|\Re z_j| > \max_j|x_j|$. Therefore, such a tuple $(z_j)_{j=1}^m\notin\mathbb{R}^m$ is not an optimal point. □

**Proposition 5.3.** *Suppose $a,b\in\mathbb{R}$ and $b>a^2/m$. Let $k=\left[\frac{1}{2}\left(m-\sqrt{ma^2/b}\right)\right]$, and $D=m-1+(m-2k-1)^2$. Then*

$$L(S_m(a,b)) = D^{-1}\left(\sqrt{bD-(m-1)a^2}+|a|(m-2k-1)\right).$$

**Proof.** By Lemmas 5.1 and 5.2, computing $L(S_m(a,b))$ under the condition $b>a^2/m$ reduces to the following optimization problem:

$$L(S_m(a,b)) = \min \quad r_l$$
$$\text{subject to} \begin{cases} y+(m-2l-1)r_l = a,\\ y^2+(m-1)r_l^2 = b,\\ r_l, y\in\mathbb{R},\\ -r_l\leqslant y\leqslant r_l,\\ l\in\{0,1,\ldots,m-1\}.\end{cases}$$

Here we omit the elementary but tedious procedure to solve this optimization problem. □

**Proposition 5.4.** *Let $m$ be an even positive integer. Suppose $a, b \in \mathbb{R}$ and $b \leqslant a^2/m$. Then*

$$L(S_m(a,b)) = m^{-1}\sqrt{2a^2 - mb}$$

*and the optimal points for $S_m(a,b)$ are $(z_j)_{j=1}^m$, where $z_j = \left(a/m \pm \sqrt{b/m - a^2/m^2}\right)$ and $z_{2j-1} = \overline{z_{2j}}$.*

**Proof.** Let $z_0 = a/m$. For any $(z_j)_{j=1}^m \in S_m(a,b)$, we have

$$\max_j |z_j| \geqslant \sqrt{\frac{\sum_j |z_j|^2}{m}} = \sqrt{|z_0|^2 + \frac{\sum_j |z_j - z_0|^2}{m}} \geqslant \sqrt{|z_0|^2 + \frac{\left|\sum_j (z_j - z_0)^2\right|}{m}} = \frac{\sqrt{2a^2 - mb}}{m}. \quad (7)$$

The former inequality in (7) collapses to equality if and only if $|z_j| = |z_l| (1 \leqslant j, l \leqslant m)$, and the latter inequality in (7) collapses to equality if and only if $z_j - z_0 (1 \leqslant j \leqslant m)$ are colinear on the complex plane. Then the optimal points are characterized as above. □

$$L(S_3(a,b)) = \begin{cases} (a^2 - b)(2|a|)^{-1}, & \text{if } 0 < b \leqslant a^2/3, \\ \sqrt{a^2 + 2|b|}/2, & \text{if } b \leqslant 0. \end{cases}$$

**Proposition 5.5.** *Let $m$ be odd and $m > 3$. Then*

$$L(S_m(a,b)) = \begin{cases} (m^2 - 3m)^{-1}\left(\sqrt{2a^2(m-1)(m-2) - bm(m-1)(m-3)} - 2|a|\right), \\ \quad \text{if } -2a^2(m-3)(m+1)^{-2} \leqslant b \leqslant a^2/m; \\ \sqrt{(m-1)^{-1}(2a^2(m+1)^{-1} - b)}, \quad \text{if } b \leqslant -2a^2(m-3)(m+1)^{-2}. \end{cases}$$

**Proof.** Suppose $(z_j)_{j=1}^m$ to be an optimal point. We have $z_m \in \mathbb{R}$, as $m$ is odd. By Lemma 5.1, $(z_j)_{j=1}^{m-1}$ has to be an optimal point in $S_{m-1}(a - z_m, b - z_m^2)$, and therefore are characterized as in Proposition 5.4. Then it only remains to solve the optimization problem

$$L(S_m(a,b)) = \min \quad \sqrt{x^2 + y^2}$$
$$\text{subject to} \begin{cases} (m-1)x + z_m = a, \\ (m-1)(x^2 - y^2) + z_m^2 = b, \\ x, y, z_m \in \mathbb{R}, \\ |z_m| \leqslant \sqrt{x^2 + y^2}. \end{cases}$$

Here we omit the detail solution to this optimization problem. □


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## References


[1] H.D. Brunk, Note on two papers of K.R. Nair, J. Indian Soc. Agricultural Statist. 11 (1959) 186–189.
[2] P. Henrici, Bounds for iterates, inverses, spectral variation and fields of values of non-normal matrices, Numer. Math. 4 (1962) 24–40.
[3] Bill G. Horne, Lower bounds for the spectral radius of a matrix, Linear Algebra Appl. 263 (1997) 261–273.

[4] T.Z. Huang, L. Wang, Improving bounds for eigenvalues of complex matrices using traces, Linear Algebra Appl. 426 (2007) 841–854.
[5] J.K. Merikoski, A. Virtanen, Bounds for eigenvalues using the trace and determinant, Linear Algebra Appl. 264 (1997) 101–108.
[6] J.K. Merikoski, A. Virtanen, Best possible bounds for ordered positive numbers using their sum and product, Math. Inequal. Appl. 4 (1) (2001) 67–84.
[7] J.K. Merikoski, A. Virtanen, The best possible lower bound for the Perron root using traces, Linear Algebra Appl. 388 (2004) 301–313.
[8] O. Rojo, R. Soto, H. Rojo, Bounds for sums of eigenvalues and applications, Comput. Math. Appl. 39 (2000) 1–15.
[9] D.J. Uherka, A.M. Sergott, On the continuous dependence of the roots of a polynomial on its coefficients, Amer. Math. Monthly 84 (5) (1977) 368–370.
[10] H. Wolkowicz, G.P.H. Styan, Extensions of Samuelson's inequality, Amer. Stat. 33 (1979) 143–144.
[11] H. Wolkowicz, G.P.H. Styan, Bounds for eigenvalues using traces, Linear Algebra Appl. 29 (1980) 471–506.
[12] H. Wolkowicz, G.P.H. Styan, More bounds for eigenvalues using traces, Linear Algebra Appl. 31 (1980) 1–17.