Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 309 (2009) 3580–3587
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Circumferences and minimum degrees in 3-connected claw-free graphs[☆]

MingChu Li[a], Yongrui Cui[a], Liming Xiong[b,*], Yuan Tian[a], He Jiang[a], Xu Yuan[a]

[a] *School of Software, Dalian University of Technology, Dalian 116620, China*
[b] *Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China*

Received 29 October 2005; accepted 18 December 2007
Available online 20 February 2008

Abstract

In this paper, we prove that every 3-connected claw-free graph G on n vertices contains a cycle of length at least $\min\{n, 6\delta - 15\}$, thereby generalizing several known results.

Keywords: Claw-free graph; Circumference; 3-connectedness; Minimum degree

1. Introduction

We use [1] for terminology and notation not defined here, and consider loopless finite simple graphs only. Let G be a graph. We denote by C_n an n-cycle and denote by $O(G)$ the set of all vertices in G with odd degrees. A graph G is *eulerian* if $O(G) = \emptyset$ and G is connected. A circuit C of G is a connected eulerian subgraph. A cycle is a connected circuit with all vertices of degree 2. Let C be a circuit of a graph G. We use $\bar{E}(C)$ to denote the set of edges in G which are incident with some vertex in C. Let $\bar{e}(C) = |\bar{E}(C)|$. The *minimum degree* and the *edge independence number* of G are denoted by $\delta(G)$ and $\alpha'(G)$, respectively. An edge $e = uv$ is called a *pendant edge* if either $d_G(u) = 1$ or $d_G(v) = 1$. A subgraph H of G (denoted by $H \subseteq G$) is *dominating* if $G - V(H)$ is edgeless. For $x \in V(G)$, let $N_H(x) = \{v \in V(H) : vx \in E(G)\}$ and $d_H(x) = |N_H(x)|$. If $S \subseteq V(G)$, $G[S]$ is the subgraph induced in G by S. For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, let $N_H(A) = \cup_{v \in A} N_H(v)$, $E_G[A, B] = \{uv \in E(G) | u \in A, v \in B\}$, and $G - A = G[V(G) - A]$. When $A = \{v\}$, we use $G - v$ for $G - \{v\}$. If $H \subseteq G$, then for an edge subset $X \subseteq E(G) - E(H)$, we write $H + X$ for $G[E(H) \cup X]$. For an integer $i \geq 1$, define $D_i(G) = \{v \in V(G) | d_G(v) = i\}$.

Let $X \subseteq E(G)$. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If K is a subgraph of G, then we write G/K for $G/E(K)$. If K is a connected subgraph of G, and if v_K is the vertex in G/K onto which K is contracted, then K is called the *preimage* of v_K, and is denoted by $PI(v_K)$. A vertex v in a contraction of G is *nontrivial* if $PI(v)$ has at least one

☆ Supported by the Natural Science Foundation of China under grant No. 60673046 and 90715037 (M. Li) and 10671014 (L. Xiong), by the Natural Science Foundation Project of Chongqing, CSTC under grant No. 2007BA2024 (M. Li) and by the Excellent Young Scholars Research Fund of Beijing Institute of Technology (No. 000Y07-28) (L. Xiong).
* Corresponding author.
E-mail addresses: li_mingchu@yahoo.com (M. Li), lmxiong@eyou.com (L. Xiong).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.12.062

edge. A complete bipartite graph $K_{1,p}$ with two disjoint vertex sets V_1 and V_2 such that $|V_1| = 1$ and $|V_2| = p$ and $p \geq 1$ is called a star, and the vertex of V_1 is called the center of the star.

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. Let H be the line graph $L(G)$ of a graph G. Then the order $|V(H)|$ of H is equal to the number $m(G)$ of edges of G, and $\delta(H) = \min\{d_G(x) + d_G(y) - 2 : xy \in E(G)\}$. Let C be a circuit of G. Then the circumference of $L(G)$ is at least $\bar{e}(C)$. If $L(G)$ is k-connected, then G is *essentially k-edge-connected*, which means that the only edge-cut sets of G having less than k edges are the sets of edges incident with some vertex of G. Harary and Nash-Williams showed that there is a closed relationship between a graph and its line graph as regards hamiltonian cycles.

Theorem 1.1 (*Harary and Nash-Williams [4]*). *The line graph $L(G)$ of a graph G is hamiltonian if and only if G has a dominating eulerian subgraph.*

A graph H is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. A vertex $v \in H$ is *locally connected* if $H[N_H(v)]$ is connected. In [9], Ryjáček defined the *closure $cl(H)$* of a claw-free graph H to be one obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H, as long as this is possible.

Theorem 1.2 (*Ryjáček [9]*). *Let H be a claw-free graph and $cl(H)$ its closure. Then*

(i) *$cl(H)$ is well defined, and $\kappa(cl(H)) \geq \kappa(H)$,*
(ii) *there is triangle-free graph G such that $cl(H) = L(G)$,*
(iii) *the two graphs H and $cl(H)$ have the same circumference.*

Many works have been done to give circumferences for a claw-free graph H in terms of its minimum degree $\delta(H)$. These conditions depend on the connectivity $\kappa(H)$. For $\kappa(H) = 2$, Matthews and Sumner [8] proved that every 2-connected claw-free graph on n vertices contains a cycle of length at least $\min\{n, 2\delta+4\}$. For $\kappa(H) = 3$, the following result was proved.

Theorem 1.3 (*Li [6]*). *If H is a 3-connected claw-free graph on n vertices, then H has a cycle of length at least min $\{n, 5\delta - 5\}$.*

Favaron and Fraisse [3] proved the following result on hamiltonian cycles.

Theorem 1.4 (*Favaron and Fraisse [3]*). *If H is a 3-connected claw-free simple graph with order ν, and if $\delta(H) \geq \frac{\nu+37}{10}$, then H is hamiltonian.*

Lai, Shao and Zhan [5] improved the result above as follows.

Theorem 1.5 (*Lai, Shao and Zhan [5]*). *If H is a 3-connected claw-free graph on n vertices with $n \geq 196$, and if $\delta(H) \geq \frac{n+5}{10}$, then either H is hamiltonian, or $\delta(H) = \frac{n+5}{10}$ and $cl(H)$ is the line graph of G obtained from the Petersen graph PTS_{10} by adding $\frac{n-15}{10}$ pendant edges at each vertex of PTS_{10}.*

Let $\mathcal{J}_1 = \{H$: H is a 3-connected non-hamiltonian claw-free graph and its Ryjáček's closure $cl(H)$ is the line graph of the graph obtained from the Petersen graph PTS_{10} by adding at least one pendant edge at each vertex of PTS_{10} and by subdividing m edges of PTS_{10} for $m = 0, 1, 2, \ldots, 15\}$. Theorem 1.4 and Theorem 1.5 have been improved as follows by Li [7] in 2006.

Theorem 1.6 (*Li [7]*). *If H is a 3-connected claw-free graph on $n \geq 220$ vertices, and if $\delta(H) \geq \frac{n+23}{11}$, then either H is hamiltonian, or $H \in \mathcal{J}_1$.*

In this paper, our purpose is to make use of the proof techniques of [3,5,7] to improve Theorem 1.3. That is, we prove the following theorem.

Theorem 1.7. *Every 3-connected claw-free graph on n vertices contains a cycle of length at least min $\{n, 6\delta - 15\}$.*

Finally, we make the following conjecture. Note that the bound $9\delta - 6$ is best possible, and the example can be found in [5].

Conjecture 1.8. *Every 3-connected claw-free graph on n vertices contains a cycle of length at least min $\{n, 9\delta - 6\}$.*

2. Lemmas

In this section, we provide some lemmas needed in the proof of Theorem 1.7. We start with the following lemma.

Lemma 2.1 (*Chen, Lai, Li, Li and Mao [2]*). *Let G be a 3-edge-connected graph and let $S \subseteq V(G)$ be a vertex subset such that $|S| \leq 12$. Then either G has a circuit C such that $S \subseteq V(C)$, or G can be contracted to the Petersen graph in such a way that the preimage of each vertex of the Petersen graph PTS_{10} contains at least one vertex in S.*

Lemma 2.2. *Let S be a set of vertices of a graph G and C a maximal circuit of G containing S. Assume that some component A of $G - V(C)$ is not an isolated vertex and is related to C by at least r edges. Then*

(i) *C contains a matching T of r edges such that at most r edges of G are adjacent to two distinct edges of T.*
(ii) $\bar{e}(C) \geq r\delta(L(G))$.

Proof. We use similar techniques to Favaron and Fraisse's proof [3]. We first have the following claim.

Claim 1. *For any vertex x on C, $d_A(x) \leq 1$. At least r vertices on C are adjacent to some vertex of A.*

Proof. If some vertex x of C has two distinct neighbors y_1 and y_2 in A, then there is a path P in A joining y_1 and y_2, and so $C + xy_1Py_2x$ is a circuit of G longer than C, a contradiction. Thus $d_A(x) \leq 1$ for any vertex x on C. Since there are at least r edges between C and A, there are at least r vertices on C adjacent to some vertex of A. Thus Claim 1 is true.

We fix an orientation of C, and so induce a set of transitions at each vertex of C and an orientation of each edge. If a_1 is the end-vertex on C of some edge between A and C, then we choose a successor a_1' of a_1 on the oriented circuit C and describe C following its orientation. Let $a_2, \ldots, a_r$ be the extremities on C, encountered in this order, of $r - 1$ other edges between C and A. For $i = 2, \ldots, r$, let a_i' be the successor of a_i. Since A is connected, there is a path P_{ij} between a_i and a_j whose internal vertices are in A. Let $T = \{a_i'a_i \in E(C) : i = 1, 2, \ldots, r\}$. Then we further have the following claim.

Claim 2. *T is a matching of C and at most r edges are joined to two distinct edges of T.*

Proof. Obviously, the edges $a_i'a_i$ for $i = 1, 2, \ldots, r$ are on C. Note that all vertices a_i are distinct. Assume that $a_i = a_j'$ for $i \neq j$. Then, replacing the edge a_ja_j' by the path P_{ij}, we obtain a new circuit C' longer than C, which contradicts the maximality of C. If $a_i' = a_j'$ for $i \neq j$, then we obtain a new circuit $C' = a_iP_{ij}a_ja_i$. By the orientation of a_ia_i' and a_ja_i', the circuit C contains a path between a_i' and a_i, and a path between a_j' and a_j, avoiding the edges a_ia_i' and a_ja_i'. Hence, the deletion of these two edges does not disconnect C at a_i', and thus $C + C'$ is a circuit contradicting the maximality of C. Thus T is a matching of C. By a similar proof to that for Case 2 of Lemma 1 in [3] (Page 300), we obtain that at most r edges are joined to two different edges of T. So Claim 2 is true.

Now we complete the proof of Lemma 2.2.

From Claims 1 and 2, there are at least $|T|\delta(L(G))$ edges of G not belonging to T and adjacent to at least one edge of T. Among them, at most r edges are adjacent to two distinct edges of T, and are thus counted twice. Thus $\bar{e}(C) \geq |T|\delta(L(G)) - r + |T| = |T|\delta(L(G))$ with $|T| = r$. So Lemma 2.2 is proved. □

3. Proof of Theorem 1.7

In this section, we will provide the proof of our result. In our proof, we use proof techniques similar to those from [3] by Favaron and Fraisse, [5] by Lai, Shao and Zhan, and [7] by Li.

Proof of Theorem 1.7. Let H be a 3-connected claw-free graph. Then, by Theorem 1.2, we can assume, without loss of generality, that $H = cl(H)$. Hence H is the line graph of a triangle-free graph G, and H is 3-connected. Obviously, G does not contain a dominating circuit; otherwise H is hamiltonian. Let C' be a longest cycle of H. If $|V(C') \geq 6\delta(H) - 15$, then we are done. Thus $|V(C')| \leq 6\delta(H) - 16$, and so, by Theorem 1.3, we have $\delta(H) \geq 11$ since $5\delta(H) - 5 \leq |V(C')| \leq 6\delta(H) - 16$.

Let $B = \{v \in V(G) | d_G(v) = 1, 2\}$. Since H is 3-connected, the sum of degrees of the two ends of each edge in G is at least 5 and thus B is independent. Let $X_0 = N_G(B)$. We name the vertices of X_0 as $x_1, x_2, \ldots, x_p$ in the following way. Assume the vertices $x_1, \ldots, x_i$ are already defined or else put $i = 0$. Let y_{i+1} denote a vertex of B which is adjacent to some vertex of $X_0 - \{x_1, \ldots, x_i\}$. Either y_{i+1} has exactly one neighbor in $X_0 - \{x_1, \ldots, x_i\}$ and we name it x_{i+1}, or y_{i+1} has exactly two neighbors in $X_0 - \{x_1, \ldots, x_i\}$ and we name them x_{i+1} and x_{i+2} and put $y_{i+2} = y_{i+1}$. Let $Y_0 = \{y_1, \ldots, y_p\}$. We note that if $1 \le i < j \le p$, then $y_i y_j \notin E(G)$ and $y_i x_j \notin E(G)$, except for the edges $y_i x_{i+1}$ when $y_i = y_{i+1}$; and that the components of the subgraph induced by the edges $x_i y_i$, $1 \le i \le p$, are paths of length 1 or 2.

Consider now a matching M of G formed by $q - p$ edges $x_i y_i$ of G, $p + 1 \le i \le q$, considered in this order and such that

(i) the sets $X_0, Y_0, X = \{x_{p+1}, \ldots, x_q\}$ and $Y = \{y_{p+1}, \ldots, y_q\}$ are pairwise disjoint,
(ii) for $p + 1 \le i < j \le q$, $y_i y_j, y_i x_j \notin E(G)$.

We choose this matching as large as possible subject to the conditions (i) and (ii). Note that by the definition of X_0 and Y_0, the whole set B is disjoint from $X \cup Y$ and that Property (ii) holds for any i and j with $1 \le i < j \le q$ except for the edges $y_i x_{i+1}$, $1 \le i \le p$, when $y_i = y_{i+1}$.

Let J be the set of indices j between $p + 1$ and q such that y_j is adjacent to some vertex $z \notin X_0 \cup Y_0 \cup X \cup Y$ with $y_k z \notin E(G)$ for $1 \le k < j$. For each $j \in J$ we choose such a vertex z_j and we put $I = \{p + 1, \ldots, q\} - J$. Let $X_I = \{x_i \in X | i \in I\}$, $X_J = \{x_i \in X | i \in J\}$, $Y_I = \{y_i \in Y | i \in I\}$ and $Y_J = \{y_i \in Y | i \in J\}$, and let $S = X_0 \cup X_I \cup Y_J$. Then we have the following claim.

Claim 1. *S is not contained in any circuit of G.*

Proof. Suppose Claim 1 is false and let C be a maximal circuit of G containing $S = X_0 \cup X_I \cup Y_J$ and $R = V(G) - V(C)$. Since G has no dominating circuit, at least one component A of $G[R]$ has two vertices, and so A is disjoint from Y_0 since the vertices of Y_0 are isolated in $G[R]$. Let r denote the number of edges between A and C.

If every vertex of A has a neighbor in C, then $r \ge d_C(u) + d_C(v) + |A| - 2$, where u and v are two end-vertices of some edge uv in A. Since G is triangle-free, $d_A(u) + d_A(v) \le |A|$ and $d_G(u) + d_G(v) = d_C(u) + d_C(v) + d_A(u) + d_A(v) \le d_C(u) + d_C(v) + |A|$. Hence $r \ge d_G(u) + d_G(v) - 2 \ge \delta(H) \ge 11$. By Lemma 2.2, $\bar{e}(C) \ge r\delta(H) \ge 11\delta(H)$. Thus A contains a vertex z such that $N_C(z) = \emptyset$. It follows that $z \notin X_0 \cup Y_0 \cup X \cup Y$ and the neighbors of z are all in $Y_I \cup X_J \cup (R - (Y_0 \cup Y_I \cup X_J))$.

If z has a neighbor in Y_I, let i be the least index such that $y_i \in Y_I$ and $zy_i \in E(G)$. Since z has no neighbor in Y_J, $zy_k \notin E(G)$ for all $k < i$, in contradiction to the definition of I. Hence z has no neighbor in Y_I, and thus in Y.

If z has a neighbor in X_J, let x_j be the vertex of $N_G(z) \cap X_J$ with the largest index. Consider the ordered sets $X' = \{x_{p+1}, \ldots, x_{j-1}, x_j, z_j, x_{j+1}, \ldots, x_q\}$ and $Y' = \{y_{p+1}, \ldots, y_{j-1}, z, y_j, y_{j+1}, \ldots, y_q\}$. Then vertex z is neither adjacent to any x_k with $k > j$, by the definition of x_j and since z has no neighbor in X_I, nor to any vertex of Y, as said above. The vertex z_j is not adjacent to any vertex y_k with $k < j$ by the choice of z_j. If $zz_j \notin E(G)$, then the sets X' and Y' define a matching M' which satisfies (i) and (ii), and thus which contradicts the maximality of M. If $zz_j \in E(G)$, then the eulerian subgraph $G[(E(C) - E(C')) \cup (E(C') - E(C))]$, with $C' = y_j z_j z x_j y_j$, satisfies $V(C) \cap V(C') = \{y_j\}$ since z has no neighbor in C, and thus contradicts the maximality of C. Hence $N_G(z) \cap X_J = \emptyset$ and z has no neighbor in X.

Finally if z has a neighbor t in $R - (Y_0 \cup Y_I \cup X_J)$, then the matching M'' corresponding to the ordered sets $X'' = \{t, x_{p+1}, \ldots, x_q\}$ and $Y'' = \{z, y_{p+1}, \ldots, y_q\}$ satisfies the conditions (i) and (ii) since z has no neighbor in $X \cup Y$. This contradicts the maximality of M and achieves the proof of Claim 1. □

Claim 2. *G is not contractible to the Petersen graph PTS_{10}.*

Proof. Suppose that G can be contracted to the Petersen graph PTS_{10} (see Fig. 1). Let $v_1, v_2, \ldots, v_{10}$ be the ten vertices of the Petersen graph PTS_{10}, and W_i be the preimage of v_i ($i = 1, 2, \ldots, 10$). Let

$$\mathcal{SV} = \{v \in V(G) : d_G(v) \ge 7\} \quad \text{and} \quad \mathcal{SW} = \{W_i : i = 1, 2, \ldots, 10\}.$$

Since $d_G(u) + d_G(v) - 2 \ge \delta(H) \ge 11$ for every edge $e = uv \in E(G)$, we have either $d_G(u) \ge 7$ or $d_G(v) \ge 7$. So we have the following fact.

Claim 2.1. *For every edge $e = uv \in E(G)$, either $u \in \mathcal{SV}$ or $v \in \mathcal{SV}$. Equivalently, if $u, v \notin \mathcal{SV}$, then $uv \notin E(G)$.*

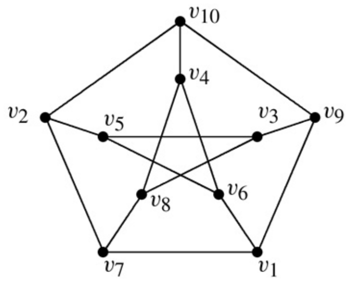

Fig. 1. Petersen graph PTS_{10}.

Let $W \in \mathcal{SW}$ and let W' be a graph obtained from W by deleting the vertices of degree 1 for $W \in \mathcal{SW}$. Then we have the following claim.

Claim 2.2. *If $\alpha'(W) = 1$, then $W = K_{1,p}$ for some $p \geq 1$. That is, W is a star.*

Claim 2.3. *If $\alpha'(W) \geq 2$, then $E(W') \neq \emptyset$ and W' is 2-edge-connected, and contains some cycle.*

Proof. Assume that W is the preimage of some vertex v_i, and that $E_G[V(W), V(G) - V(W)] = \{e_1, e_2, e_3\}$, where e_1, e_2, e_3 are edges adjacent to v_i in PTS_{10}. Let $\alpha'(W) = t \geq 2$ and let $\{z_i y_i : i = 1, 2, \ldots, t\}$ be a matching of W. Without loss of generality assume that $z_i \in \mathcal{SV}$ for $i = 1, 2, \ldots, t$. If $E(W') = \emptyset$, then $d_W(y) = 1$ for $y \in V(W) - \{z_1, z_2, \ldots, z_t\}$ and $z_i z_j \notin E(G)$ for $i \neq j$ and $i, j = 1, 2, \ldots, t$. It is easy to see that at least one edge of $\{e_1, e_2, e_3\}$ is a cut-edge of G, a contradiction. Thus $E(W') \neq \emptyset$.

Suppose that W' contains a cut-edge $e = z_1 z_2$. Then e is also a cut-edge of W. Let (U_1, V_1) be the partition of $V(W)$ such that $E_W[U_1, V_1] = \{e\}$ and $z_1 \in U_1$ and $z_2 \in V_1$. Since $z_1, z_2 \in V(W')$, we have $d_W(z_1) \geq 2$ and $d_W(z_2) \geq 2$. Thus $E(G[U_1]) \neq \emptyset$ and $E(G[V_1]) \neq \emptyset$. Note that $E_G[V(W), V(G) - V(W)] = \{e_1, e_2, e_3\}$. We may assume that the number of edges joining U_1 and $V(G) - V(W)$ is 1, call it e_1. Then $\{e_1, e\}$ is an essential edge-cut in G, a contradiction. So Claim 2.3 holds. □

By the definition of contraction, without loss of generality assume that $v_i \in V(G)$ for $i = 1, 2, \ldots, 10$. Let $w_1^i, w_2^i, w_3^i \in N_{W_i}(V(G) - V(W_i))$ for $i \in \{1, 2, 3, \ldots, 10\}$ for $\alpha'(W_i) \geq 1$. Then we have the following fact.

Claim 2.4. *If $\alpha'(W_i) \geq 2$, then*

(I) *$d_{W_i}(w_j^i) \geq 2$ if $w_j^i = w_{j+1}^i$ for $j = 1, 2$, and $d_{W_i}(w_1^i) \geq 2$ if $w_1^i = w_3^i$;*

(II) *there are paths P_j with at least two vertices in W_i connecting w_j^i and w_{j+1}^i for $j = 1, 2$ and path P_3 with at least two vertices in W_i connecting w_1^i and w_3^i.*

Proof. (I) If $w_1^i = w_2^i$, then $d_{W_i}(w_1^i) \geq 2$ since otherwise $\{w_1^i x, w_3^i v_j\}$ is a cut-edge set of two edges of G (where $x \in W_i$, $v_j \in V(PTS_{10})$), a contradiction. Similarly, we can prove other parts of (I).

(II) Let $W_i' = W_i - D_1(W_i)$. If $w_1^i = w_2^i$, then, by (I), $d_{W_i}(w_1^i) \geq 2$. Thus $w_1^i \in W_i'$. By Claim 2.3, W_i' is 2-edge-connected. Let zw_1^i be an edge of W_i'. Then there is a cycle C_i' in W_i' containing the edge zw_1^i. That is, there is a path P_1 connecting w_1^i and w_2^i. Since G is triangle-free, $|V(P_1)| \geq 4$.

Assume that $w_1^i \neq w_2^i$. If $w_1^i w_2^i$ is an edge of W_i, then $P_1 = w_1^i w_2^i$ is the path that we required. Thus $w_1^i w_2^i \notin E(G)$. Adding the new edge $w_1^i w_2^i$ into W_i, we obtain that $W_i'' = W_i + \{w_1^i w_2^i\} - D_1(W_i)$ is 2-edge-connected. Thus W_i'' has a cycle C' containing the edge $w_1^i w_2^i$. That is, W_i has a path P_1 with at least three vertices connecting w_1^i and w_2^i. Similarly, we can prove the other parts of (II). Thus Claim 2.4 is true. □

Now we complete the proof of Claim 2.

Let $Z = \{v_i | v_i$ is a trivial vertex in $PTS_{10}\}$. Then, by claim 2.1, Z is independent. Since $\alpha(PTS_{10}) = 4$, we have $0 \leq |Z| \leq 4$. If $|Z| \geq 1$, then without loss of generality assume that $v_1 \in Z$. We know that $PTS_{10} - \{v_1\}$ has a spanning cycle C'. Since $|Z| \leq 4$, C' contains at least six vertices (such as $v_5, v_6, \ldots, v_{10}$) which do not belong to Z. Without loss of generality assume that $C' = (v_2 v_5 v_6 v_4 v_{10} v_9 v_3 v_8 v_7 v_2)$ is a 9-cycle in $PTS_{10} - \{v_1\}$ (see Fig. 1). Let M be the set of edges of G that will be defined recursively as follows:

Initialize $M = \emptyset$; then we add edges into M according to the value of $\alpha'(W_i)$ for $i = 1, 2, \dots, 10$ in the following.

If $\alpha'(W_i) = 1$, then, by Claim 2.2, $W_i = K_{1,p_i}$ for some $p_i \ge 1$ is a star. Let $V(W_i) = V_1 \cup V_2$ such that $V_1 = \{v_i\}$ and $|V_2| = p_i$. Then v_i is the center of the star W_i. If $w_1^i, w_2^i, w_3^i \in V_2$, then $w_j^i \ne w_k^i$ for $\{j, k\} \subset \{1, 2, 3\}$ and $j \ne k$ since otherwise, for example, $w_1^i = w_2^i$, $\{v_i w_1^i, v_i w_3^i\}$ is an essential cut of G, a contradiction. Similarly, if two vertices of $\{w_1^i, w_2^i, w_3^i\}$ belong to V_2, then the two vertices must be distinct. It is easy to see that there is a (w_j^i, w_k^i)-path $P^i(j, k)$ containing v_i connecting w_j^i and w_k^i for $\{j, k\} \subset \{1, 2, 3\}$ and $j \ne k$, and there is a vertex $v_i' \in W_i$ such that $e_i = viv_i' \in E(W_i)$ and $v_i' \notin \{w_1^i, w_2^i, w_3^i\}$. Note that it is possible that $w_j^i = w_k^i = v_i$. Let $M = M \cup \{e_i\}$.

If $\alpha'(W_i) \ge 2$, then, by claim 2.3, W_i is 2-edge-connected and contains some cycle. If $w_1^i = w_2^i = w_3^i$, then $d_{W_i}(w_1^i) \ge 3$ since otherwise, $N_{W_i}(w_1^i)$ is an essential edge-cut of G with size at most 2, a contradiction. Thus there is a (w_j^i, w_k^i)-path $P^i(j, k)$ containing v_i connecting w_j^i and w_k^i for $\{j, k\} \subset \{1, 2, 3\}$ and $j \ne k$, and there is a vertex $v_i' \in W_i$ such that $e_i = v_i v_i' \in E(W_i)$ and $v_i' \notin \{w_1^i, w_2^i, w_3^i\}$. Note that it is possible that $w_j^i = w_k^i = v_i$. Let $M = M \cup \{e_i\}$.

If $w_1^i = w_2^i = w_3^i$ is not true, that is, $|N_{W_i}(V(G) - V(W_i))| \ge 2$, then it is easy to see that there is a (w_j^i, w_k^i)-path $P^i(j, k)$ with at least one vertex in W_i connecting w_j^i and w_k^i for $\{j, k\} \subset \{1, 2, 3\}$ and $j \ne k$. If $|V(P^i(j, k))| = 1$, then $w_j^i = w_k^i$, and there is a vertex $v_i' \in W_i$ such that $e_i = w_j^i v_i' \in E(W_i)$ and $v_i' \notin \{w_1^i, w_2^i, w_3^i\}$. Let $M = M \cup \{e_i\}$. If $|V(P^i(j, k))| \ge 2$, then pick up one edge e_i in $P^i(j, k)$ such that $e_i \in E(W_i)$. Let $M = M \cup \{e_i\}$.

For the convenience of the proof, we can assume that $|Z| = 4$ and $Z = \{v_1, v_2, v_3, v_4\}$ (and the proofs of other cases are similar). Since $C' = (v_2v_5v_6v_4v_{10}v_9v_3v_8v_7v_2)$ is a 9-cycle in $PTS_{10} - \{v_1\}$ (see Fig. 1), without loss of generality assume that $v_2w_1^5, w_1^6w_2^5 \in E_G[V(W_5), G - V(W_5)]$, $w_2^5w_1^6, v_4w_2^6 \in E_G[V(W_6), G - V(W_6)]$, $v_2w_1^7, w_2^7w_2^8 \in E_G[V(W_7), G - V(W_7)]$, $v_3w_1^8, w_2^7w_2^8 \in E_G[V(W_8), G - V(W_8)]$, $v_3w_1^9, w_1^{10}w_2^9 \in E_G[V(W_9), G - V(W_9)]$, and $w_2^9w_1^{10}, v_4w_2^{10} \in E_G[V(W_{10}), G - V(W_{10})]$. Let

$$F_1 = \{v_2w_1^5, w_2^5w_1^6, w_2^6v_4, v_4w_2^{10}, w_1^{10}w_2^9, w_1^9v_3, v_3w_1^8, w_2^8w_2^7, w_1^7v_2\}.$$

Then $C = F_1 + \sum_{i=5}^{10} P^i(1, 2)$ is a circuit of G and M is a match of G. Note that $|M| \ge 6$ since $|Z| \le 4$. Let $d_G(e)$ denote the value of $d_G(u_e) + d_G(v_e) - 2$ for an edge e of G. Then $d_G(e) \ge \delta(H)$, and so $\bar{e}(C) \ge \sum_{e \in M} d_G(e) - 15 + 6 \ge 6\delta(H) - 9$. Note that the edges of the Petersen graph PTS_{10} may be counted twice, and 6 is the cardinality of M. This contradiction shows that Claim 2 is true. □

Now we complete the proof of our theorem.

Let G^1 be the graph or multigraph obtained from G by deleting the vertices of degree 1 or 2 and replacing each path ayb where $d_G(y) = 2$ by the edge ab. Since G is essentially 3-edge-connected, G^1 is 3-edge-connected. Moreover, to each circuit subgraph C of G^1 corresponds a circuit of G containing $V(C)$. Since $S \cap B = \emptyset$, S is contained in $V(G^1)$. Since S is not contained in any circuit of G by Claim 1, S is not contained in any circuit of G^1. By Lemma 2.1, $|S| = |X_0 \cup X_I \cup X_J| \ge 13$. Let $F = \{x_iy_i | 1 \le i \le 12\}$ such that $P = \{x_i | 1 \le i \le 12\}$ contains as many vertices of X_0 as possible, and $Q = \{y_i | 1 \le i \le 12\}$. Then $P \subset X_0$ or $X_0 \subset P$ or $X_0 = P$. We suppose that F consists of l paths of length 2 with $0 \le l \le 6$ and $12 - 2l$ edges of a matching. Then $|P| = 12$ and $|Q| = 12 - l$. We know that Q is independent, that $y_ix_j \notin E(G) - F$ for any $y_i \in Q$ and $x_j \in P$ with $1 \le i < j \le 12$, and that G is triangle-free. Hence, two different edges of F are joined by at most one edge of G which is of type x_ix_j or x_iy_j with $1 \le i < j \le 12$. More precisely, we can give an upper bound on the number μ of edges of G which are adjacent to two different edges of F. For a given value of l, this number can be maximum if the l paths of F occur with smaller indices than those of the $12 - 2l$ edges of the matching. This is due to the fact that the l vertices y_i belonging to paths of length 2 have degree 2 and thus they cannot be adjacent by an edge not in F to any vertex x_i with $i < j$. When this condition is fulfilled, there are at most l^2 edges between the vertices $x_1, x_2, \dots, x_{2l}$ (since the number of edges of a triangle-free graph of order $2l$ is at most $(2l)^2/4$), $2l(12 - 2l)$ edges of type x_iy_j between the sets $\{x_1, x_2, \dots, x_{2l}\}$ and $\{y_{2l+1}, y_{2l+2}, \dots, y_{12}\}$, and $\frac{(12-2l)(12-2l-1)}{2}$ edges of type x_ix_j or x_iy_j with $i < j$ between the vertices of the set $\{x_{2l+1}, \dots, x_{12}, y_{2l+1}, \dots, y_{12}\}$. Then $\mu \le l^2 + 2l(12 - 2l) + \frac{(12-2l)(12-2l-1)}{2} = l - l^2 + 66$.

By Lemma 2.1 and Claim 2, P is contained in a circuit C of G. If P is a subset of X_0 or $P = X_0$, then, counting the edges of $G - F$ adjacent to some edge of F, we find at least $(12 - 2l)\delta(H)$ edges adjacent to an edge of a matching of F and $2l(\delta(H) - 1)$ edges adjacent to an edge of a path of length 2 (since each vertex

y_i on such a path has degree 2 in G). At most $l - l^2 + 66$ of these edges have their two end-vertices in $P \cup Q$ and are thus counted twice. Hence $\bar{e}(C) \geq (12 - 2l)\delta(H) + 2l(\delta(H) - 1) - (l - l^2 + 66) + 12$, that is, $\bar{e}(C) \geq 12\delta(H) + l^2 - 3l - 54 = 6\delta(H) - 14 + (6\delta + l^2 - 3l - 40)$ since l is an integer between 0 and 6, a contradiction. Thus X_0 is a subset of P and $P \neq X_0$. It follows that $p = |X_0| < 12$. If $l \geq 3$, then we easily prove that $\bar{e}(C) \geq 2l\delta(H) - l^2 \geq 6\delta - 14$, a contradiction. Thus $l \leq 2$.

If p is even, assume that

$$P' = X_0 \cup \{x_{p+1}, y_{p+1}, \dots, x_{p+(6-\frac{p}{2})}, y_{p+(6-\frac{p}{2})}\}$$

and

$$F = \left\{x_i y_i | 1 \leq i \leq 6 + \frac{p}{2}\right\}.$$

Then $|P'| = 12$ and F consists of l paths of length 2 with $0 \leq l \leq 2$ and $(p - 2l) + (6 - \frac{p}{2})$ edges of a matching. If p is odd, assume that

$$P' = X_0 \cup \{x_{p+1}, y_{p+1}, \dots, x_{p+(6-(p+1)/2)}, y_{p+(6-(p+1)/2)}\}$$

and

$$F = \left\{x_i y_i | 1 \leq i \leq 5 + \frac{p-1}{2}\right\}.$$

Then $|P'| = 11$, and F consists of l paths of length 2 with $0 \leq l \leq 2$ and $(p - 2l) + (6 - \frac{p+1}{2})$ edges of a matching. Recall that $X = \{x_{p+1}, \dots, x_q\}$ and $Y = \{y_{p+1}, \dots, y_q\}$ and X_0, X and Y are pairwise disjoint. Thus $P' \subset X_0 \cup X \cup Y$.

Thus P' is contained in a circuit of G by Lemma 2.1 since $|P'| \leq 12$. Now we estimate the upper bound on the number μ. It is easy to see that there are at most l^2 edges between the vertices $x_1, x_2, \dots, x_{2l}$ (since the number of edges of a triangle-free graph of order $2l$ is at most $(2l)^2/4$).

If p is even, then $2l(6 + p/2 - 2l)$ edges of type $x_i y_j$ between the sets $\{x_1, x_2, \dots, x_{2l}\}$ and $\{y_{2l+1}, y_{2l+2}, \dots, y_{p+(6-p/2)}\}$, and $\frac{(6+p/2-2l)(6+p/2-2l-1)}{2}$ edges of type $x_i x_j$ or $x_i y_j$ with $i < j$ between the vertices of the set $\{x_{2l+1}, \dots, x_{p+(6-p/2)}, y_{2l+1}, \dots, y_{p+(6-p/2)}\}$. Then

$$\mu \leq l^2 + 2l\left(6 + \frac{p}{2} - 2l\right) + \frac{(p + p/2 - 2l)(p + p/2 - 2l - 1)}{2} = l - l^2 + 15 + \frac{11p}{4} + \frac{p^2}{8}.$$

So

$$\bar{e}(C) \geq \left(6 + \frac{p}{2} - 2l\right)\delta(H) + 2l(\delta(H) - 1) - \left(l - l^2 + 15 + \frac{11}{4}p + \frac{p^2}{8}\right) + 6 + \frac{p}{2}.$$

That is,

$$\begin{aligned}\bar{e}(C) &\geq 6\delta(H) + \frac{p}{2}\delta(H) + l^2 - 9 - \frac{9}{4}p - \frac{p^2}{8} - 3l \\ &= 6\delta(H) - 9 + \frac{p}{2}\left(\delta(H) - \frac{p}{4} - \frac{9}{2}\right) + l^2 - 3l \geq 6\delta(H) - 11.\end{aligned}$$

Note that $l^2 - 3l \geq -2$ for $l \leq 2$ and $\delta(H) - \frac{p}{4} - \frac{9}{2} \geq \frac{44-10-18}{4} = 4$ for $p < 12$ even and $\delta(H) \geq 11$. Similarly, if p is odd, then we also get a contradiction. Thus the proof of Theorem 1.7 is completed. □

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York, 1976.

[2] Z.-H. Chen, H.-J. Lai, X. Li, D. Li, J. Mao, Eulerian subgraphs in 3-edge-connected graphs and hamiltonian line graphs, J. Graph Theory 42 (2003) 308–319.

[3] O. Favaron, P. Fraisse, Hamiltonicity and minimum degree in 3-connected claw-free graphs, J. Combin. Theory Ser. B 82 (2001) 297–305.

[4] F. Harary, C.St.J.A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs, Canad. Math. Bull. 8 (1965) 701–709.

[5] H.-J. Lai, Y. Shao, M. Zhan, Hamiltonicity in 3-connected claw-free graphs, J. Combin. Theory Ser. B 96 (2006) 493–504.

[6] MingChu Li, Hamiltonian properties of claw-free graphs, Ph.D. Thesis, University of Toronto, 1998.
[7] MingChu Li, Cycles and minimum degrees in 3-connected claw-free graphs, 2006 (Preprint).
[8] M.M. Matthews, D.P. Sumner, Longest cycles paths in $K_{1,3}$-free graphs, J. Graph Theory 9 (1985) 269–277.
[9] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B 70 (1997) 217–224.