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New Proofs of Some Results of Nielsen

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0. INTRODUCTION

Given a compact oriented surface M with negative Euler characteristic and a homeomorphism $\tau: M^2 \to M^2$, what is the best representative $\varphi: M^2 \to M^2$ in the isotopy class of τ? Thurston [8] answered this question by defining pseudo-Anosov homeomorphisms (a generalization of Anosov diffeomorphisms on T^2) which may be characterized as follows. The homeomorphism φ is pseudo-Anosov iff there is a pair of transverse measured geodesic laminations Λ^s and Λ^u which intersect each closed non-peripheral geodesic in M, which are (topologically) preserved by φ and which have their respective transverse measures multiplied under the action of φ by $1/\lambda$ and λ for some $\lambda > 1$.

Thurston showed that $\varphi: M^2 \to M^2$ can be pieced together from homeomorphisms of subsurfaces which are either periodic or pseudo-Anosov. More precisely, Thurston proved Theorem 0.1. The preprint [8] provides an introduction to this Theorem, describing many of the useful properties of pseudo-Anosov diffeomorphisms. The proof outlined in [8] is contained in [11].

THEOREM 0.1. *Let $\tau: M^2 \to M^2$ be a homeomorphism of a compact oriented surface of negative Euler characteristic. Then $\tau \simeq \varphi$ such that either:*

(1.i) $\varphi^n = $ *identity for some* $n > 0$

(1.ii) φ *is pseudo-Anosov, or*

(1.iii) *there is a finite collection $\tau = \{\gamma_1,\ldots,\gamma_k\}$ of simple disjoint closed curves such that φ permutes disjoint open regular neighborhoods η_i of γ_i. Let $S_1,\ldots,S_m$ be the components of $M - \bigcup_{i=1}^k \eta_i$ and n_j the least positive integer such that $\varphi^{n_j}(S_j) = S_j$. Then $\varphi^{n_j} \mid S_j$ satisfies* (i) *or* (ii).

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Thurston remarked in [8] that Nielsen's work, seen from a modern viewpoint, contains parts of Theorem 0.1. This was made explicit in [3], which contains a listing of the relevant results of [4–7]. (See also [2].) Miller extended Nielsen's techniques to prove Theorem 0.2 below; he also showed (Sect. 11 of [3]) how the techniques used in [11] to construct Markov partitions can be applied to strengthen (2.ii) to (1.ii).

THEOREM 0.2. *Let* $\tau\colon M^2 \to M^2$ *be a homeomorphism of a compact oriented surface of negative Euler characteristic. Then* $\tau \simeq \varphi$ *such that either*

(2.i) φ^n *is isotopic to the identity for some* $n > 0$

(2.ii) φ *preserves a pair of transverse geodesic laminations* Λ^s *and* Λ^u *which intersect every closed nonperipheral geodesic in* M *and which are minimal in the sense that each leaf of the lamination is dense in the lamination.*

(3.iii) *Same as* (1.iii).

Remark. Miller quotes Nielsen as having proved (1.i) rather than (2.i). Nielsen's proof is incorrect [10]. The known proofs that a map τ which is periodic on the level of homotopy is isotopic to a periodic map ([10], for example) require more sophisticated machinery (e.g., Teichmüller space and Smith theory) then we use in this paper.

The advantage of the Nielsen approach to pseudo-Anosov diffeomorphisms is that it involves only very elementary arguments. The disadvantage of [3] is that it relies on Nielsen's proofs which are not only cumbersome but which add considerable length to the full exposition. The purpose of this paper is to prove Theorem 0.2 using only the skeleton of Nielsen's program rather than his case by case detailed arguments. For the most part, the content of this paper is complementary to that of [3]; some of Section 4 is contained in [3] and is included here for completeness. Sections 10 and 11 of [3] involve replacing the geodesic laminations Λ^s and Λ^u by measured foliations, and finding invariant transverse measures on Λ^s and Λ^u, respectively; we have not repeated these arguments here.

The rest of the paper is organized as follows. Section 1 contains notation and an exposition of some of the basic ideas of Nielsen. In Section 2 we construct Γ of (iii) and decompose τ into its "irreducible" pieces. Section 3 contains the definition of Λ^s and Λ^u and a proof that they are indeed geodesic laminations. In Section 4 we verify that Λ^s and Λ^u are transverse, minimal, and fill up M and then construct φ.

The reader interested in abstracting the basic ideas of this paper should concentrate on Corollaries 1.2 and 1.4, the introduction to Section 3 including the definition of Λ^s and Λ^u, and Theorem 3.2.

We would like to thank R. D. Edwards, D. Epstein, and R. T. Miller for several helpful conversations.

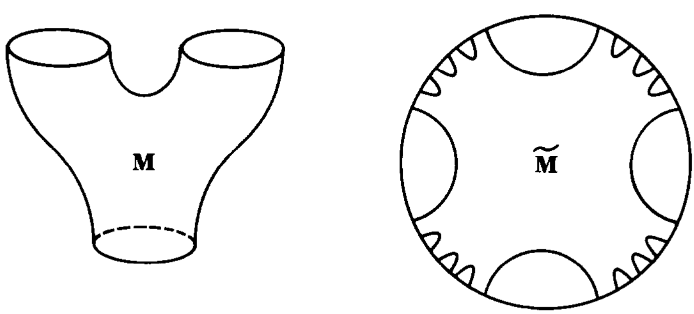
1. Notation and the Nielsen Setting

Let $\tau: M \to M$ be a homeomorphism of a compact connected orientable surface of negative Euler characteristic. In this section we establish preliminary results about the "circle at infinity" associated to M and the homeomorphisms induced on it by lifts of τ to the universal cover of M.

We use the Poincare disk model for the hyperbolic plane H. Namely, $H = \operatorname{int} D^2$ with the hyperbolic metric $dx/\sqrt{1-r^2}$, where dx is the Euclidean metric on D^2 and r is the distance to the origin. We compactify H by adding the "circle at infinity" $S_\infty = \partial D^2$ and decreeing that the neighborhoods of $P \in S_\infty$ in $H \cup S_\infty$ are exactly those of P in D^2. (The relationship of this compactification to the hyperbolic metric on H is considered in Lemma 1.1.) The geodesics in this model are segments of Euclidean circles and straight lines which meet S_∞ orthogonally. The isometries are the restrictions of linear fractional transformations of $\mathbb{C}$ which map D^2 onto itself.

We restrict our attention to the hyperbolic isometries of H (see Chap. 4 of [9] for a classification of the isometries of H and other background material) since these are the only ones which occur as covering translations for the universal cover of a compact surface. Each hyperbolic isometry f fixes (setwise) a unique geodesic $A(f)$ called the axis of f. The extension of f over $H \cup S_\infty$ pointwise fixes only the endpoints of $A(f)$. One of these points (f^-) is a source and the other (f^+) is a sink. For every other $x \in H \cup S_\infty$, $\lim_{n \to \pm\infty} f^n(x) = f^\pm$.

A hyperbolic structure on M is a metric of constant curvature -1; every surface of negative Euler characteristic has many such metrics. The choice of a hyperbolic structure on M identifies (up to an isometry of H) the universal cover $\tilde{M}$ with a convex suset of H. If M is closed, then $\tilde{M}$ is identified with all of H but if $\partial M \neq \emptyset$ then $\tilde{M}$ is the convex hull of a Cantor set in S_∞. Our notation will follow the closed case; in particular, we will write $H \cup S_\infty$ for the compactified universal cover of M. Our proofs apply equally well in the case that $\partial M \neq \emptyset$.



The identification of $\tilde{M}$ with H induces a representation of $\pi_1(M)$ into the isometry group of H. The image $\tilde{\pi}$ is a discrete group of hyperbolic isometries acting as covering translations. If $f \in \tilde{\pi}$ then $[f]$ denotes the geodesic in M which lifts to the axis of f.

Unless otherwise stated, a properly embedded geodesic arc $\alpha \subset M$ is assumed to meet ∂M orthogonally.

We will use the following elementary facts about geodesics in H and M, all of which may be found in [9]: (i) each nontrivial free homotopy class of closed curves or properly embedded arcs contains a unique geodesic; (ii) distinct axes have disjoint endpoints; and (iii) two hyperbolic isometries commute if and only if they have the same axis.

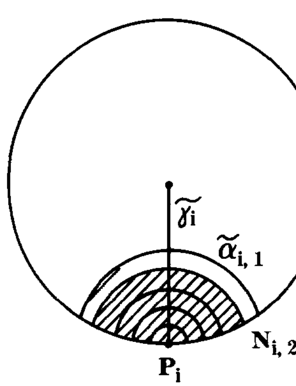
If α is a closed geodesic or a properly embedded geodesic arc in M, then $\tau_*(\alpha)$ denotes the unique geodesic freely homotopic to $\tau(\alpha)$.

A geodesic lamination $\Lambda \subset M$ (see Chap. 8 of [9] for examples and discussion) is a closed set which is foliated by geodesics.

The following lemma provides a neighborhood system for $P \in S_\infty$ in terms of simple geodesics on M.

LEMMA 1.1. *For each $P \in S_\infty$ there is a simple closed geodesic or properly embedded geodesic arc $\alpha \subset M$ and lifts $\tilde{\alpha}_j \subset H$ such that $P = \bigcap_{j=1}^{\infty} \bar{N}_j$, where N_j is the component of $H - \tilde{\alpha}_j$ whose closure in $H \cup S_\infty$ contains P. For $P_1, P_2 \in S_\infty$ we may assume that α_1 and α_2 are disjoint or equal.*

Proof. Choose a basepoint in H and let $\tilde{\gamma}_i$ be the half-infinite geodesic connecting this basepoint to P_i, $i = 1, 2$. It suffices to find disjoint or equal $\alpha_i \subset M$ such that γ_i (= projection of $\tilde{\gamma}_i$ to M) crosses α_i infinitely many times. Given such α_i, we note that each intersection of α_i with γ_i determines a lift $\tilde{\alpha}_{i,j}$ of α_i which intersects $\tilde{\gamma}_i$. By ordering the $\tilde{\alpha}_{i,j}$ appropriately we may assume that $\bar{N}_{i,1} \supset \cdots \supset \bar{N}_{i,j} \supset \bar{N}_{i,j+1} \supset \cdots \supset P$. Since there is a lower bound to the hyperbolic distance between $\tilde{\alpha}_{i,j}$ and $\tilde{\alpha}_{i,j+1}$, the hyperbolic distance between $\tilde{\alpha}_{i,1}$ and $\tilde{\alpha}_{i,j}$ goes to ∞. This implies that the Euclidean diameter of $\bar{N}_{i,j}$ goes to 0 and hence that $P_i = \bigcap_{j=1}^{\infty} \bar{N}_{i,j}$ as desired.



If $\partial M \neq \varnothing$, then M contains a collection of disjoint properly embedded geodesic arcs which partition M into disks. Since no half-infinite geodesic in M can be contained in one of these disks, each γ_i intersects some partitioning arc α_i infinitely often.

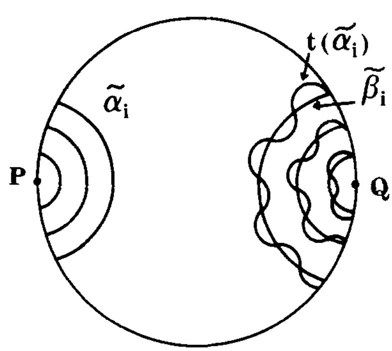
If $\partial M = \varnothing$, then M contains a collection of nondisjoint simple closed geodesics which partition M into disks. Since γ_2 is asymptotic to at most one closed curve, we may assume that γ_2 is not asymptotic to any of the geodesics in this collection. As above, γ_1 intersects some α_1 infinitely often. If γ_2 intersects α_1 infinitely often we are done. If not, then γ_2 is eventually contained in a component M' of $M - \alpha_1$. Choose a collection of nondisjoint simple closed geodesics in M' which partition M' into disks and peripheral annuli. Since γ_2 is not asymptotic to α_1, γ_2 does not eventually lie in a peripheral annulus. It follows that γ_2 intersects some simple closed geodesic $\alpha_2 \subset M' = M - \alpha_1$ infinitely often. ▌

The following Corollary of Lemma 1.1 is fundamental.

COROLLARY 1.2. *Every lift $t: H \to H$ of $\tau: M \to M$ extends to a unique homeomorphism (also called) $t: H \cup S_\infty \to H \cup S_\infty$.*

Proof. Let $\bar{U}_1 \supset \bar{U}_2 \supset \cdots \supset P$ be a neighborhood system in $H \cup S_\infty$ for $P \in S_\infty$. Choose α, $\tilde{\alpha}_i$, and $N(\tilde{\alpha}_i)$ as in Lemma 1.1 so that after passing to subsequences, $U_i \supset N(\tilde{\alpha}_i) \supset U_{i+1}$.

Lifting an isotopy of M which carries $\tau(\alpha)$ to $\tau_*(\alpha) \equiv \beta$, we see that there is a uniform bound to the hyperbolic distance between $t(\tilde{\alpha}_i)$ and the corresponding lift $\tilde{\beta}_i$ of β. In particular, $\tilde{\beta}_i$ and $t(\tilde{\alpha}_i)$ have the same endpoints.



Since there is a lower bound to the hyperbolic distance between $\tilde{\beta}_i$ and $\tilde{\beta}_{i+1}$ there exists $k > 0$ such that $N(\tilde{\beta}_{i+2k}) \subset t(N(\alpha_{i+k})) \subset N(\tilde{\beta}_i)$ for all i. Thus $\bigcap_{i=1}^{\infty} \overline{t(U_i)} = \bigcap_{i=1}^{\infty} \overline{t(N(\alpha_i))} = \bigcap_{j=1}^{\infty} \overline{N(\tilde{\beta}_j)} = Q \in S_\infty$ is a single point.

Define $t(P)=Q$. Since $\{U_i\}$ was arbitrary, t is well defined. Continuity of t follows from the observation that for all n, $\bar{U}_1 \supset \cdots \supset \bar{U}_n$ extends to a neighborhood system of any point in the interior of $\bar{U}_n \cap S_\infty$. To see that the extension of t is a homeomorphism, we simply apply this construction to t^{-1}. ∎

Remarks. Every map of H which moves points a bounded hyperbolic distance extends continuously by the identity on S_∞. In particular, if τ and τ' are homotopic homeomorphisms of M, and t and t' are lifts of τ and τ' coming from a lift of the homotopy, then $t \mid S_\infty = t' \mid S_\infty$.

Corollary 1.2 allows us to extend τ_* over nonclosed geodesics as follows. For each geodesic $\lambda \subset M$, let $\tilde{\lambda} \subset H$ be a lift of λ and let t be a lift of τ. Define $t_*(\tilde{\lambda})$ to be the geodesic in H whose endpoints are the t images of the endpoints of $\tilde{\lambda}$, and define $\tau_*(\lambda) \subset M$ to be the projected image of $t_*(\tilde{\lambda})$. Since the endpoints of axes are dense in S_∞, τ_* is completely determined by its action on closed geodesics.

The following lemma is useful in identifying free homotopy classes of closed curves which are fixed by some iterate of τ. Let $\mathrm{Fix}(t)$ be the fixed point set of t.

LEMMA 1.3. *If $f^+ \in \mathrm{Fix}(t)$ for some $f \in \tilde{\pi}$ and lift t of τ^n, then $ft = tf$. In this case, $f^- \in \mathrm{Fix}(t)$, $\tau^n_*[f]=[f]$ and if $\mathrm{Fix}(t) \neq \{f^+, f^-\}$, then f^+ and f^- are nonisolated in* $\mathrm{Fix}(t)$.

Proof. We may assume that f is indivisible. For each $x \in H$, $\lim_{m\to\infty}(tft^{-1})^m(x) = t(\lim_{m\to\infty} f^m(t^{-1}x)) = tf^+ = f^+$. Thus $(tft^{-1})^+ = f^+$ and the axes of f and tft^{-1} have a common endpoint. This implies that these axes are equal and hence that $tft^{-1}=f^k$ for some $k>0$. Thus $f = (t^{-1}ft)^k$ and (by the indivisibility of f) $k=1$. It follows immediately that t fixes f^- and that $\tau^n_*[f]=[f]$. Finally, if $P \in \mathrm{Fix}(t) - \{f^+, f^-\}$ then $f^m(P) \in \mathrm{Fix}(t)$ for all $m \in \mathbb{Z}$. Since $\lim_{m\to\pm\infty} f^m(P) = f^\pm$, the proof is completed. ∎

To establish Theorem 0.2, we will use the fact that τ^n is isotopic to the identity if and only if $\tau^n_* = \text{identity}$. This is contained in the remark following Corollary 1.2 and the following corollaty of Lemma 1.3.

COROLLARY 1.4. *If $\tau\colon M^2 \to M^2$ and $\sigma\colon M^2 \to M^2$ have lifts $t\colon H \cup S_\infty \to H \cup S_\infty$ and $s\colon H \cup S_\infty \to H \cup S_\infty$ such that $t \mid S_\infty = s \mid S_\infty$, then $\tau \simeq \sigma$.*

Proof. Choose a basepoint $x \in M$ and a lift $\tilde{x} \in H$ of x. After an isotopy of τ, we may assume that $s^{-1}t$ fixes $\tilde{x}$. Since $s^{-1}t \mid S_\infty = \text{identity}$, Lemma 1.3 implies that $s^{-1}t$ fixes $f\tilde{x}$ for every $f \in \tilde{\pi}$. It follows that $\sigma^{-1}\tau$

induces the identity map on $\pi_1(M, x)$, and hence that $\sigma^{-1}\tau$ is isotopic to the identity. ∎

2. Reducing τ

In this section we make a preliminary isotopy which decomposes τ into a union of homeomorphisms on simpler surfaces.

Let Γ' be the collection of simple closed nonperipheral geodesics in M which have finite τ_*-order (i.e., $\tau_*^n(\gamma')=\gamma'$ for some $n>0$). We focus on the subset Γ of elements in Γ' which are isolated in the sense that they are disjoint from all other elements of Γ'. Then $\Gamma=\{\gamma_1,..., \gamma_m\}$ is a finite collection of disjoint simple closed geodesics which is invariant under the action of τ_*. Let $\eta(\gamma_i)$, $i=1,..., m$, be disjoint open product neighborhoods of the γ_i's. After an isotopy of τ, we may assume that τ permutes the $\eta(\gamma_i)$'s and hence that τ permutes the components $\{S_j\}$ of $M-\bigcup_{i=1}^m \eta(\gamma_i)$.

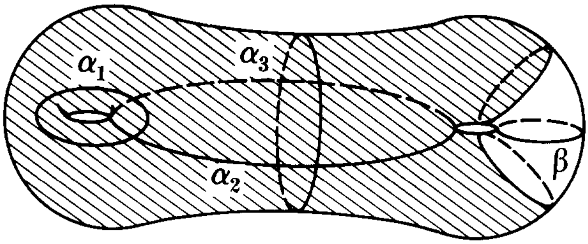
Let n_j be the least positive integer such that $\tau^{n_j}(S_j)=S_j$. Then $\Gamma(\tau^{n_j} \mid S_j)=\varnothing$ and, as Lemma 2.2 shows, we are reduced to proving

Theorem 2.1. *Let $\tau\colon M^2\to M^2$ be a homeomorphism such that no closed nonperipheral closed geodesics have finite τ_*-order. Then there exists a homeomorphism $\varphi\simeq\tau$ and a pair of transverse minimal geodesic laminations Λ^s and Λ^u which intersect every nonperipheral closed geodesic in M and which are preserved by φ.*

Lemma2.2. *If $\tau\colon M^2\to M^2$ is a homeomorphism such that $\tau_*^n\neq$ identity for any $n>0$ and such that $\Gamma(\tau)=\varnothing$, then no closed nonperipheral geodesics have finite τ_*-order.*

Proof. Let $\mathscr{C}=\{$nonperipheral subsurfaces $S\subset M$ such that each component of ∂S is essential in M and such that for some $n>0$, τ_*^n (thought of as taking free homotopy classes in M to free homotopy classes in M) is the identity on free homotopy classes in $S.\}$ There is a partial ordering on the elements of $\mathscr{C}$ defined as follows: $S_1<S_2$ if and only if $S_1\subset S_2$ and $S_1 \not\simeq S_2$. Note that $S_1<S_2$ implies that $\chi(S_1)>\chi(S_2)\geqslant\chi(S)$ so that every ascending chain is finite.

If $\alpha_1,..., \alpha_k\subset M$ are closed geodesics, define the augmented regular neighborhood $N(\alpha_1,..., \alpha_k)$ to be the union of a regular neighborhood $N'(\alpha_1,..., \alpha_k)$ of $\alpha_1\cup\cdots\cup\alpha_k$ and the disk components of $M-N'(\alpha_1,..., \alpha_k)$. We cannot in general isotop $N(\alpha_1,..., \alpha_k)$ so that its boundary components are all geodesics. We therefore normalize $N(\alpha_1,..., \alpha_k)$ by assuming that each boundary component is at a small constant distance ε from the geodesic in its free homotopy class.

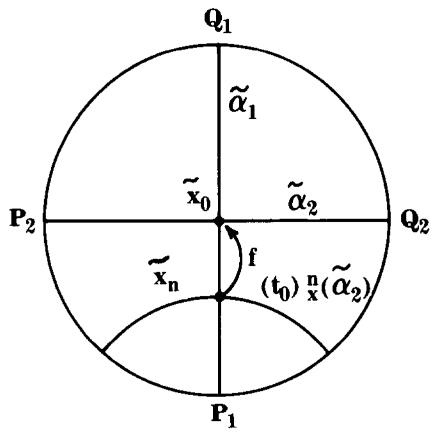


$\mathbf{N}(\alpha_1, \alpha_2, \alpha_3)$

We show below that if $\alpha_1,..., \alpha_k$ have finite τ_*-order, then $N(\alpha_1,..., \alpha_k) \in \mathscr{C}$. Thus if some nonperipheral closed curve has finite τ_*-order, then $\mathscr{C} \neq \emptyset$ and there is a nontrivial maximal element $S \in \mathscr{C}$. We assume without loss that $S = N(\alpha_1,..., \alpha_k)$ for some closed geodesics $\alpha_1,..., \alpha_k$ of finite τ_*-order. Since $\tau_*^n \mid M \neq$ identity, ∂S contains at least one nonperipheral component β. As $\Gamma(\tau) = \emptyset$, there is a closed geodesic γ of finite τ_*-order such that $\gamma \cap \beta \neq \emptyset$. It follows that $S_1 = N(\alpha_1,..., \alpha_k, \gamma) \in \mathscr{C}$ and $S_1 > S$ in contradiction to the assumption that S is maximal.

It therefore suffices to show that if $\alpha_1,..., \alpha_k$ have finite τ_*-order, then $N(\alpha_1,..., \alpha_k) \in \mathscr{C}$. For notational convenience we will assume that $k = 1$, that $\alpha = \alpha_1$ is a non-simple geodesic without triple points and that α is fixed by τ_*. The general case is similar.

Choose nondisjoint lifts $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ of α and a lift t_0 of τ which fixes the endpoints $\{P_1, Q_1\}$ of $\tilde{\alpha}_1$.

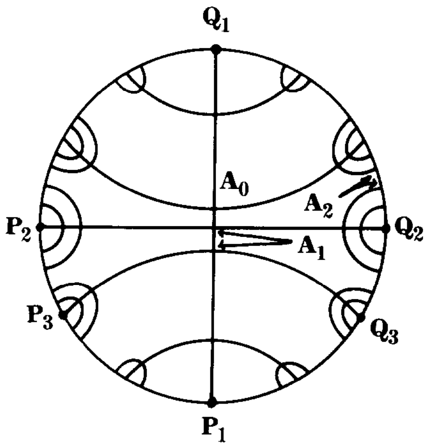


Since τ_* fixes α, t_* permutes the lifts of α. Thus each $(t_0)^n(\tilde{\alpha}_2)$ is a lift of α which intersects $(t_0)^n_*(\alpha_1) = \alpha_1$. The intersection $\tilde{x}_n = (t_0)^n_* \tilde{\alpha}_2 \cap \tilde{\alpha}_1$ projects to a self-intersection point of α. Since there are only finitely many such self-intersection points, there exists $n > 0$ and $f \in \tilde{\pi}$ such that $f\tilde{x}_n = \tilde{x}_0$. Let

$t = f \cdot t_0^n$; then t_* either fixes or permutes $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. Replacing t by t^2 if necessary, we may assume that t_* fixes both $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

Let A_1 be the collection of lifts of α which intersect $A_0 = \tilde{\alpha}_1$. Using the fact that t fixes the endpoints $\{P_2, Q_2\}$ of $\tilde{\alpha}_2$, we will show that t fixes the endpoints $\{P_3, Q_3\}$ of every $\tilde{\alpha}_3 \in A_1$. Arguing as above, we see that there exists $m > 0$ and $g \in \tilde{\pi}$ such that $(t')_* = (g \circ t_0^m)_*$ fixes both $\tilde{\alpha}_1$ and $\tilde{\alpha}_3$. The covering translation $h = t^m \cdot (t')^{-n}$ fixes P_1 and Q_1 and therefore commutes with t and t'. If h is nontrivial, then P_3 satisfies $\bigcup_{k=-\infty}^{\infty} t^{mk}(P_3) = \bigcup_{k=-\infty}^{\infty} (h \cdot (t')^n)^k P_3 = \bigcup_{k=-\infty}^{\infty} h^k P_3$ has accumulation set $\{P_1, Q_1\}$. This contradicts the fact that $\{P_2, Q_2\} \subset \mathrm{Fix}(t \mid S_\infty)$ links $\{P_1, Q_1\}$. Thus $(t')^n = t^m$ and Fix $(t \mid S_\infty) = \mathrm{Fix}(t' \mid S_\infty) \supset \{P_3, Q_3\}$.

Define A_{n+1} to be the collection of lifts of $\tilde{\alpha}$ which intersect some element of A_n. Arguing inductively we see that t_* fixes every element of $\bigcup_{n=0}^{\infty} A_n$.



Now $\bigcup_{n=0}^{\infty} A_n$ is a component of the union of all lifts of α. Let $\tilde{N}'$ be the convex hull in H of the closure of the endpoints of elements of $\bigcup_{n=0}^{\infty} A_n$; let $\tilde{N}$ be the complement in $\tilde{N}'$ of the ε-neighborhood of $\partial \tilde{N}'$. It is an exercise to check that $\tilde{N}$ is a component of the full preimage of $N(\alpha)$. Since t fixes each endpoint in $\tilde{N}$, Lemma 1.3 implies that $N(\alpha) \in \mathscr{C}$. ∎

3. Construction of Λ^s, Λ^u

For the remainder of this paper we assume that τ_* does not act finitely on any nonperipheral closed geodesics.

In this section we construct a pair of τ_*-invariant geodesic laminations Λ^s and Λ^u which will be the stable and unstable laminations for φ. This is motivated as follows. Corollary 1.4 implies that if $t: H \cup S_\infty \to H \cup S_\infty$ is a lift of τ and if $\tilde{\varphi}: H \cup S_\infty \to H \cup S_\infty$ is any equivariant homeomorphism which agrees with $t \mid S_\infty$, then $\tilde{\varphi} \mid H$ descends to a homeomorphism $\varphi: M \to M$ which is isotopic to τ. Thus we find a good representative φ in the isotopy class of τ by finding a good equivariant representative $\tilde{\varphi}$ in the isotopy class rel S_∞ of t. (This is analogous to the case $M = T^2$, where φ is chosen so that $\tilde{\varphi}$ is a linear map of $\mathbb{R}^2$.) It is therefore natural to look for a characteristic set of geodesics in H. It turns out that there are two such and that the union of their projections decomposes M into disks and peripheral annuli.

The remainder of this section is organized as follows. After defining Λ^s and Λ^u as the closure of a certain collection of geodesics, we prove (Lemma 3.1) that they are nonempty. This is an exercise in elementary fixed point theory. We then show (Theorem 3.2) that they are in fact geodesic laminations. This argument, which replaces a lengthy and detailed one of Nielsen, is the heart of the paper. A more detailed description of Λ^s and Λ^u is given in Section 4.

We begin with a list of definitions. Let T be the collection of all lifts of all positive iterates of τ. For each $t \in T$, $\delta^s(t) \subset S_\infty$ is the set of fixed points of $t \mid S_\infty$ which are not isolated sources, $\tilde{\Delta}^s(t) \subset H$ is the convex hull of $\delta^s(t)$, and $\Delta^s(t) \subset M$ is the projected image of $\tilde{\Delta}^s(t)$. Nielsen called $\tilde{\Delta}^s(t)$ the principal region of t. Define

$$\Lambda^s = \overline{\bigcup_{t \in T} \partial\tilde{\Delta}^s(t)} - \partial\tilde{M}$$

where $\partial\tilde{\Delta}^s(t)$ is the frontier of $\tilde{\Delta}^s(t)$, closure is taken in H and $\partial\tilde{M}$ is the full pre-image of ∂M; let $\Lambda^s \subset M$ be the projected image of $\tilde{\Lambda}^s$.

We define Λ^u similarly replacing $\delta^s(t)$ by $\delta^u(t)$, the set of fixed points of $t \mid S_\infty$ which are not isolated sinks.

LEMMA 3.1. *Λ^s and Λ^u are nonempty.*

Proof. This is an extension of the proof that every homeomorphism of a compact surface of negative Euler characteristic has periodic points. After showing that some Nielsen class of periodic points has negative Nielsen number (and is therefore nonempty), we show that this Nielsen class determines a lift $t \in T$ such that $\delta^s(t)$ and $\delta^u(t)$ contain at least two points which are not the endpoints of a common peripheral axis. Since $t \mid S_\infty \neq$ identity, $\partial\tilde{\Delta}^s(t) - \partial\tilde{M}$ and $\partial\tilde{\Delta}^u(t) - \partial\tilde{M}$ are nonempty.

Replacing τ by τ^2 if necessary, we may assume that τ is orientable.

We begin by fixing an integer n such that $L(\tau^n) \leqslant \chi(M^2) < 0$, where the Lefschetz number $L(\tau_n) = \operatorname{tr}(\tau_0^n) - \operatorname{tr}(\tau_1^n) + \operatorname{tr}(\tau_2^n)$ is the alternating sum of the traces of the maps τ_i^n induced by τ^n on the ith homology group of M.

We need only choose n so that $\operatorname{tr}(\tau_1^n) \geqslant \operatorname{rank} H_1(M)$. There exists $n>0$ so that the argument of each eigenvalue of τ_1^n is close to zero. If each eigenvalue of τ_1 has unit modulus, then $\operatorname{tr}(\tau_1^n)$ is close to, and hence equal to, rank $H_1(M)$. In any other case, τ_1 has an eigenvalue with modulus greater than one, and $\operatorname{tr}(\tau_1^n)$ is unbounded.

Since $\{t \mid S_\infty : t \in T\}$ is unaffected by isotopies of τ, we may assume that $\operatorname{Fix}(\tau^n) = \{x_1, \ldots, x_m\}$ is a finite set contained in the interior of M. Recall that if $\sigma: M^2 \to M^2$ is any homeomorphism with isolated fixed points $\{x_1, \ldots, x_s\}$, then $L(\sigma) = \sum_{i=1}^s I(\sigma, x_i)$, where $I(\sigma, x_i)$ is the fixed point index of x_i [1]. (If $x_i \in \partial M$, then $I(\sigma, x_i) = \frac{1}{2} I(D\sigma, x_i)$ where $D\sigma$ is the homeomorphism on the double of M which agrees with σ on each copy of M.) Thus $\sum_{i=1}^m I(\tau^n, x_i) < 0$.

For each x_i there is a (nonunique) $\tilde{x}_i$ covering x_i and a lift t_i of τ^n which fixes $\tilde{x}_i$. If t_i and t_j can be chosen so that $t_i = t_j$, we say that x_i and x_j are Nielsen equivalent; we write $\langle x_i \rangle$ for the Nielsen class containing x_i. If t_i does not commute with any nontrivial covering translations, then covering projection induces a one-to-one index preserving correspondence between $\operatorname{Fix}(t_i \mid H)$ and $\langle x_i \rangle$. In this case we say that $\langle x_i \rangle$ is nonperipheral.

We first consider the case in which there is a nonperipheral Nielsen class, say $\{x_1, \ldots, x_k\}$, such that $\sum_{i=1}^k I(\tau^n, x_i) < 0$. Let $t = t_1$. Then

$$\begin{aligned}
\sum_{\operatorname{Fix}(t \mid S_\infty)} I(t, \cdot) &= L(t) - \sum_{\operatorname{Fix}(t \mid H)} I(t, \cdot) \\
&= \chi(D^2) - \sum_{i=1}^k I(t, \tilde{x}_i) \\
&= \chi(D^2) - \sum_{i=1}^k I(\tau^n, x_i) > 1.
\end{aligned}$$

Since $\operatorname{Fix}(t \mid S_\infty)$ cannot contain the endpoints of an axis, it suffices to show that $\delta^s(t)$ and $\delta^u(t)$ contain at least two points. This reduces to showing that if $z \in S_\infty$ is an isolated fixed point of t which is a (source, sink, saddle node) for $t \mid S_\infty$, then $I(t, z) = (\frac{1}{2}, \frac{1}{2}, 0 \text{ or } 1)$.

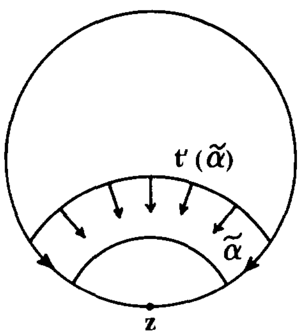
The key to computing $I(t, z)$ is the observation [5] that for any $K>0$, there is a neighborhood N of z such that t moves each $\tilde{y} \in N \cap H$ by a hyperbolic distance greater than K. To prove this, suppose that there exists $\tilde{y}_i \to z$ and $K>0$ such that hyp. dist. $(t\tilde{y}_i, \tilde{y}_i) < K$ for all i. Since t is uniformly continuous there is no loss in assuming that the $\tilde{y}_i$'s are all lifts of some $y \in M$. Let $f_i \in \tilde{\pi}$ satisfy $\tilde{y}_i = f_i \tilde{y}_1$. Then

$$\begin{aligned}
\text{hyp. dist.}(f_i^{-1} t f_i \tilde{y}_1, \tilde{y}_1) &= \text{hyp. dist.}(t f_i \tilde{y}_1, f_i \tilde{y}_1) \\
&= \text{hyp. dist.}(t\tilde{y}_i, \tilde{y}_i) < K \text{ for all } i.
\end{aligned}$$

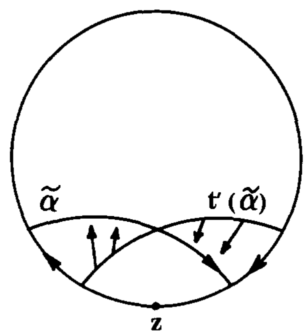
Hence there exists $i \neq j$ such that $f_i^{-1} t f_i = f_j^{-1} t f_j$. This implies that $t(f_i f_j^{-1}) = (f_i f_j^{-1})t$, which contradicts the assumption that t commutes with no nontrivial covering translations.

Choose a simple closed geodesic curve or arc α and lifts $\tilde{\alpha}_i$ determining decreasingly small neighborhoods N_i of z as in Lemma 1.1. Lifting an isotopy of $\tau(\alpha)$ to $\tau_*(\alpha)$, we see that for all $i \in \mathbb{Z}$ there is an isotopy of $t(\tilde{\alpha}_i)$ to $t_*(\tilde{\alpha}_i)$ which moves no point more than a uniform hyperbolic distance C. We may assume without loss that t moves each $\tilde{y} \in N_1 \cap H$ by a hyperbolic distance greater than $3C$ and that $N_1 \supset N_2 \cup t(N_2)$. In particular, there is a homotopy of $t_1 = t \mid N_2$ through maps $t_u \colon N_2 \to N_1$ whose only fixed point is z, such that the resulting map $t_0 = t' \colon N_2 \to N_1$ satisfies $t'(\tilde{\alpha}_2) = t_*(\alpha_2)$.

It follows [1] that $I(t, z) = I(t', z)$. The desired calculation of $I(t', z)$ is immediate from the definition of the index.



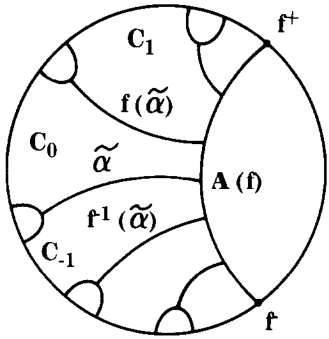
$\mathbf{I(t', z) = 1/2}$



$\mathbf{I(t', z) = 1}$

We now consider the case that $\sum_{j=1}^{r} I(\tau^n, x_j) < 0$ for some peripheral Nielsen class $\{x_1, ..., x_r\}$. There is an indivisible peripheral covering translation f which commutes with $t = t_1$. Thus $\{f^{\pm}\} \subset \mathrm{Fix}(t \mid S_\infty)$. If $\mathrm{Fix}(t \mid S_\infty) \neq \{f^{\pm}\}$, then $\delta^s(t)$ and $\delta^u(t)$ are infinite and we are done; suppose then that $\mathrm{Fix}(t \mid S_\infty) = \{f^{\pm}\}$. This implies that t commutes with $g \in \tilde{\pi}$ if and only if g is an iterate of f, and hence that $\mathrm{Fix}(t) = \{f^{\pm}\} \cup \bigcup_{l=-\infty}^{\infty} \bigcup_{j=1}^{r} f^l(\tilde{x}_j)$.

Choose a geodesic arc α which has at least one endpoint on $[f] \subset \partial M$ and which is disjoint from $\mathrm{Fix}(\tau)$. Let $\tilde{\alpha}$ be a lift of α which has one endpoint on $A(f)$, let C_l be the region of $H \cup S_\infty$ between $f^l(\tilde{\alpha})$ and $f^{l+1}(\tilde{\alpha})$, and let $D_l = \{f^+\} \cup \bigcup_{i=l}^{\infty} C_i$. We may assume that $C_l \cap \mathrm{Fix}(t) = \bigcup_{j=1}^{r} f^l(\tilde{x}_j)$. Since $t \mid D_l$ is conjugate to $t \mid D_{l+1}$, the total index contribution of D_l equals that of D_{l+1}. This implies that the index contribution of C_l is zero, in contradiction to the assumption that $\sum_{\mathrm{Fix}(t \mid C_l)} I(t, \cdot) = \sum_{j=1}^{r} I(\tau^n, x_j) < 0$. ∎

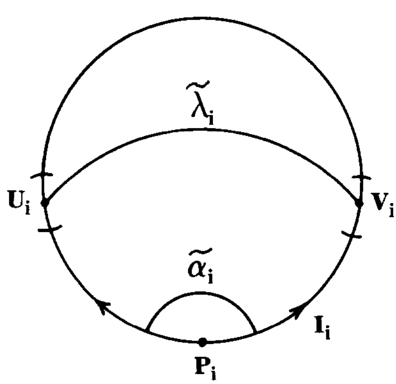


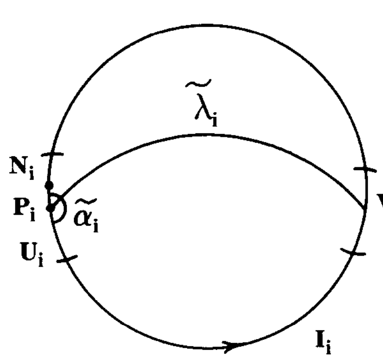
THEOREM 3.2. *Λ^s and Λ^u are τ_*-invariant geodesic laminations.*

Proof. We consider only Λ^s, the argument for Λ^u being similar. For each $f \in \tilde{\pi}$ and $t \in T$, $f(\delta^s(t)) = \delta^s(ftf^{-1})$. Moreover, if t_0 is a lift of τ, then $t_0\,\delta^s(ft_0^n) = \delta^s(t_0 ft_0^{n-1})$. Thus $\tilde{\Lambda}^s$ is invariant under the action of $\tilde{\pi}$ and T_*, and Λ^s is invariant under the action of τ_*.

Suppose that $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are geodesics in $\tilde{\Lambda}^s$ which intersect transversely. Then the endpoints of $\tilde{\lambda}_1$ are separated in S_∞ by the endpoints of $\tilde{\lambda}_2$. Choose neighborhoods U_i and V_i of the endpoints of $\tilde{\lambda}_i$ such that U_1 and V_1 are separated in S_∞ by U_2 and V_2. If $\tilde{\gamma}_i$ $(i=1,2)$ are any geodesics with endpoints in U_i and V_i, then $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ intersect transversely. We may therefore assume that $\tilde{\lambda}_i$ is a component of some $\partial\tilde{\Delta}^s(t_i)$, $i=1,2$.

Let I_i be an interval in S_∞ whose endpoints are those of $\tilde{\lambda}_i$ and which contains no other points in $\delta^s(t_i)$. There are two cases to consider. If I_i contains a fixed point P_i of t_i in its interior, then P_i is a source and t_i move all points of int $I_i - P_i$ from P_i toward ∂I_i. If the interior of I_i contains no fixed points of t_i, then t_i moves all points of int I_i from one point in ∂I_i toward the other. Let P_i be the repelling endpoint of ∂I_i and assume that $P_i \subset U_i$. Since P_i is not an isolated source, there is a point $N_i \in U_i$ which is not moved away from P_i by the action of t_i.





Lemma 1.1 implies that there are disjoint or equal simple closed geodesic curves or arcs α_i and lifts $\tilde{\alpha}_i$ determining small neighborhoods of P_i. In the former case we assume that $\partial\tilde{\alpha}_i \subset I_i$ and in the latter case we assume that $\partial\tilde{\alpha}_i \subset U_i$ and separates N_i from P_i. For sufficiently large n, $\tilde{\gamma}_i = (t_i)^n_*(\tilde{\alpha}_i)$ has endpoints in U_i and V_i. Since $\tilde{\gamma}_1$ are $\tilde{\gamma}_2$ are lifts of $\tau^n_*(\alpha_1)$ and $\tau^n_*(\alpha_2)$, respectively, they do not intersect transversely. This contradicts our choice of U_i and V_i and proves that Λ^s is a union of disjoint simple geodesics.

To prove that Λ^s is closed, it suffices to show that ∂M is not contained in $\bar{\Lambda}^s$. Each lift of a component of ∂M is the axis $A(f)$ of a covering translation f. Lemma 1.3 implies that $A(f)$ cannot share a single endpoint with $\tilde{\lambda} \subset \partial\tilde{\Delta}^s(t_i)$ for any $t_i \in T$. If $\lambda \subset \Lambda^s$ had endpoints close to but disjoint from those of $A(f)$, then $f(\tilde{\lambda}) \cap \tilde{\lambda} \neq \varnothing$, contradicting the fact that $\tilde{\lambda}$ is simple. Together these imply that $A(f) \not\subset \bar{\tilde{\Lambda}}^s$. ▮

4. Further Properties of Λ^s and Λ^u and the Construction of φ

The first part of this section completes our picture of Λ^s and Λ^u; we show that they are minimal and transverse and fully describe their complementary components. The second part of this section is the construction (as given in [3]) of an equivariant homeomorphism $\tilde{\varphi}: H \cup S_\infty \to H \cup S_\infty$ which agrees with $t \mid S_\infty$ for some fixed lift t of τ and which preserves $\tilde{\Lambda}^s$ and $\tilde{\Lambda}^u$. The induced homeomorphism $\varphi: M \to M$ is isotopic to τ and preserves Δ^s and Δ^u.

We begin with some general results about laminations.

Lemma 4.1. *Le $\Lambda \subset M$ be a geodesic lamination. Then*:

(i) *$M - (\Lambda \cup \partial M)$ has finitely many components, $\Delta_1, \ldots, \Delta_l$, each being the* 1–1 *immersed image of the interior of a hyperbolic surface* (*with geodesic boundary*) *S_i with finite area.*

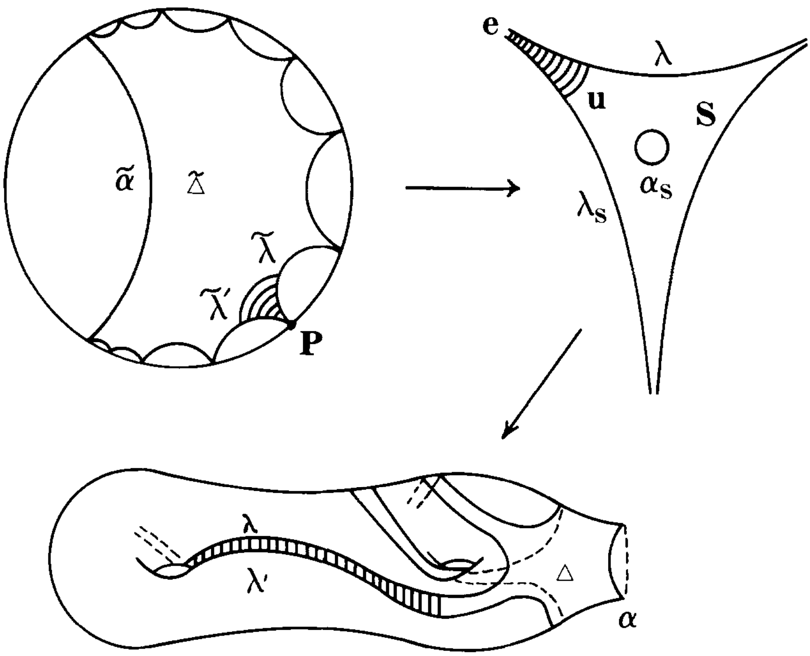
(ii) int $\Lambda = \varnothing$.

(iii) *Λ contains only finitely many sublaminations.*

(iv) *If Λ is τ^k_*-invariant for some $k > 0$ and if $\Lambda \cap \partial M = \varnothing$, then each S_i is an asymptotic polygon or a crown* (= *an asymptotic polygon with an open disk removed from its interior*); *the crowns are in* 1–1 *correspondence with the components of ∂M.*

Proof. (i) Let Δ be a component of $M - (\Lambda \cup \partial M)$, $\tilde{\Delta} \subset H$ the closure of a component of $H - (\tilde{\Lambda} \cup \partial\tilde{M})$ which projects onto Δ, and $\delta \subset S_\infty$ the vertex set for $\tilde{\Delta}$. The projection of $\tilde{\Delta}$ into M factors through the hyperbolic surface (with geodesic boundary) $S = \tilde{\Delta}/\{f \in \tilde{\pi}: f(\tilde{\Delta}) = \tilde{\Delta}\}$ and induces a

1–1 area preserving immersion of int S onto Δ. The Gauss–Bonnet theorem implies that the area of S is bounded below and that the number of boundary components of S is bounded above. In particular, $M-(\Lambda \cup \partial M)$ has finitely many components. ∎



Let $\tilde{\lambda}$ ve any leaf in $\partial\tilde{\Delta}$ that is not an axis and let P be one of its endpoint. Then $\tilde{\lambda}$ projects to a noncompact component λ_S of ∂S and P determines an end e of S. A neighborhood U of e intersects ∂S in two asymptotic half-infinite geodesics. As U is contractible and $\tilde{\Delta}$ is the universal cover of S, there is a neighborhood $\tilde{U}$ of P in $\tilde{\Delta}$ which intersects $\partial\tilde{\Delta}$ in half-infinite segments of two geodesics $\tilde{\lambda}$ and $\tilde{\lambda}'$, which have P as an endpoint. Thus P is isolated in δ. Moreover (since both endpoints of $\tilde{\lambda}$ are isolated in δ), $\tilde{\lambda}$ is not the limit, on its int $\tilde{\Delta}$-side, of geodesics which are disjoint from $\partial\tilde{\Delta}$. We use these observations in proving (ii)–(iv).

(ii) Given a geodesic lamination Λ' with nonempty interior, let $\Lambda=\Lambda'-\mathrm{int}(\Lambda')$. Then each component of int Λ' is also a component of $M-(\Lambda \cup \partial M)$; let Δ be such a component. Since $\tilde{\Lambda}' \mid \tilde{\Delta}$ fills up $\tilde{\Delta}$, each $\lambda \subset \partial\tilde{\Delta}$ must be an axis. This implies that S is a compact surface with $\chi(S)<0$ in contradiction to the hypothesis that S supports the nonsingular line field generated by Δ'_S, the projected image of Λ'. ∎

(iii) Let $\Lambda' \subset \Lambda$ be a sublamination and let $\tilde{\lambda}' \subset \partial\tilde{\Delta}'_i$ for some component Δ_j' of $M-(\Lambda' \cup \partial M)$. If $\tilde{\lambda}'$ is not an axis, then it is isolated in Λ on at least one side (since no leaf of Λ can intersect $\partial\Delta_j'$ transversely). In particular, $\lambda' \subset \bigcup_{i=1}^{l} \partial\Delta_i$. Now Λ' is determined by $\bigcup_{j=1}^{l'} \partial\Delta_j'$ and $\Lambda' \cap \partial M$ and hence by a subset of $\bigcup_{i=1}^{l} \partial\Delta_i \cup \{\text{closed geodesics in } \Lambda\}$. As this union is finite, there are only finitely many Λ'. ▮

(iv) Suppose that Λ is τ_*-invariant (the $k>1$ case is similar) and that $\Lambda \cap \partial M = \varnothing$; in particular each component of ∂M is contained in the frontier of some Δ_i. It suffices to show that each S is an asymptotic polygon or peripheral crown.

As $\Delta \neq M$, $\partial\tilde{\Delta}$ contains a leaf $\tilde{\lambda}$ which is not an axis. Since each t_* permutes the components and boundary components of $H-(\tilde{\Lambda} \cup \partial\tilde{M})$ and since there are only finitely many such up to covering translation, there exists $t \in T$ such that $t_*(\tilde{\Delta})=\tilde{\Delta}$ and $t_*(\tilde{\lambda})=\tilde{\lambda}$. The endpoints $\{Q_0, Q_1\}$ of $\tilde{\lambda}$ are isolated in δ. The leaves of $\partial\tilde{\Delta}$ which share an endpoint with $\tilde{\lambda}$ must also be nonaxes and thus their endpoints are also isolated in δ. Arguing inductively, se see that either δ is finite or δ contains an infinite collection $\{Q_i\}$ of adjacent isolated points, all of which must be fixed by t. In the former case, $S=\tilde{\Delta}$ is an asymptotic polygon. In the latter case, there is an indivisible covering translation f such that $f(\tilde{\Delta})=\tilde{\Delta}$ and $f(Q_0)=Q_k$ for some $k>0$. It follows that $f(Q_i)=Q_{i+k}$ for all i and hence that f fixes P and R, the first nonisolated point of δ to the left and right of Q_0, respectively. Note that f^+ and f^- are both nonisolated points in δ so that P and R are necessarily distinct; clearly $\{P, R\}=\{f^\pm\}$. Since t also fixes P and R, $A(f)$ is a peripheral axis and $S=\tilde{\Delta}/\{f\}$ is a peripheral crown. ▮

In characterizing Λ^s and Λ^u we will need the following (Nielsen style) result.

LEMMA 4.2. *If* $\mathrm{Fix}(t_1 \mid S_\infty)$ *and* $\mathrm{Fix}(t_2 \mid S_\infty)$ *are distinct but non disjoint, then one of these sets equals* $\{f^\pm\}$ *for some peripheral* $f \in \tilde{\pi}$.

Proof. Suppose that t_i is a lift of τ^{n_i}. Then $f=t_1^{n_2}t_2^{-n_1}$ is a nontrivial covering translation which fixes the nonempty set $\mathrm{Fix}(t_1 \mid S_\infty) \cap \mathrm{Fix}(t_2 \mid S_\infty)$. Lemma 1.3 implies $\mathrm{Fix}(t_1 \mid S_\infty) \cap \mathrm{Fix}(t_2 \mid S_\infty)=\{f^\pm\}$ and (by irreducibility) that f is peripheral. In particular, $\mathrm{Fix}(t_1 \mid S_\infty)-\{f^\pm\}$ and $\mathrm{Fix}(t_2 \mid S_\infty)-\{f^\pm\}$ lie in the same component of $S_\infty-\{f^\pm\}$. If neither of these sets is empty, then the argument given in Lemma 2.2 to show that $\mathrm{Fix}(t \mid S_\infty)=\mathrm{Fix}(t' \mid S_\infty)$ applies here to show that $\mathrm{Fix}(t_1 \mid S_\infty)=\mathrm{Fix}(t_2 \mid S_\infty)$. ▮

The following Proposition gives a detailed description of Λ^s and Λ^u.

PROPOSITION 4.3. (i) *Λ^s and Λ^u are transverse minimal laminations which intersect every nonperipheral closed curve in M.*

(ii) *{components of* $M-(\Lambda^s \cup \partial M)\} = \{\text{int}\, \Delta^s(t)\colon t\in T\}$; *similarly for* Λ^u. *For each* $t \in T$, *either* $\text{Fix}(t \mid S_\infty)$ *is the endpoint set of a peripheral axis, or the isolated points of* $\text{Fix}(t \mid S_\infty)$ *are sources and sinks.*

Proof. (ii) Lemma 4.2 implies that each nonempty int $\tilde{\Delta}^s(t)$ is a component of $H-(\tilde{\Lambda}^s \cup \partial\tilde{M})$. To prove the converse, suppose that $\tilde{\Delta}$ is a component of $H-(\tilde{\Lambda}^s \cup \partial\tilde{M})$ with vertex set δ. While proving Lemma 4.1(iv) we showed that $\delta \subset \text{Fix}(t \mid S_\infty)$ for some $t \in T$. We now show that neither δ nor $\delta^s(t)$ contains adjacent points $P, Q \in \text{Fix}(t \mid S_\infty)$ unless P and Q bound a peripheral axis.

Suppose to the contrary. Then the geodesic $\tilde{\lambda}$ connecting P and Q belongs to $\tilde{\Lambda}^s$. Let I be the interval in S_∞ such that $I \cap \text{Fix}(t \mid S_\infty) = \partial I \cap \text{Fix}(t \mid S_\infty) = \{P, Q\}$. Then t moves all points in I from one endpoint of I toward the other. This implies that any geodesic which is close to $\tilde{\lambda}$ and which has endpoints in int I, intersects its t_*-image. In particular no such geodesic can be in $\tilde{\Lambda}^s$. Lemma 4.2 implies that there are no leaves of $\tilde{\Lambda}^s$ which are close to $\tilde{\lambda}$ and which have one endpoint in int I. Thus $\tilde{\lambda} \subset \partial\tilde{\Delta}'$ for some component $\tilde{\Delta}'$ of $H-(\tilde{\Lambda}^s \cup \partial\tilde{M})$ whose vertex set δ' is contained in I. As above, $\delta' \subset \text{Fix}(t' \mid S_\infty)$ for some $t' \in T$. By Lemma 4.2, $\text{Fix}(t' \mid S_\infty) = \text{Fix}(t \mid S_\infty)$ in contradiction to our choice of I.

As $\text{Fix}(t \mid S_\infty)$ is not the endpoint set of a peripheral axis, the isolated points of $\text{Fix}(t \mid S_\infty)$ must all be sources and sinks. Since $\partial\tilde{\Delta}^s(t)$ and $\partial\tilde{\Delta}$ cannot intersect transversely, $\delta \subset \delta^s(t)$. This implies that the components int $\tilde{\Delta}$ and int $\tilde{\Delta}^s(t)$ are nondisjoint and hence equal (Lemma 4.2). ∎

(i) Let $\Lambda' \subset \Lambda^s$ be a sublamination. Lemma 4.1(iii) implies that Λ' is τ_*^k-invariant for some $k>0$. The proof of Lemma 4.3(ii) applies to (Λ', τ^k) and shows that $\{\text{components of } M-(\Lambda' \cup \partial M)\} = \{\text{int}\, \Delta^s(t');\ t' \text{ is a lift of } \tau^{nk} \text{ for some } n>0\}$. Since this latter collection of sets equals $\{\text{int}\, \Delta^s(t);\ t \in T\}$, Λ and Λ' have identical complements and must therefore be equal.

Transversality of Λ^s and Λ^u is a consequence of their being distinct and minimal. If $\tilde{\lambda}$ is a geodesic which is disjoint from $\tilde{\Lambda}^s$, then its endpoints are contained in a vertex set $\delta^s(t)$ for some $t \in T$. If λ is closed, then by Lemma 1.3 and irreducibility, it is peripheral. ∎

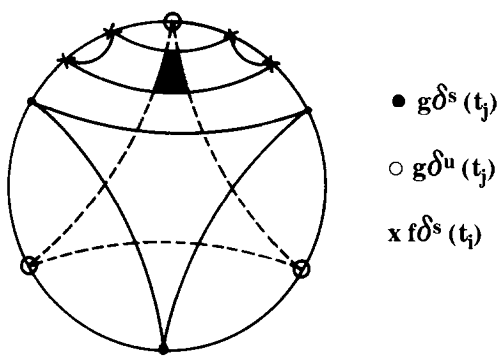
Remark 4.4. Before turning to the construction of $\tilde{\varphi}$ as described in [3], we list two additional properties of $\tilde{\Lambda}^s$ and $\tilde{\Lambda}^u$ which are used in that construction. We use the convention that $\tilde{\lambda} \subset \tilde{\Lambda}^s$ and $\tilde{\gamma} \subset \tilde{\Lambda}^u$.

4.4(i) If $\tilde{\lambda} \cap \tilde{\gamma}$ is close to $P \in S_\infty$, then either $\tilde{\lambda}$ or $\tilde{\gamma}$ is close to P.

Proof. Since $\tilde{\Lambda}^s$ and $\tilde{\Lambda}^u$ have no common leaves there is a lower bound to the angle of intersection between $\tilde{\lambda}$ and $\tilde{\gamma}$. ∎

4.4(ii) There exist $t_1, \ldots, t_m \in T$ such that each component $\tilde{C}$ of $H-(\tilde{\Lambda}^s \cup \tilde{\Delta}^u \cup \partial\widetilde{M})$ equals $f\tilde{\Delta}^s(t_i) \cap g\tilde{\Delta}^u(t_j)$ for some $f, g \in \tilde{\pi}$ and $1 \leqslant i, j \leqslant m$. If $i \neq j$ or $f \neq g$, then $\tilde{C}$ is a quadrilateral.

Proof. The first part is just a restatement of Lemma 4.3(ii). Suppose that $i \neq j$ or $f \neq g$. Then $f\delta^s(t_i)$ lies in a single component of $S_\infty - g\delta^s(t_j)$. As this component contains only one point of $g\delta^u(t_i)$, $\tilde{C}$ is a quadrilateral.



▮

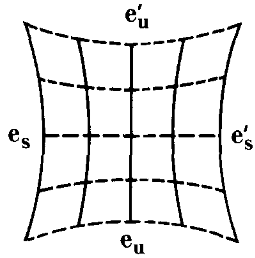
We now present the construction of $\tilde{\varphi}$; for further details, see [3]. Fix a lift t of τ and let $\tilde{\varphi} \mid S_\infty = t \mid S_\infty$. We extend $\tilde{\varphi}$ over successive skeleta of the decomposition of H given by $\tilde{\Lambda}^s \cup \tilde{\Lambda}^u \cup \partial\widetilde{M}$.

If $\tilde{\lambda} \subset \tilde{\Lambda}^s$ and $\tilde{\gamma} \subset \tilde{\Lambda}^u$ intersect, then so do $t_*(\tilde{\lambda})$ and $t_*(\tilde{\gamma})$. As these intersections are single points, we may define $\tilde{\varphi} \mid \tilde{\Lambda}^s \cap \tilde{\Lambda}^u$ by $\tilde{\varphi}(\tilde{\lambda} \cap \tilde{\gamma}) = t_*(\tilde{\lambda}) \cap t_*(\tilde{\gamma})$. Injectivity and continuity on $\tilde{\Lambda}^s \cap \tilde{\Lambda}^u$ are easy to verify. Continuity of $\tilde{\varphi} \mid S_\infty \cup (\tilde{\Lambda}^s \cap \tilde{\Lambda}^u)$ at $P \in S_\infty$ follows from Remark 4.4(i).

Let $\tilde{C}$ be a component of $H-(\tilde{\Lambda}^s \cup \tilde{\Lambda}^u \cup \partial\widetilde{M})$. An edge e_s of $\partial\tilde{C}$ that is contained in $\tilde{\Lambda}^s$ is a segment of some $\tilde{\lambda}$ and is bounded by $\tilde{\lambda} \cap \tilde{\gamma}_1$ and $\tilde{\lambda} \cap \tilde{\gamma}_2$ for some $\tilde{\gamma}_1, \tilde{\gamma}_2 \subset \tilde{\Lambda}^u$. Define $\tilde{\varphi}$ on e_s so that $\tilde{\varphi}(e_s)$ is the subinterval of $t_*(\tilde{\lambda})$ bounded by $t_*(\tilde{\lambda}) \cap t_*(\tilde{\gamma}_1)$ and $t_*(\tilde{\lambda}) \cap t_*(\tilde{\gamma}_2)$ and so that $\tilde{\varphi}$ uniformly expands distances on e_s. Define $\tilde{\varphi}$ on an edge $e_u \subset \tilde{\Lambda}$ similarly. To complete the definition of $\tilde{\varphi}$ on the 1-skeleton, define $\tilde{\varphi} \mid \partial\widetilde{M}$ to be equivariant and a homeomorphism on each component.

To extend $\tilde{\varphi}$ over a quadrilateral $\tilde{C} = f\tilde{\Delta}^s(t_i) \cap g\tilde{\Delta}^u(t_j)$ ($f \neq g$ or $i \neq j$) we coordinatize $\tilde{C}$ as follows. Let $e_s \subset \tilde{\lambda}$ and $e'_s \subset \tilde{\lambda}'$ (resp. $e_u \subset \tilde{\gamma}$ and $e'_u \subset \tilde{\gamma}'$) be the edges of $\partial\tilde{C}$ in $\tilde{\Lambda}^s$ (resp. $\tilde{\Lambda}^u$) and let l_s and l'_s (resp. l_u and l'_u) be their lengths. Define $\tilde{E}^u(\tilde{C})$ to be the family of geodesics whose endpoints are at a distance αl_s and $\alpha l'_s$ from $e_s \cap e_u$ and $e'_s \cap e_u$, respectively, $0 \leqslant \alpha \leqslant 1$. Define $\tilde{E}^s(\tilde{C})$ similarly.

Let $\tilde{\varphi} \mid \tilde{C}$ be the unique homeomorphism which agrees with $\tilde{\varphi} \mid \partial C$ and which maps geodesics in $\tilde{E}^s(\tilde{C})$ and $\tilde{E}^u(\tilde{C})$ to geodesics in $\tilde{E}^s(t_*\tilde{C})$ and



$\tilde{E}^s(t_*\tilde{C})$, where t_*C is the quadrilateral in $H-(\tilde{\Lambda}^s\cup\tilde{\Lambda}^u\cup\partial\tilde{M})$ which is bounded by subintervals of $t_*(\tilde{\lambda})$, $t_*(\tilde{\lambda}')$, $t_*(\tilde{\gamma})$, and $t_*(\tilde{\gamma}')$.

Finally, extend $\tilde{\varphi}$ over $\bigcup_{f\in\tilde{\pi}}\bigcup_{i=1}^{m} f\tilde{\Delta}^s(t_i)\cap f\tilde{\Delta}^u(t_i)$ so that it is equivariant and so that is a homeomorphism on each component. This completes the definition of $\tilde{\varphi}$. It is straightforward to check that $\tilde{\varphi}$ is continuous and hence a homeomorphism. ∎

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